Stochastic solutions of nonlinear pde's: McKean versus superprocesses

Rui Vilela Mendes

Centro de Matemática e Aplicações, UL, Lisbon

Instituto de Plasmas e Fusão Nuclear, IST, Lisbon

(http://label2.ist.utl.pt/vilela/)

FCT, EURATOM (Mobility), CPT (CNRS)

Stochastic solutions

- **Stochastic solution** = a stochastic process which, started from a particular point in the domain, generates after time t a boundary measure which, integrated over the initial condition at $t = 0$, provides the solution at x and time t.
- **Example**: the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad \text{with} \qquad u(0,x) = f(x)$$

the process is Brownian motion, $dX_t = dB_t$, and the solution

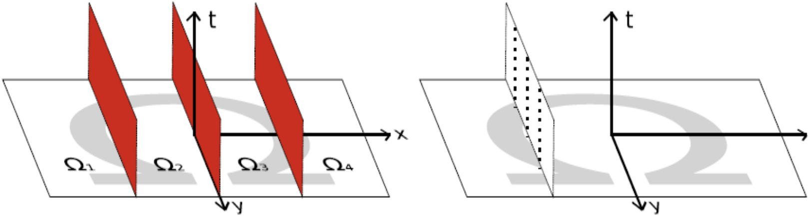
$$u(t,x) = \mathbb{E}_x f(X_t) \tag{1}$$

- The domain here is $\mathbb{R} \times [0,t)$ and the expectation value in (1) is the inner product $\langle \mu_t, f \rangle$ of the initial condition f with the *measure* μ_t *generated by the Brownian motion at the* $t-$*boundary*.
- The process should be the same for any initial condition.
- Classical results for linear pde's. Recent work in nonlinear pde's: KPP, Navier-Stokes, Poisson-Vlasov, MHD, etc.

Stochastic solutions: What are they good for?

- New **exact solutions**
- New **numerical algorithms**
  Deterministic algorithms grow exponentially with the dimension d of the space, roughly N^d ($\frac{L}{N}$ the linear size of the grid). The stochastic process only grows with the dimension d.
- Provide **localized solutions**
- Sample paths started from the same point are independent. Likewise, paths starting from different points are independent from each other.
  The stochastic algorithms are a natural choice for parallel and distributed computation.
- Stochastic algorithms handle equally well regular and complex boundary conditions.
- **Domain decomposition** using interpolation of localized stochastic solutions and then, in each small domain, a deterministic code. Avoids the communication time problem. Fully parallel.

Stochastic solutions and domain decomposition



Number of processors	PDD			ScaLAPACK
	T_{MC}	T_{INTERP}	T_{TOTAL}	T_{TOTAL}
64	2' 17"	<1"	3' 37"	-
128	2' 18"	<1"	2' 23"	7510' 20"
256	2' 18"	<1"	2' 18"	5223' 41"

(J. Acebrón, A. Rodríguez-Rozas, R. Spigler)

Stochastic solutions: Two construction methods

- **McKean's method:** a probabilistic version of the Picard series.
  First the differential equation is written as an integral equation and rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition
  Then the Picard iteration is interpreted as an evolution and branching process
  The stochastic solution is equivalent to importance sampling of a normalized Picard series.
- **The method of superprocesses:** constructs the boundary measures of a measure-valued stochastic process and obtain the solutions of the differential equation by a scaling procedure.
- Comparison of the two methods and generalization of superprocesses (to signed measures and distribution-valued processes).

The KPP equation: McKean's formulation

- $$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + v^2 - v$$

  and initial data $v\left(0, x\right) = g\left(x\right)$
- $G\left(t, x\right) =$ Green's operator for heat equation $\partial_t v(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v(t,x)$

  $$G\left(t, x\right) = e^{\frac{1}{2} t \frac{\partial^2}{\partial x^2}}$$
- KPP in integral form

  $$v\left(t, x\right) = e^{-t} G\left(t, x\right) g\left(x\right) + \int_0^t e^{-(t-s)} G\left(t-s, x\right) v^2\left(s, x\right) ds \quad (2)$$

