Self-organization and ergodic parameters

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Contents

1. Ergodic tools. Exponents and entropies

- *Invariant measures and ergodic parameters*

$$I_F(\mu) = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} F(f^n x_0)$$

- *Lyapunov and conditional exponents*

From the k x k and (m-k)x(m-k) blocks of the Jacobian , obtain the conditional exponents as the eigenvalues of the limits

$$\lim_{n\to\infty} \left(D_k f^{n*}(x)\, D_k f^n(x)\right)^{\frac{1}{2n}}$$

$$\lim_{n\to\infty} \left(D_{m-k} f^{n*}(x)\, D_{m-k} f^n(x)\right)^{\frac{1}{2n}}$$

or

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_k f^n(x) u\| = \xi_i^{(k)}$$

$$0 \neq u \in E_x^i / E_x^{i+1}$$

E_x^i is the subspace spanned by the eigenstates corresponding to eigenvalues $\leq \exp(\xi_i^{(k)})$

Existence of the conditional exponents

- First proposed by Pecora and Carroll to study the phenomenology of synchronization of chaotic systems
  PRL 64 (1990) 821 ; PRA 44 (1991) 2374

- *Theor. 1 The existence of the conditional exponents is guaranteed under the same conditions as for the Lyapunov exponents*

  Existence of a measurable map from the dynamical space V to m x m matrices

  $$T : V \to M_m$$

  and $$\int \mu(dx) \log^{+} \left\| T(x) \right\| < \infty$$

  The proof follows the same steps as for the Oseledec's theorem
  PLA 248 (1998) 167

- Regular functionals of the exponents will also be well-defined ergodic parameters

2 - Structures and self-organization

- Structure index

$$S = \frac{1}{N} \sum_{i=1}^{N_+} \left(\frac{\lambda_0}{\lambda_i} - 1 \right)$$

diverges whenever a Lyapunov exponent approaches zero from above (points where long time correlations develop)

- Self-organization (partitions $\Sigma_k = R^k \times R^{m-k}$)

$$I_\Sigma(\mu) = \sum_{k=1}^{N} \{ h_k(\mu) + h_{m-k}(\mu) - h(\mu) \}$$

$$h_k(\mu) = \sum_{\xi_i^{(k)} > 0} \xi_i^{(k)} ; h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)} > 0} \xi_i^{(m-k)} ; h(\mu) = \sum_{\lambda_i > 0} \lambda_i$$

- Self-organization concerns the dynamical relation of the whole to its parts. Therefore, $I_{\Sigma}(\mu)$ is a measure of dynamical self-organization
- It is a measure of apparent dynamical freedom (or apparent rate of information production), that each agent may infer from the local dynamics
- Self-organization occurs when local information is very different from global behavior
- These global parameters, besides providing information on structure formation and self-organization may also be used to characterize the topology of the interactions (network connectivity)

3 - Examples :

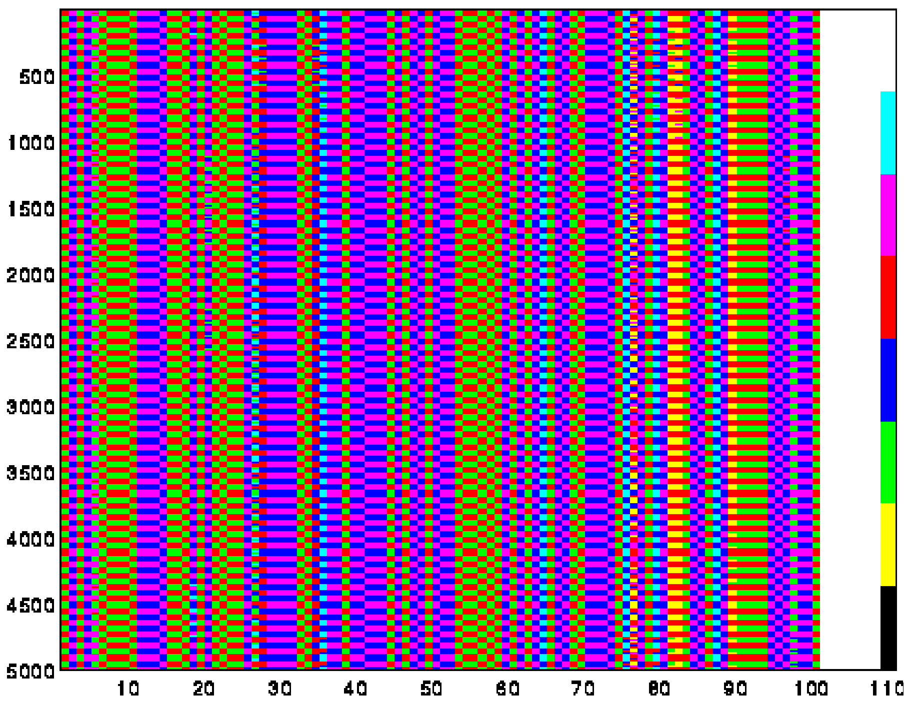
- Fully coupled system

$$x_i(t+1) = (1-c)\, f(x_i(t)) + (c/(N-1))\, \Sigma_{k\neq i}\, f(x_k(t))$$

$$f(x)=2x \pmod 1$$

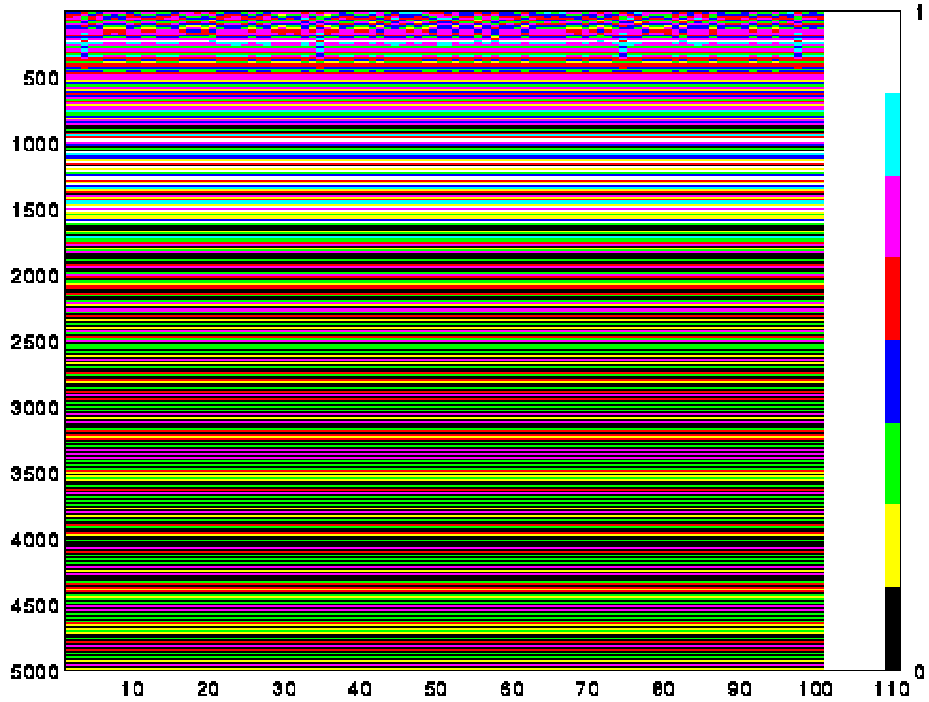
c = 0.495

Fig. 2

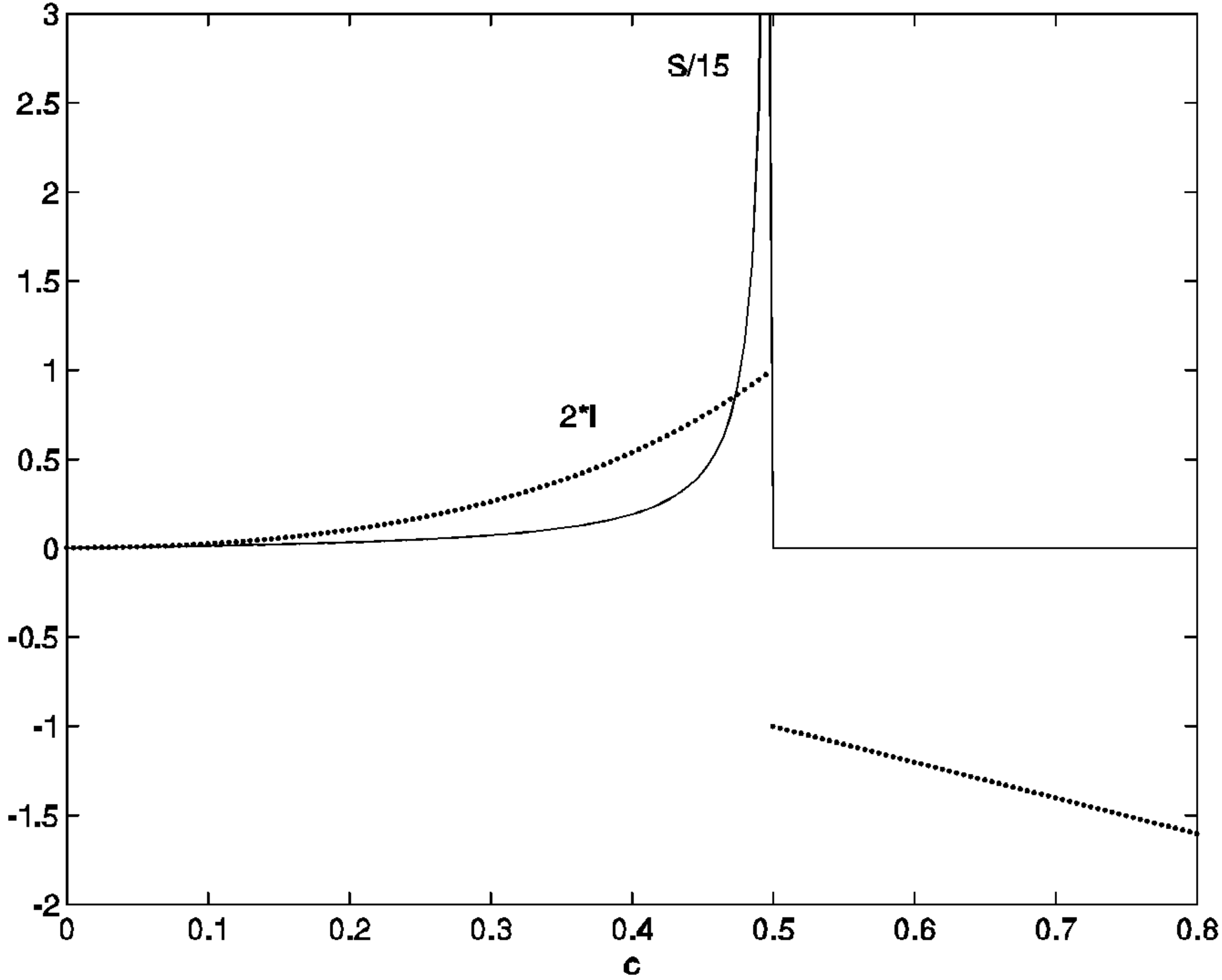


c = 0.51

Fig. 3

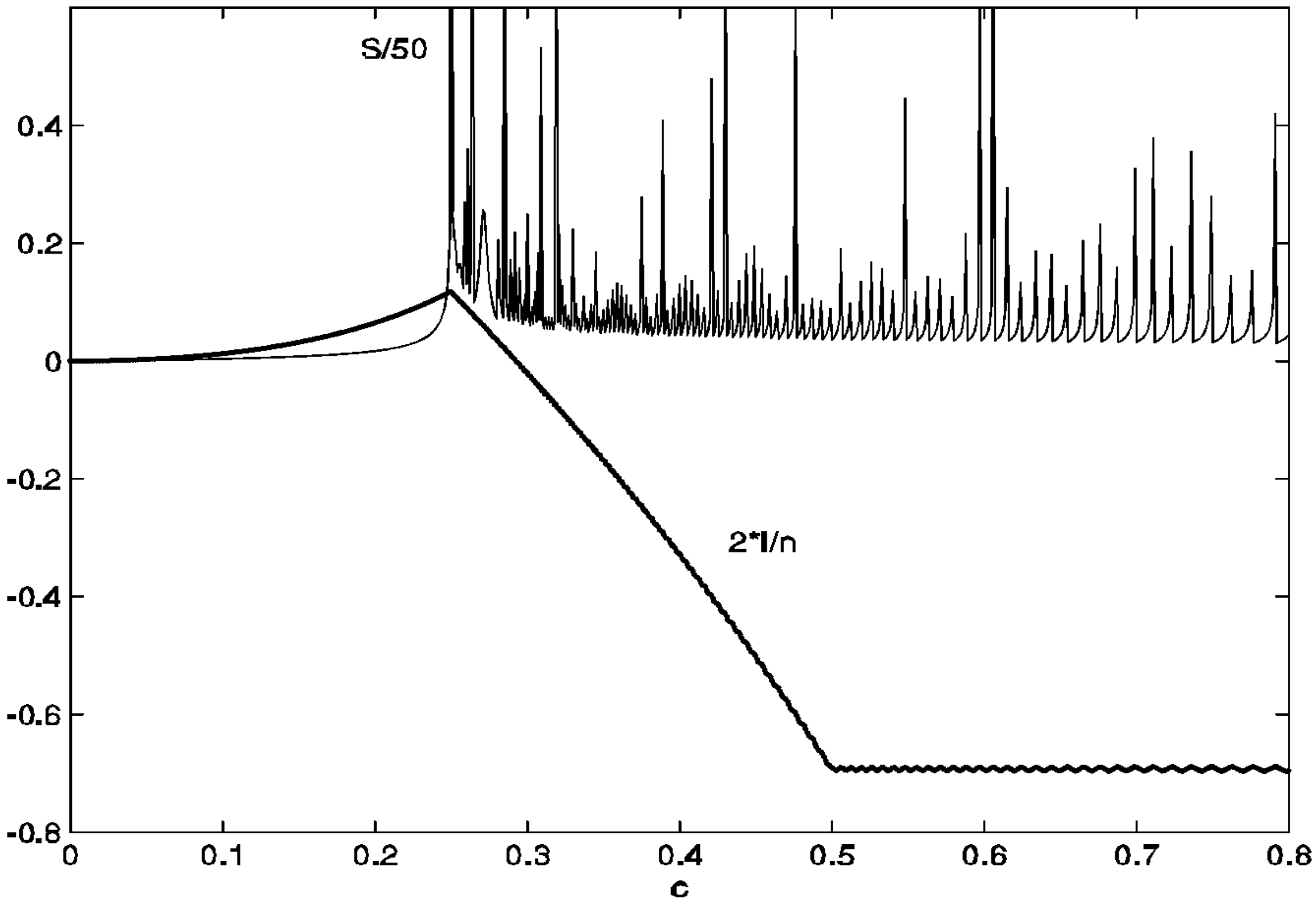


Fully coupled system. Structure and self-organization index



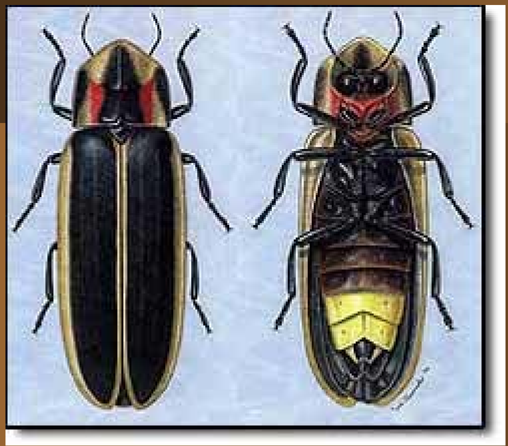
Nearest-neighbor coupling

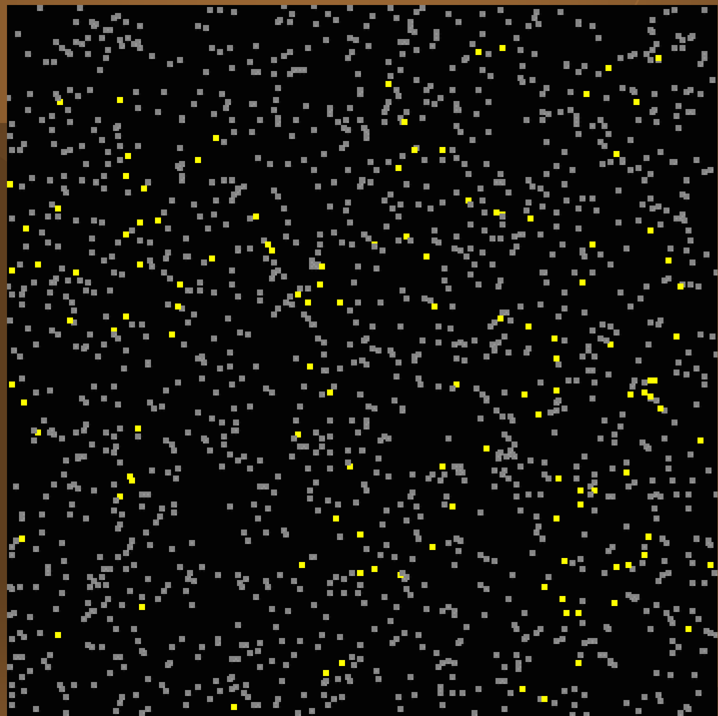
- $x_i(t+1) = (1-c)\, f(x_i(t)) + (c/2)\, (\, f(x_{i+1}(t)) + f(x_{i-1}(t))\,)$



4 - Synchronization and beyond

- Synchronous flashing of fireflies, cells, fads, ...





4 - Synchronization and beyond

- Synchronization
  (Classical mathematical example: the Kuramoto model)
  A similar, discrete-time oscillators model :

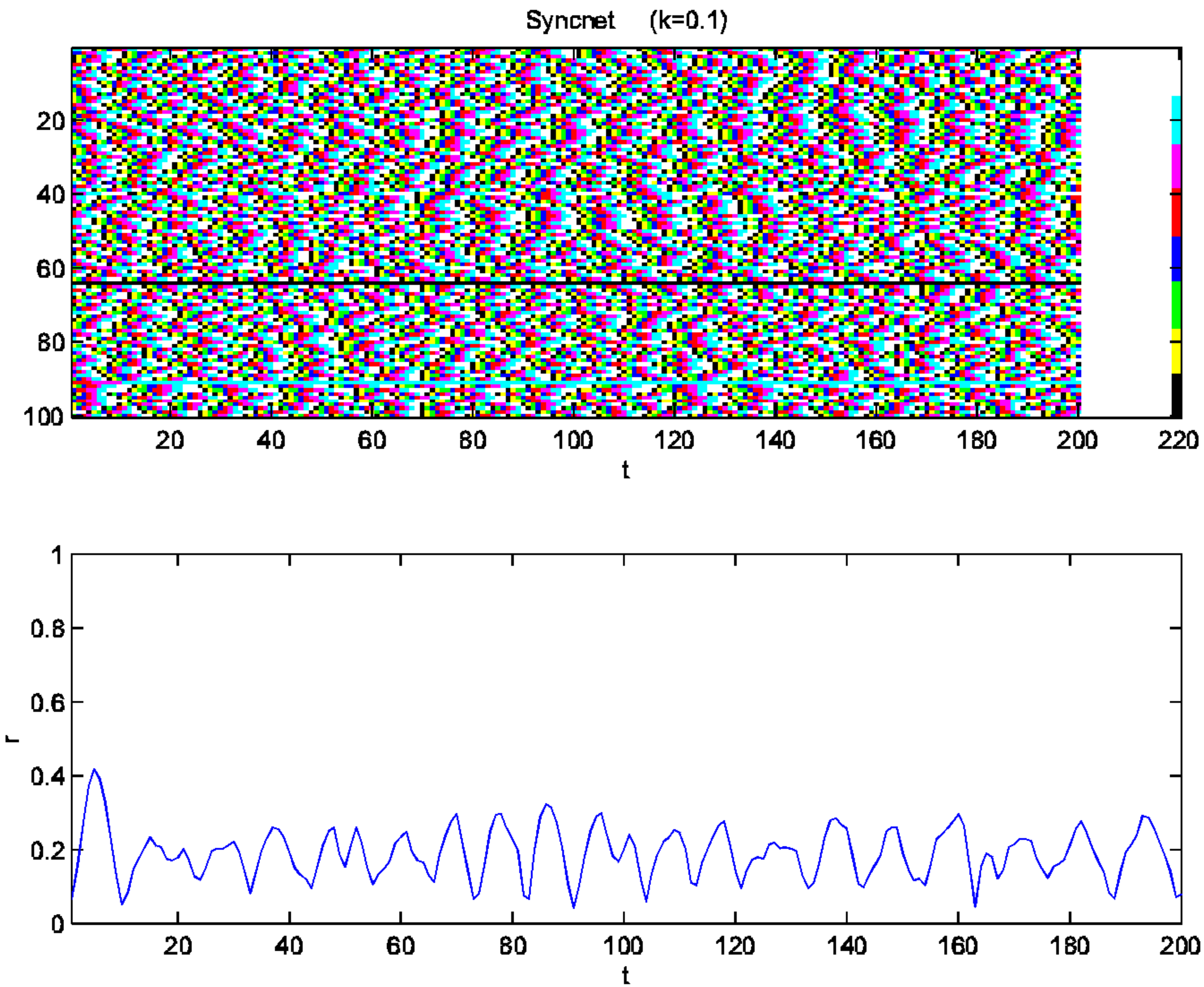
$$x_i(t+1) = x_i(t) + \omega_i + \frac{k}{N-1}\sum_{j=1}^{N} f_\alpha(x_j - x_i)$$