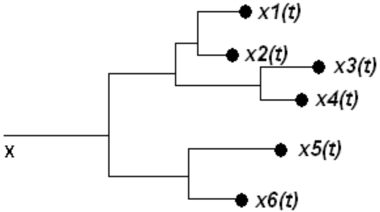
  Denoting by $\left(\xi_t, \Pi_x\right)$ a Brownian motion starting from time zero and coordinate x, Eq.(2) may be rewritten

  $$\begin{aligned} v\left(t, x\right) &= \Pi_x \left\{ e^{-t} g\left(\xi_t\right) + \int_0^t e^{-(t-s)} v^2\left(s, \xi_{t-s}\right) ds \right\} \\ &= \Pi_x \left\{ e^{-t} g\left(\xi_t\right) + \int_0^t e^{-s} v^2\left(t-s, \xi_s\right) ds \right\} \end{aligned}$$

The KPP equation: McKean's formulation

- **The stochastic solution process:** a Brownian motion plus branching process with exponential holding time T, $P(T > t) = e^{-t}$. At each branching point the particle splits into two, the new particles going along independent Brownian paths. At time $t > 0$ one has n particles located at $x_1(t), x_2(t), \cdots x_n(t)$. The solution is obtained by

$$v(t,x) = \mathbb{E}\left\{g(x_1(t))\, g(x_2(t)) \cdots g(x_n(t))\right\}$$



- An equivalent interpretation: a backwards-in-time process from time t at x. When it reaches $t = 0$ samples the initial condition. Generates a measure at the $t = 0$ boundary which is applied to $g(x) = v(0,x)$.

Superprocesses: Branching exit measures

- $(E, \mathcal{B})$ a measurable space, $M_+(E)$ the space of finite measures in E and $\left(X_t, P_{0\,,\mu}\right)$ a branching stochastic process with values in $M_+(E)$ and transition probability $P_{0,\mu}$ starting from time 0 and measure μ.
- The process satisfies a *branching property* if given $\mu = \mu_1 + \mu_2$

$$P_{0,\mu} = P_{0,\mu_1} * P_{0,\mu_2}$$

that is, after the branching, $\left(X_t^1, P_{0,\mu_1}\right)$ and $\left(X_t^2, P_{0\,,\mu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $\left(X_t, P_{r0,\mu}\right)$.
- For the *transition operator* V_t operating on functions on E this is

$$V_t f\left(\mu_1 + \mu_2\right) = V_t f\left(\mu_1\right) + V_t f\left(\mu_2\right)$$

where $e^{-\langle V_t f, \mu\rangle} \circeq P_{0,\mu} e^{-\langle f, X_t\rangle}$ or

$$V_t f(\mu) = -\log P_{0,\mu} e^{-\langle f, X_t\rangle}$$

V_t is called the *log-Laplace semigroup* associated to X_t. If the initial measure μ is δ_x one writes

$$V_t f(x) = -\log P_{0,x} e^{-\langle f, X_t\rangle}$$

Superprocesses: Branching exit measures

- In $S = [0,\infty) \times E$ consider a set $Q \subset S$ and the associated branching exit process (X_Q, P_μ) composed of a propagating Markov process in E, $\xi = (\xi_t, \Pi_{0,x})$, a set of probabilities $p_n(t,x)$ describing the branching and a parameter k defining the lifetime.

$$u(x) = V_Q f(x) = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \tag{3}$$

  $\langle f, X_Q \rangle$ is the integral of the function f on the (space-time) boundary with the boundary exit measure generated by the process.

- This branching exit process is a (ξ, ψ) *−superprocess* if $u(x)$ satisfies the equation

$$u + G_Q \psi(u) = K_Q f \tag{4}$$

  where G_Q is the Green operator,

$$G_Q f(r,x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s)\, ds$$

  K_Q the Poisson operator

$$K_Q f(x) = \Pi_{0,x} 1_{\tau<\infty} f(\xi_\tau)$$

Superprocesses: The construction

Let $\varphi(s,x;z)$ be the offspring generating function at time s and point x

$$\varphi(s,x;z) = c\sum_0^\infty p_n(s,x)z^n$$

where $\sum_n p_n = 1$ and c denotes the branching intensity.