$$p(\omega) = \frac{\gamma}{\pi\left[\gamma^2 + (\omega - \omega_0)^2\right]}$$

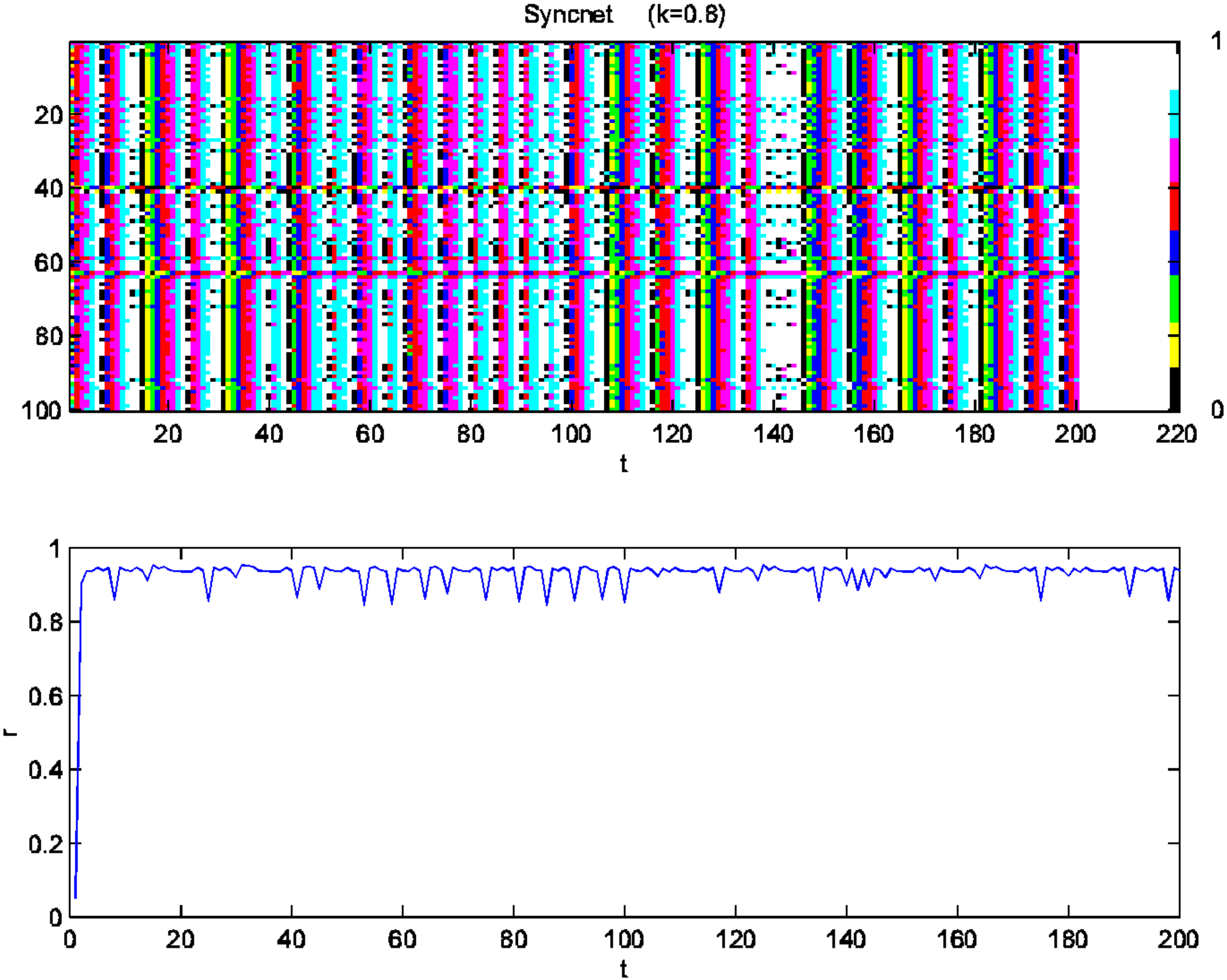
$$f_\alpha(x_j - x_i) = \alpha(x_j - x_i)\,(mod 1)$$

- Order parameter

$$r(t) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi x_j(t)}\right|$$



Syncnet (k=0.1)
20
40
60
80
100
20
40
60
80
100
120
140
160
180
200
220
t
1
0
1
0.8
0.6
0.4
0.2
0
r
20
40
60
80
100
120
140
160
180
200
t


Syncnet (k=0.8)
20
40
60
80
100
20
40
60
80
100
120
140
160
180
200
220
t
1
0
1
0.8
0.6
0.4
0.2
0
r
0
20
40
60
80
100
120
140
160
180
200
t

- The Lyapunov spectrum controls the dynamical self-organization of the system.

- In this case

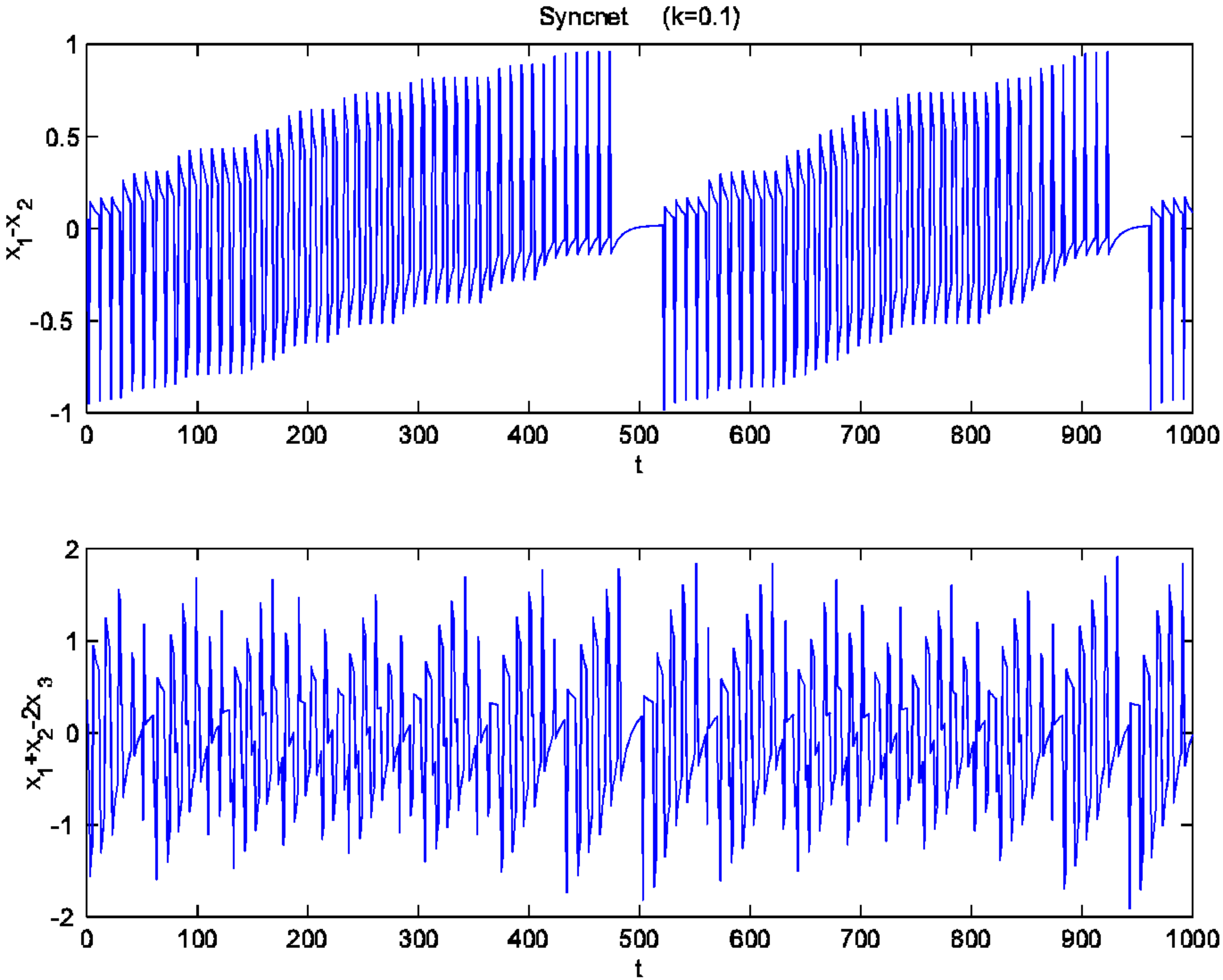
  λ_1=0 and

  λ_i=log(1-$\alpha\lambda$k(N/N-1)) (N-1) times

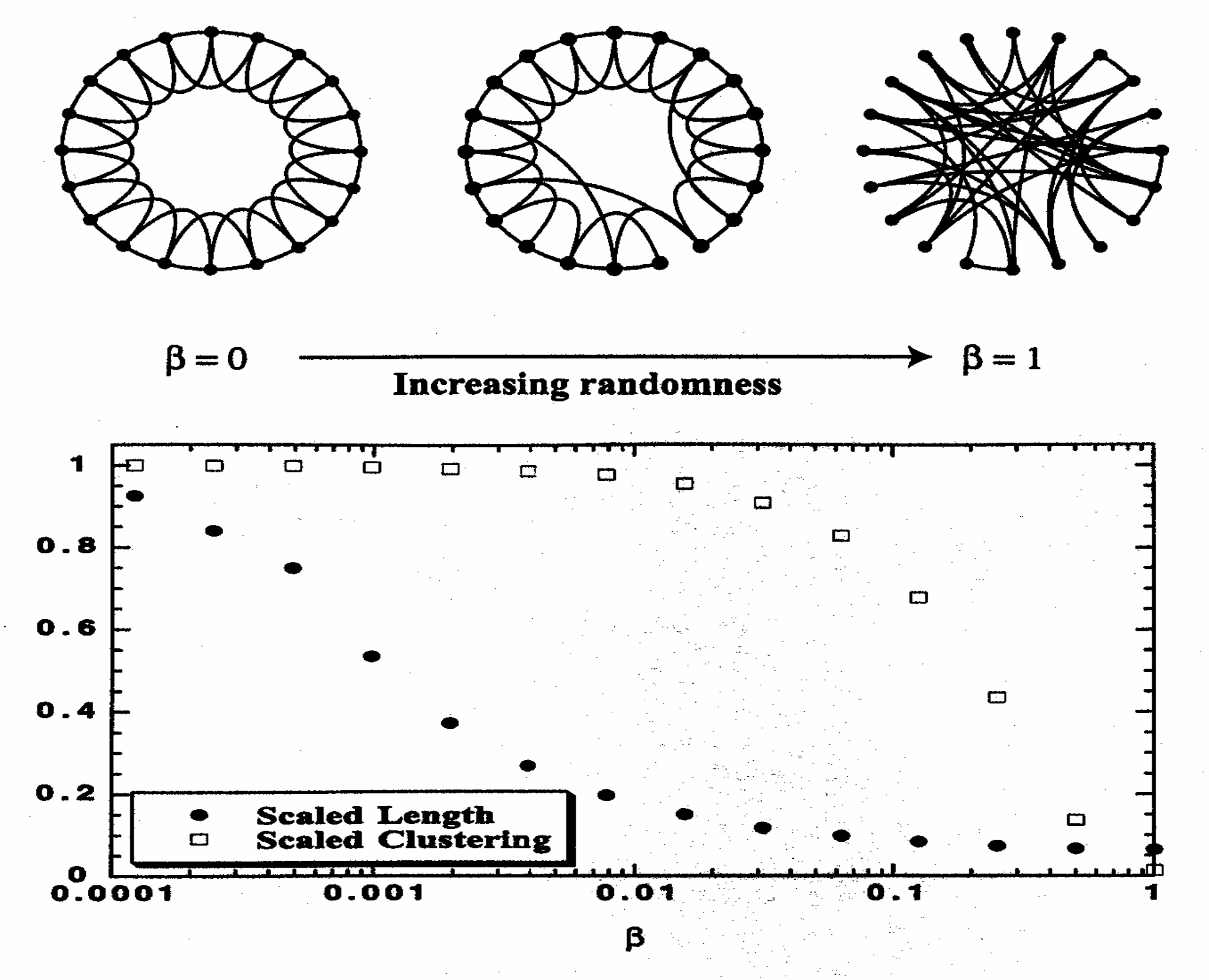
  N-1 contracting directions for k$\neq$0

  “One-dimensional” system !

- $\Rightarrow$ strong dynamical correlations even before synchronization


Syncnet (k=0.1)
1
0.5
0
-0.5
-1
x1-x2
0
100
200
300
400
500
600
700
800
900
1000
t
2
1
0
-1
-2
x1+x2-2x3
0
100
200
300
400
500
600
700
800
900
1000
t

5.Network structure and dynamics.The small world phase



Define a dynamical system on the network nodes

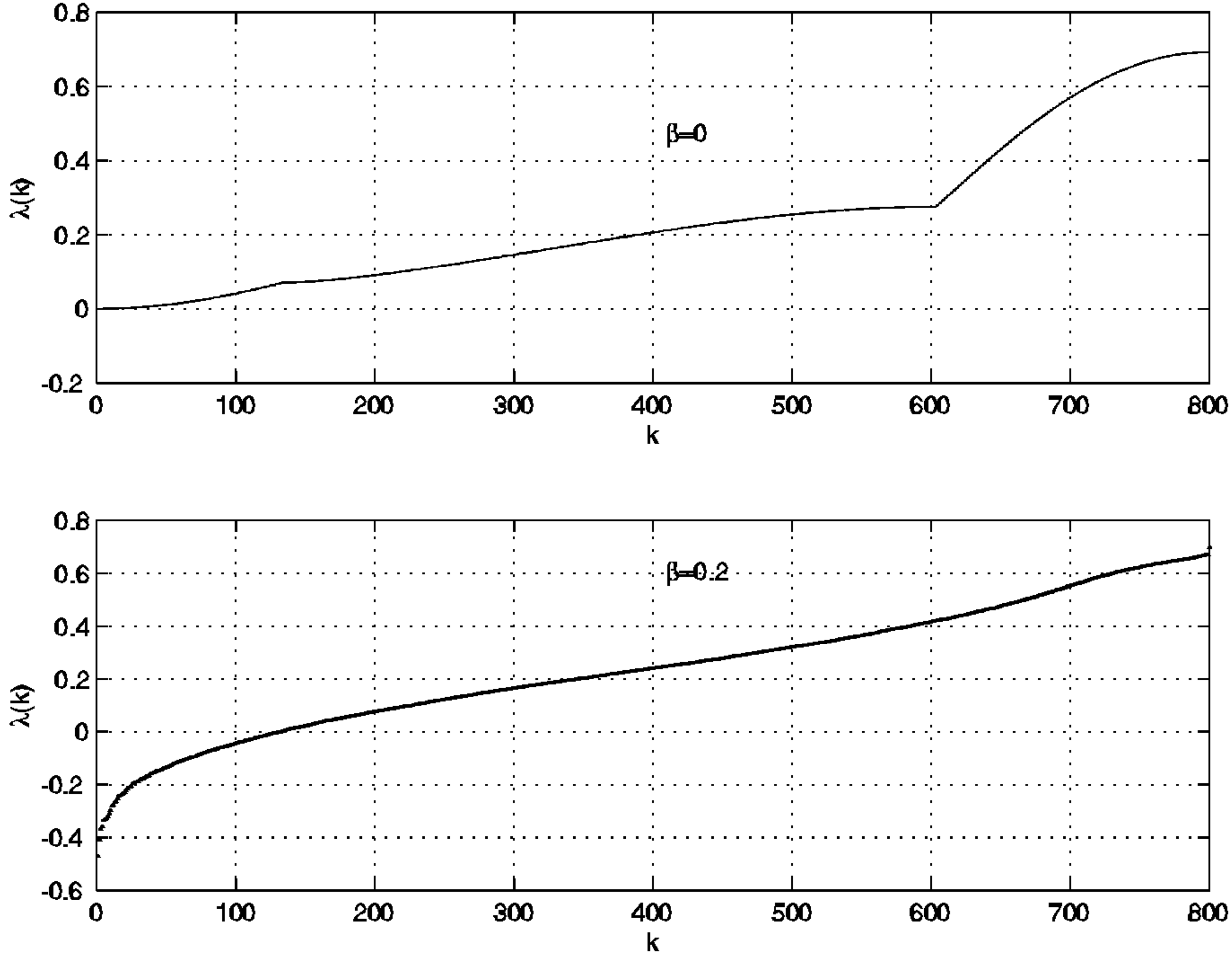
- $x_i(t+1) = \Sigma_{k=1}^{N} W_{ik} f(x_k(t))$

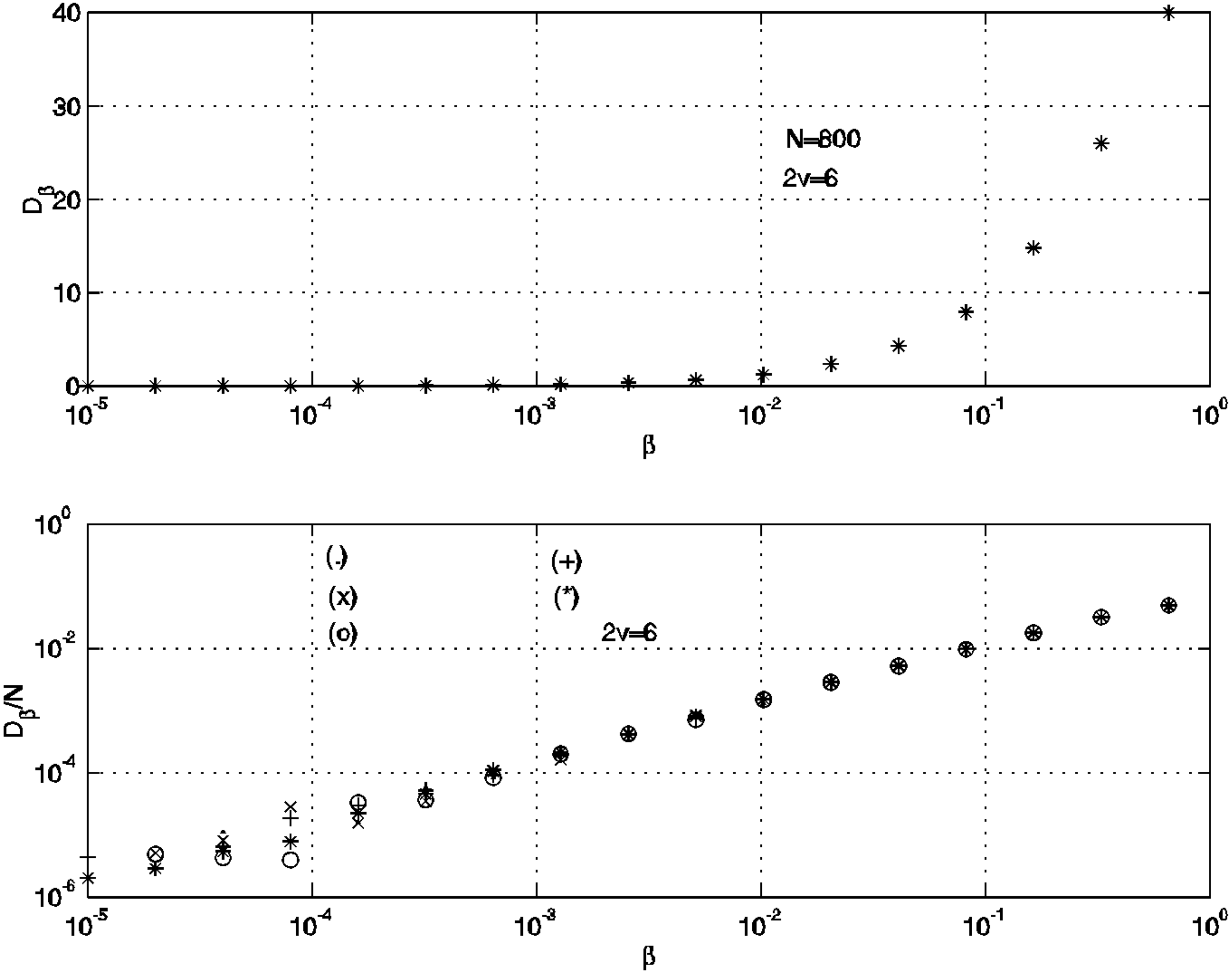
  $f(x)=\alpha x \pmod 1$

$$W_{ik} = \begin{cases} 1-\frac{n_v(i)}{2v}c & \text{if } i = k \\ \frac{c}{2v} & \text{if } i \neq k \text{ and } k \in n_v(i) \\ 0 & \text{otherwise} \end{cases}$$

- $D_\beta = - \Sigma_{\lambda_i<0} \lambda_i$

  $D_\beta = c\, N\, (\beta - \beta_{c1})^{\eta}$ $\beta_{c1} < 10^{-5}$ $\eta = 1.01 \pm 0.06$


0.8
0.6
0.4
0.2
0
-0.2
λ(k)
β=0
0
100
200
300
400
500
600
700
800
k
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
λ(k)
β=0.2
0
100
200
300
400
500
600
700
800
k