$$\begin{aligned} P_{0,x}e^{-\langle f,X_Q\rangle} &\stackrel{\circ}{=} \\ e^{-w(0,x)} &= \Pi_{0,x}\left[e^{-k\tau}e^{-f(\tau,\xi_\tau)} + \int_0^\tau dske^{-ks}\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right)\right] \end{aligned} \tag{5}$$

The measure-valued process starts from δ_x at time 0, τ is the first exit time from Q and $f(\tau,\xi_\tau)$ the value of a function in the boundary ∂Q. Using $\int_0^\tau ke^{-ks}ds = 1-e^{-k\tau}$ and the Markov property $\Pi_{0,x}1_{s<\tau}\Pi_{s,\xi_s} = \Pi_{0,x}1_{s<\tau}$, Eq.(5) for $e^{-w(0,x)}$ is converted into

$$e^{-w(0,x)} = \Pi_{0,x}\left[e^{-f(\tau,\xi_\tau)} + k\int_0^\tau ds\left[\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right) - e^{-w(\tau-s,\xi_s)}\right]\right]$$

Replace $w(0,x)$ by $\beta w_\beta(0,x)$ and f by βf. β may be interpreted as the mass of the particles and when the measure-valued process $X_Q \longrightarrow \beta X_Q$ then $P_\mu \longrightarrow P_{\frac{\mu}{\beta}}$.

$$\begin{aligned} e^{-\beta w(0,x)} &= \Pi_{0,x}\Big[e^{-\beta f(\tau,\xi_\tau)} \\ &\quad + k_\beta \int_0^\tau ds\left[\varphi_\beta\left(s,\xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\Big] \end{aligned}$$

Defining

$$u_\beta = \left(1 - e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta = \left(1 - e^{-\beta f}\right)/\beta$$

$$\psi_\beta(r,x;u_\beta) = \frac{k_\beta}{\beta}\left(\varphi(r,x;1-\beta u_\beta) - 1 + \beta u_\beta\right)$$

Superprocesses: The limiting procedure

One obtains

$$u_\beta(0,x) + \Pi_{0,x} \int_0^\tau ds \psi_\beta(s, \xi_s; u_\beta) = \Pi_{0,x} f_\beta(\tau, \xi_\tau)$$

that is

$$u_\beta + G_Q \psi_\beta(u_\beta) = K_Q f_\beta \tag{6}$$

When $\beta \to 0$, $f \to f_\beta$ and if ψ_β goes to a well defined limit ψ then u_β tends to a limit u, solution of (4) associated to a superprocess. Also in the $\beta \to 0$ limit

$$u_\beta \to w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$$

If to reproduce with (6) the equation we want it must be $\beta \to 0$ and $k_\beta \to \infty$, the superprocess would correspond to a cloud of particles for which both the mass and the lifetime tend to zero.

The KPP equation as a superprocess

- The KPP equation

$$v(t,x) = e^{-t} G(t,x)\, g(x) + \int_0^t e^{-s} G(t-s,x)\, v^2(s,x)\, ds$$

is identical to

$$P_{0,x} e^{-\langle f, X_Q\rangle} \overset{\circ}{=} e^{-w(0,x)} =$$

$$= \Pi_{0,x}\left[e^{-k\tau} e^{-f(\tau,\xi_\tau)} + \int_0^\tau ds k e^{-ks} \varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right)\right]$$

with $k=1$, $e^{-w(0,x)} = v(\tau,x)$, $e^{-f(\tau,\xi_\tau)} = g(\xi_\tau)$,
$\varphi\left(s,\xi_s; e^{-w(\tau-s,\xi_s)}\right) = v^2(\tau-s,\xi_s)$.

- The McKean probabilistic construction corresponds to an intermediate step in the superprocess construction. Summing over the exit measure, the solution is

$$v(t,x) = e^{-\langle f, X_Q\rangle} = e^{-\sum_i f\left(\xi_{\tau_i}\right)} = e^{\sum_i \log g\left(\xi_{\tau_i}\right)} = \Pi_i g\left(\xi_{\tau_i}\right)$$

essentially the same as before.