N=600
2v=6
D_β
β
(.)
(x)
(o)
(+)
(*)
2v=6
D_β/N
β

Define a dynamical system on the network nodes

- $x_i(t+1) = \Sigma_{k=1}^{N} W_{ik} f(x_k(t))$

  $f(x)=\alpha x \pmod 1$

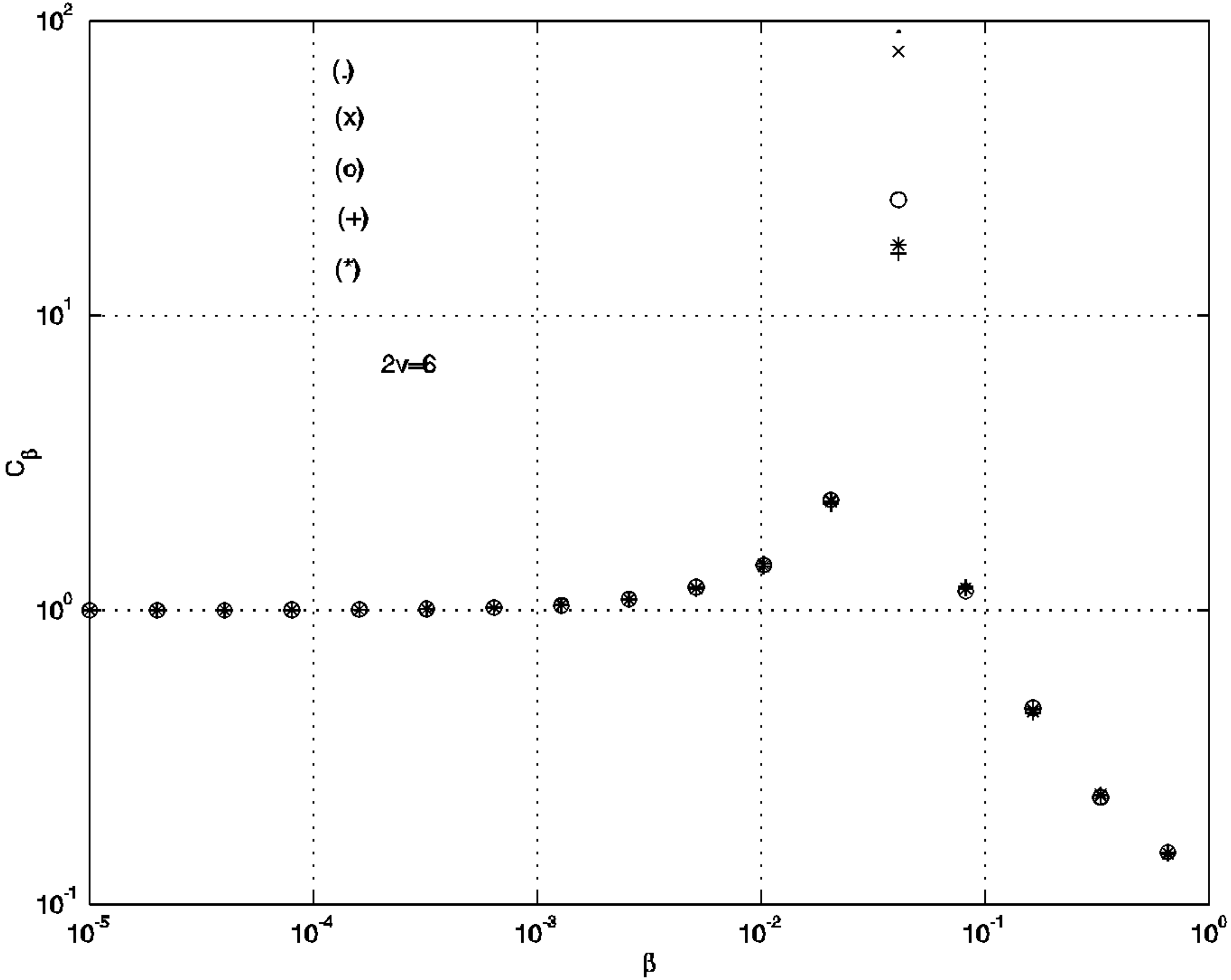
$$W_{ik} = \begin{cases} 1 - \frac{n_v(i)}{2v} c & \text{if } i = k \\ \frac{c}{2v} & \text{if } i \neq k \text{ and } k \in n_v(i) \\ 0 & 0 \text{ otherwise} \end{cases}$$

- $D_\beta = - \Sigma_{\lambda_i<0} \lambda_i$

  $D_\beta = c N (\beta - \beta_{c1})^{\eta}$ $\beta_{c1} < 10^{-5}$ $\eta = 1.01 \pm 0.06$

- $C_\beta = \left| \frac{h_0^* - h_0}{h_\beta^* - h_\beta} \right|$; $h_\beta^* = \sum_{i=1}^{N} \left(\frac{1}{d_i} \sum_{\lambda_\beta^*>0} \lambda_\beta^*(j) \right)$; $h_\beta = \sum_{\lambda_\beta>0} \lambda_\beta(j)$

  $\beta_{c2} = 0.04$ $C_\beta \sim |\beta - \beta_{c2}|^{-\delta}$ $\delta_1 = 1.14$ $\delta_2 = 0.93$



10^2
10^1
10^0
10^-1
10^-5
10^-4
10^-3
10^-2
10^-1
10^0
(.)
(x)
(o)
(+)
(*)
2v=6
C_β
β

6 - Self-organized criticality (SOC)

- A qualitative definition :
  SOC = *mechanism of slow energy accumulation and fast energy redistribution (avalanches) driving the system towards a critical state, where the distribution of avalanche sizes is a power law obtained without fine tuning, that is, there is no tunable parameter in the model*.
- Power law → no natural scale, excitations at all scales
- No tunable parameter ≠ usual critical points in phase transitions
- A critical point as an attractor ?
- Ubiquity of SOC (geophysics, cosmology evolutionary biology, ecology, economics sociology, solar physics, …)
- Objective: Characterize SOC by ergodic parameters

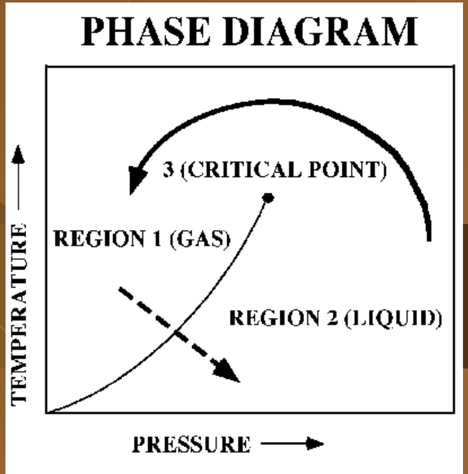


FIGURE 4

Real world manifestations

- *The Gutenberg-Richter law*

  Data from 1977-1995

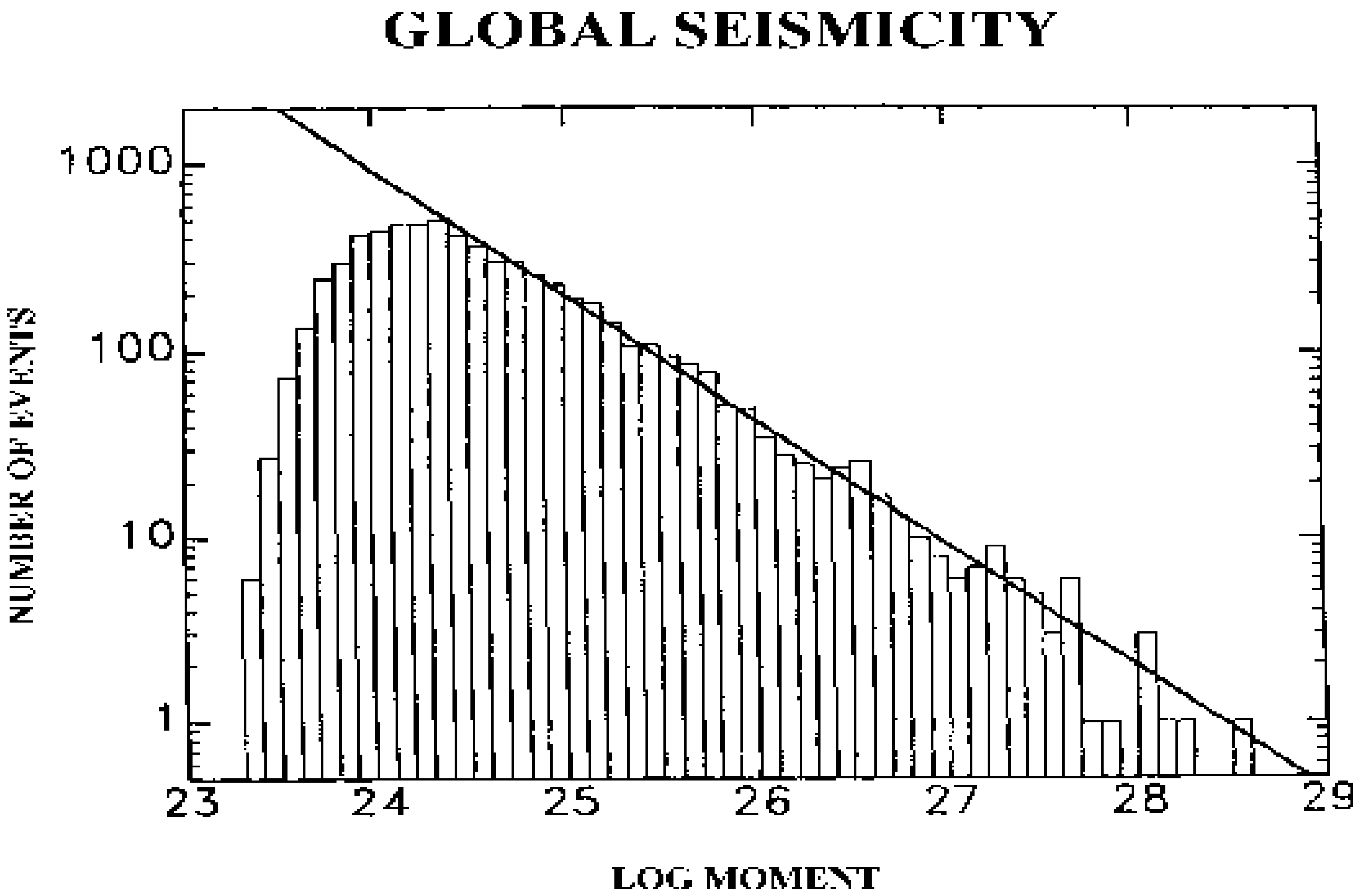
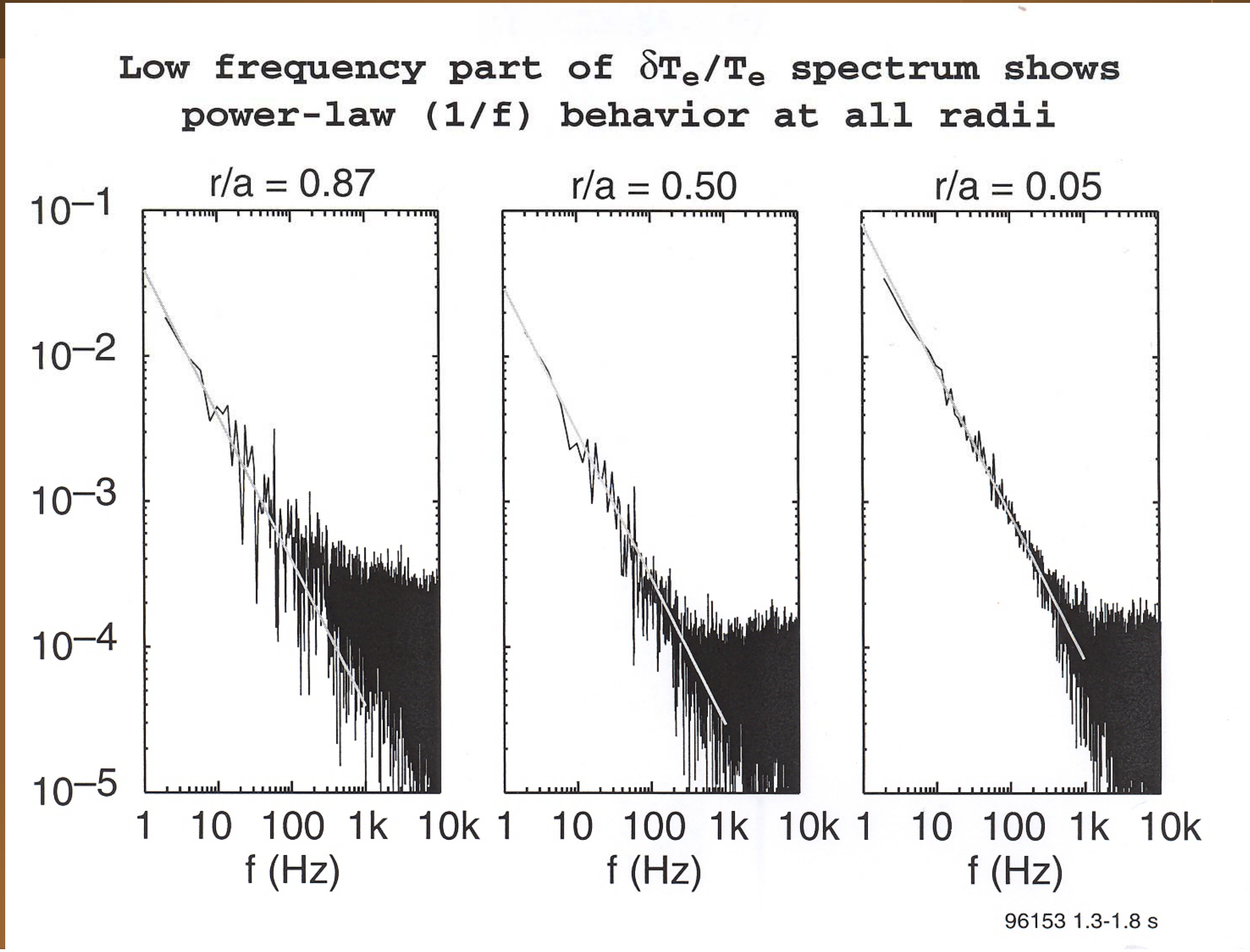


FIGURE 6

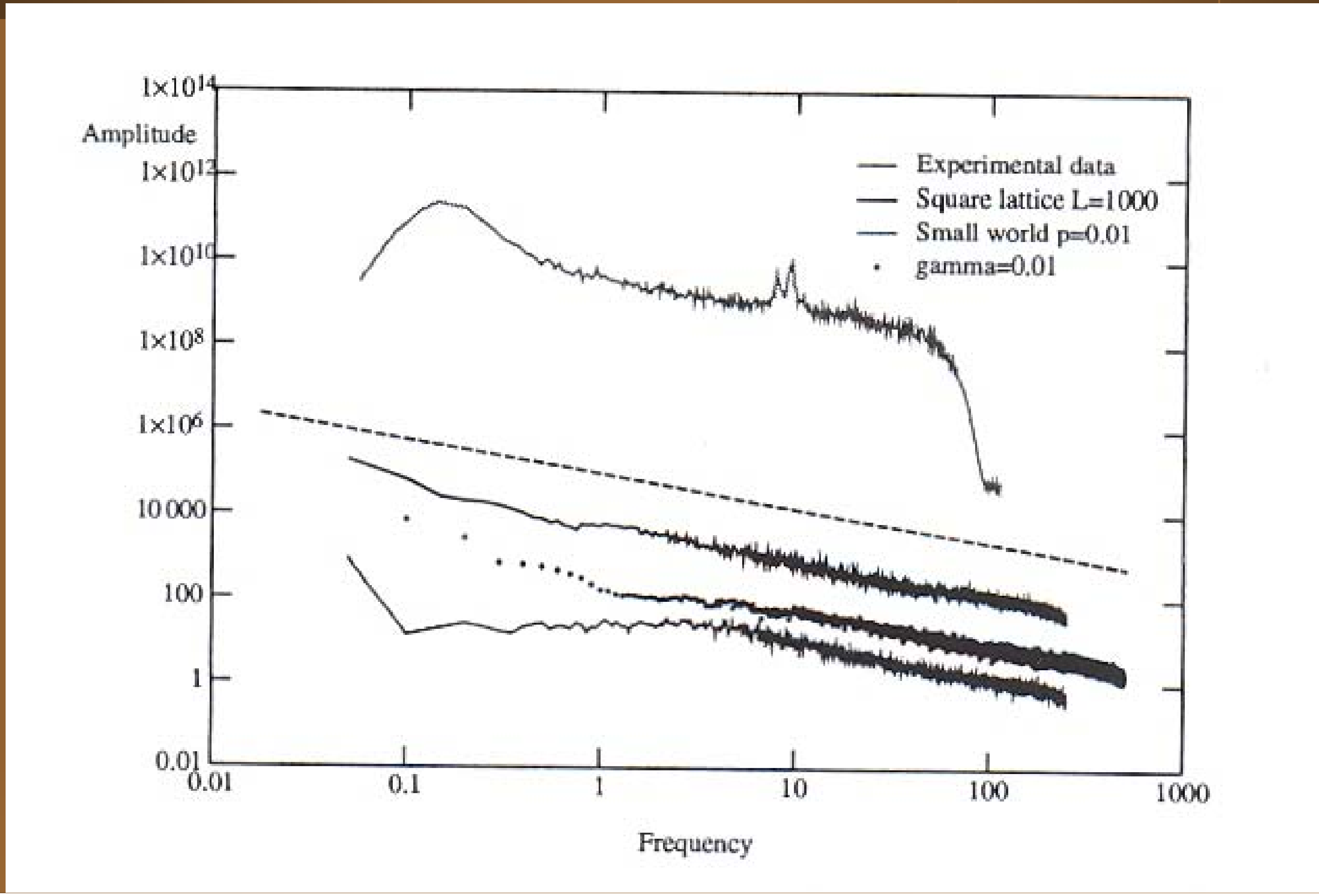
Real world manifestations

- *Electron temperature fluctuations in a magnetically confined plasma (ECE diagnostic)*
  (Politzer, PRL 84 (2000) 1192)



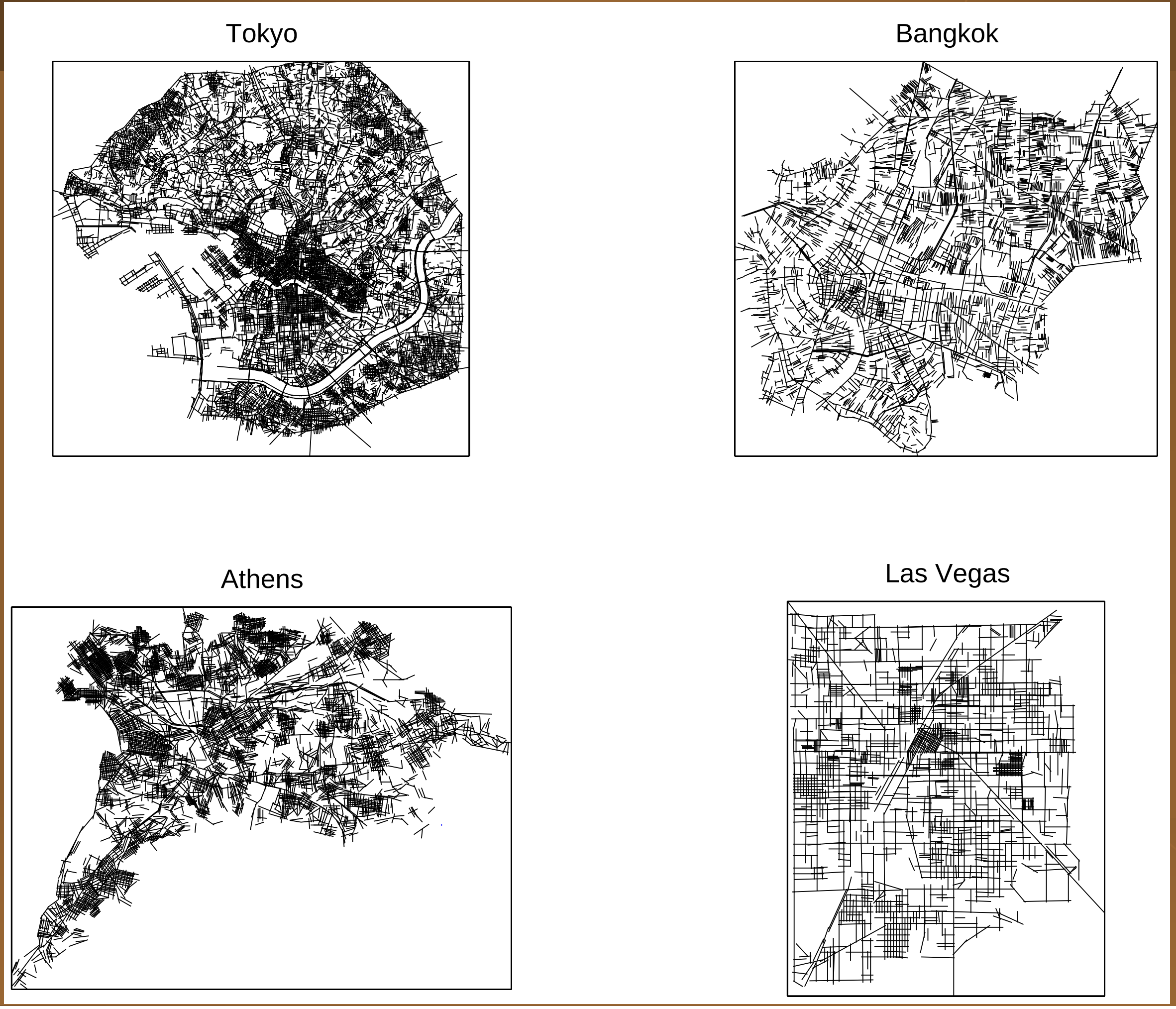
Real world manifestations

- *Avalanches in living neurons*
  Magnetoencephalography data compared with models
  (de Arcangelis et al. PRL 96 (2006) 028107)