The KPP equation as a superprocess

Let $u(t,x) = 1 - v(t,x)$, which satisfies the equations

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2 + u$$

$$u(t,x) + \Pi_x \int_0^t \left(u^2(t-s,\xi_s) - u(t-s,\xi_s)\right) ds = \Pi_x\left(1 - g(\xi_t)\right)$$

that is, for KPP, $\psi(0,x;u) = u^2 - u$

$$\begin{aligned} \psi_\beta(0,x;u_\beta) &= \frac{k_\beta}{\beta}\left(\varphi(0,x;1-\beta u_\beta) - 1 + \beta u_\beta\right) \\ &= \frac{k_\beta}{\beta}\left(c\sum p_n (1-\beta u_\beta)^n - 1 + \beta u_\beta\right) \\ &= \frac{k_\beta c}{\beta}\left(\beta^2 u_\beta^2 - \beta u_\beta\right) = u^2 - u \end{aligned}$$

with $p_n = \delta_{n,2}$. Therefore $c = \beta = 1$ and $k_\beta = 1$. That is, for KPP the superprocess is not a scaling limit. It coincides with the McKean process. In this case, because $\beta = 1$ instead of $\beta \to 0$, the solution is $(1 - e^{-W})$.

Superprocesses and nonlinear heat equation

With other limiting choices for β, stochastic solutions are constructed for other equations, in particular for equations without the natural Poisson clock which is present in the KPP equation. For example for

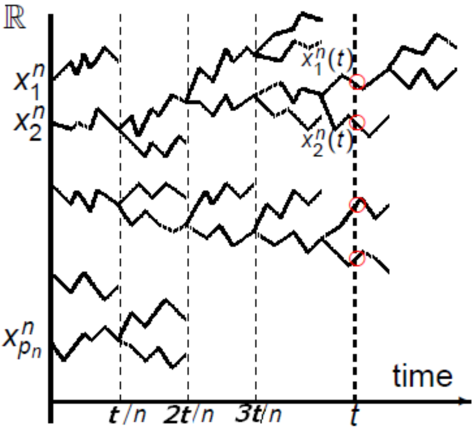
$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2$$

$$\psi\left(0, x; u\right) = u^2$$

$$\begin{aligned} \psi_\beta\left(0, x; u_\beta\right) &= \frac{k_\beta}{\beta}\left(\beta u_\beta - 1 + \sum_{n=0}^{2} p_0 + p_1\left(1-\beta u_\beta\right) + p_2\left(1-\beta u_\beta\right)^2\right) \\ &= u_\beta^2 \end{aligned}$$

leads to $p_1 = 0; \quad p_0 = p_2 = \frac{1}{2}; \quad k_\beta = \frac{2}{\beta}$
In this case, with $\beta \to 0$, the solution is given by (3) and the superprocess is a scaling limit $(n \to \infty$ in the figure$)$ where both mass and lifetime of the particles tend to zero and at each bifurcation one has equal probability of either dying without offspring or having two children.

Superprocesses and nonlinear heat equation



Superprocesses for more general interactions

The construction may be generalized for interactions u^{α} with $1 < \alpha \leq 2$. With $z = 1 - \beta u_{\beta}$ one has

$$\varphi(0, x; z) = \sum_n p_n z^n = z + u_{\beta}^{\alpha} = z + \frac{\beta}{k_{\beta}} \frac{(1-z)^{\alpha}}{\beta^{\alpha}} = z + \frac{1}{k_{\beta}\beta^{\alpha-1}}$$
$$\times \left(1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots\right)$$

Choosing $k_{\beta} = \frac{\alpha}{\beta^{\alpha-1}}$ the terms in z cancel and for $1 < \alpha \leq 2$ the coefficients of all the remaining z powers are positive and may be interpreted as branching probabilities. **It would not be so for $\alpha > 2$.**

$$p_0 = \frac{1}{\alpha}; \quad p_1 = 0; \quad \cdots \quad p_n = \frac{(-1)^n}{\alpha} \binom{\alpha}{n} \quad n \geq 2$$

with $\sum_n p_n = 1$. With this choice of branching probabilities, $k_{\beta} = \frac{\alpha}{\beta^{\alpha-1}}$ and $\beta \to 0$ one obtains a superprocess which provides a solution to the equation $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^{\alpha}$ for $1 < \alpha \leq 2$.

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