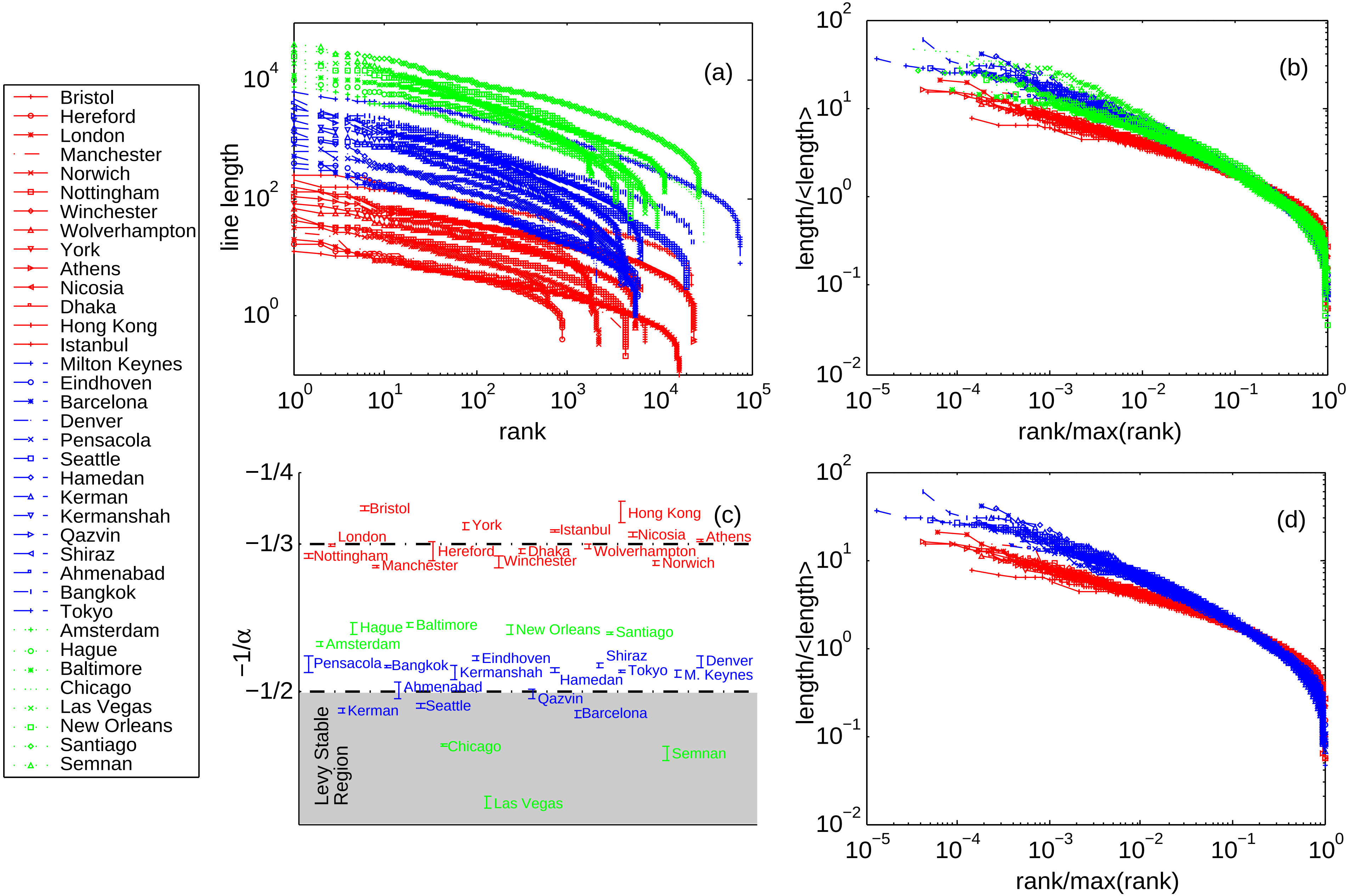


Real world manifestations

- *Distribution of lengths of open spaces in urban environments*
  (Carvalho and Penn, Physica A 332 (2004) 539)

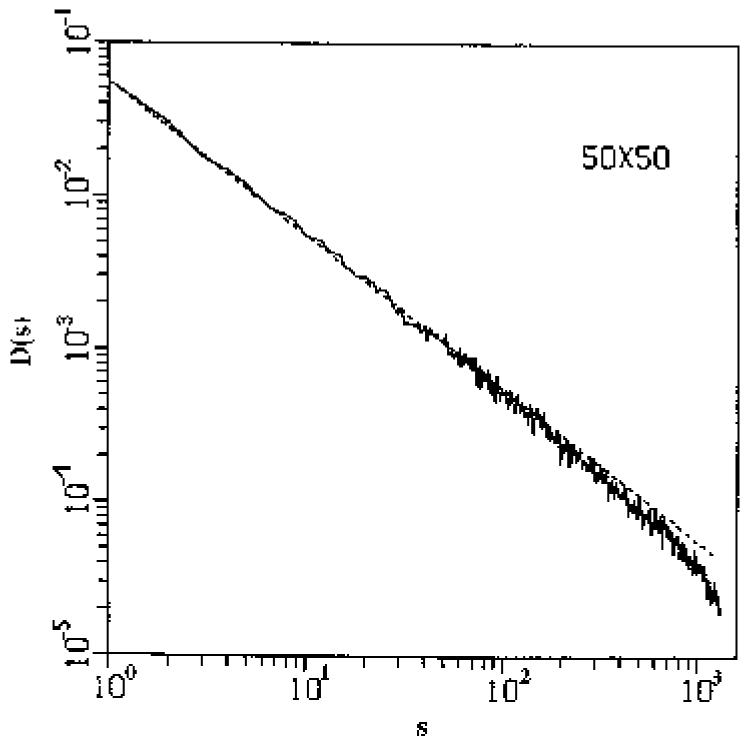


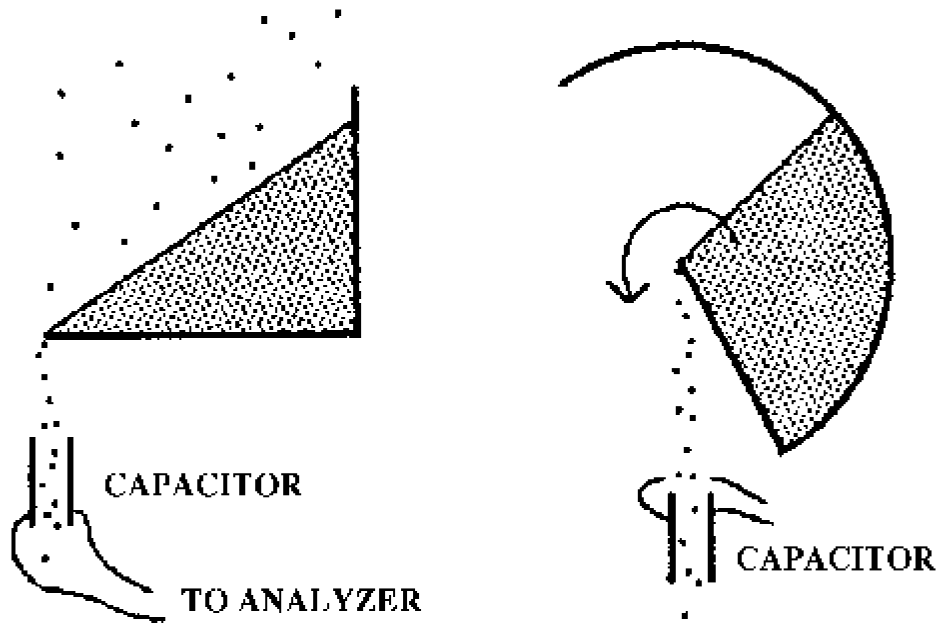
Real world manifestations



Toy models

- Sand piles on the computer and on the lab

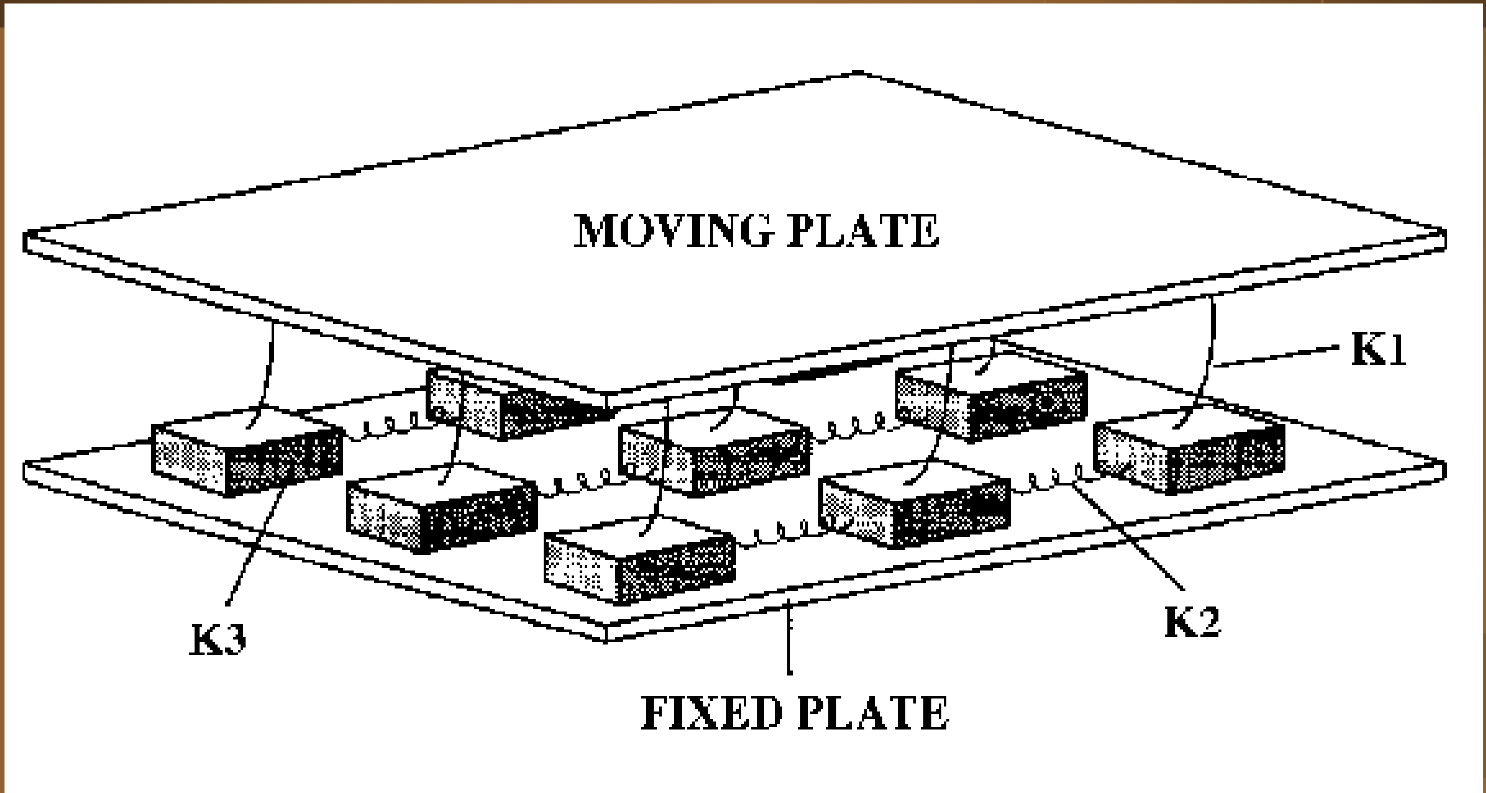




- However, the emergence of scaling laws on lab sand piles depend on grain size and shape

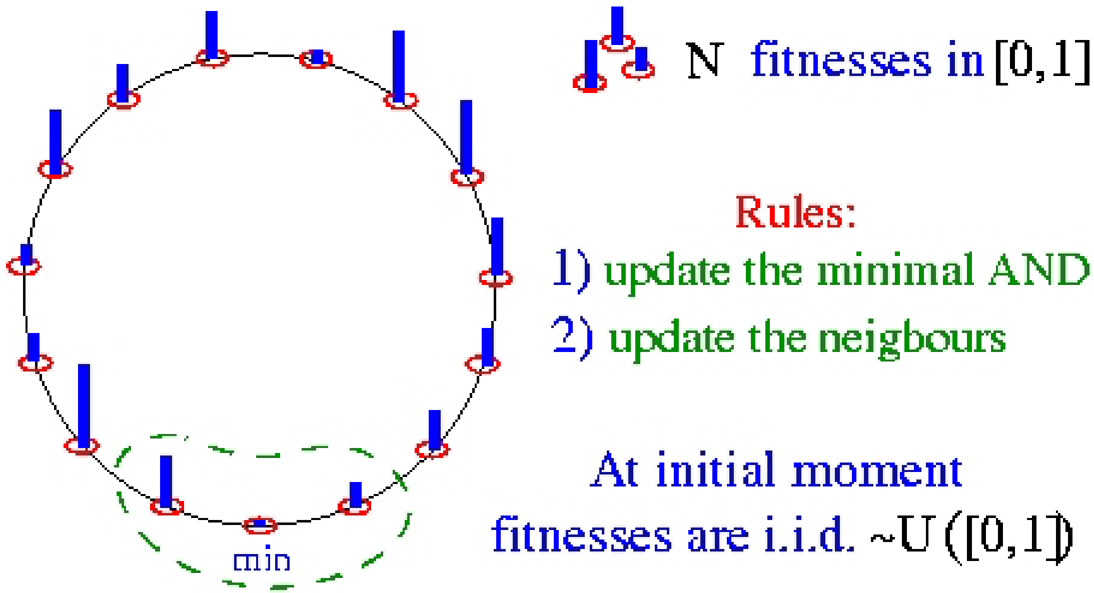
Toy models

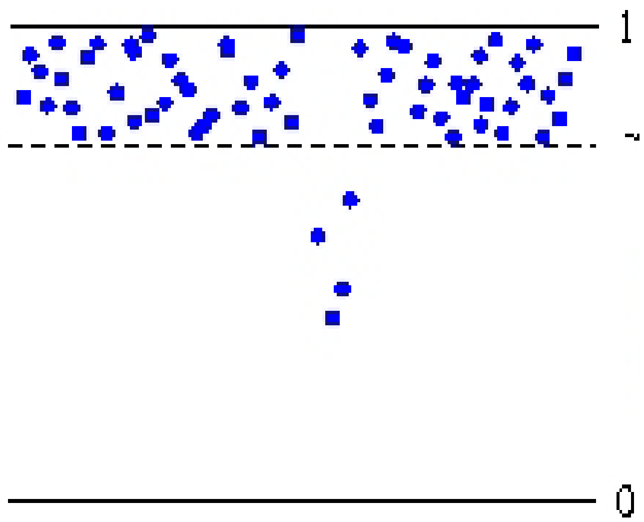
- Springer – slider block mode
  (friction of the blocks on the fixed plate)



A mathematical model: Bak-Sneppen (BS)

- *Toy model for the evolution of species*





- After a short transitory period the system self-organizes with most species having fitness above 0.667
- Avalanches show power-law behavior

8-Two features of most models and a mathematical result

- *Most SOC models display :*
  - Instable behavior of the local dynamics
  - Extremal dynamics
- *Theor. 2 If, in a N-agent model :*
  - The single-agent dynamics has positive Lyapunov exponents and
  - The global dynamics is extremal with finite range
  then, in the $N \rightarrow \infty$, the Lyapunov spectrum converges to 0^+
- In the $T \rightarrow \infty$ limit, used to compute the Lyapunov spectrum, the tangent maps have only a nontrivial finite size block during an average time of order (2r+1)T/N
- With the Lyapunov spectrum converging to 0^+ there are no dynamical scales. Thus, in the $N \rightarrow \infty$, the system is "tuned" to SOC

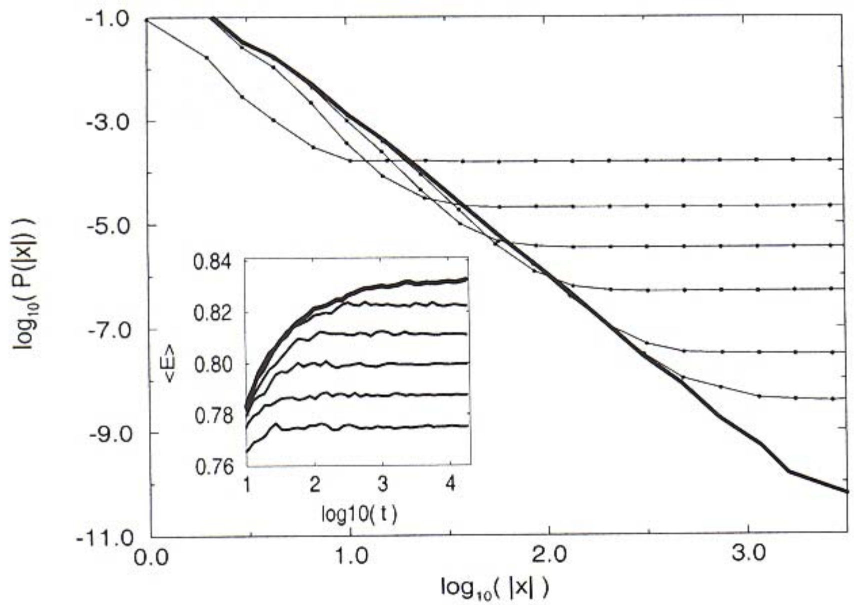
9 - Head's critique of parameter-independence in SOC

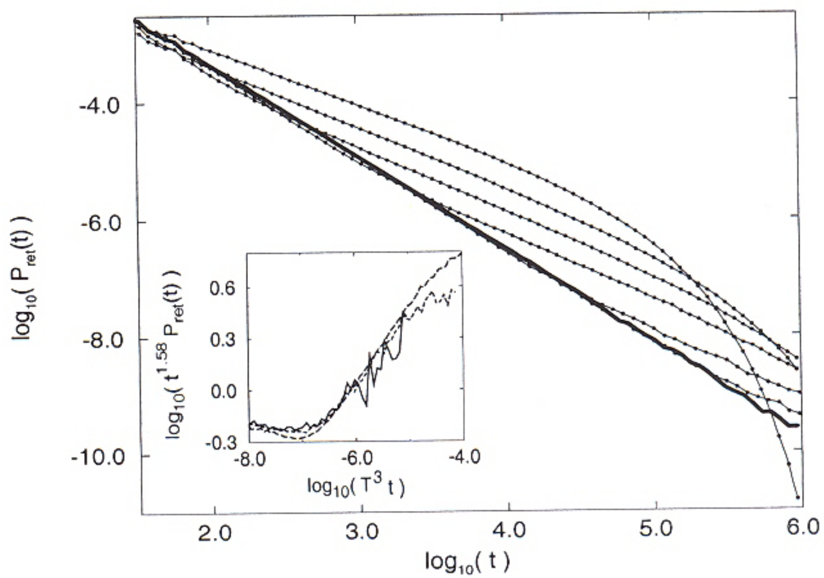
- "… SOC models *do* in fact require parameter tuning, but they had been defined in such a way that the tuning had been carried out *implicitly*."
  (Eur. Phys. J. B 17 (2000) 289)
- To make his point, he modified the Bak-Sneppen model defining the probability of activation of an element by

$$p_i = \frac{e^{-E_i/T}}{\sum_{k=1}^{N} e^{-E_k/T}}$$

- Then he finds that it is only in the T→0 limit that power laws are obtained, that is, BS is a zero temperature limit of his model

9 - Head's critique of parameter-independence in SOC





A deterministic version of B-S-Head's model

$$x_i(t+1) = \Gamma_i\left(\underset{\sim}{x}\right) x_i(t) + \left(1 - \Gamma_i\left(\underset{\sim}{x}\right)\right) f\left(x_i(t)\right) \quad (1)$$

$\underset{\sim}{x} = \{x_i\}$ is the vector of agent coordinates

$$f(x_i) = kx \quad \mod .1$$

$k = 2, 3, \cdots.$

$\Gamma_i\left(\underset{\sim}{x}\right)$ is nearly zero if i corresponds to the minimum x value or to one of its $2n_v$ neighbors and is nearly one otherwise.

$$\Gamma_i^{(1)}\left(\underset{\sim}{x}\right) = \prod_{j=i-n_V}^{j=i+n_V} \left(1 - \prod_{k \neq j} \left(1 + e^{-\alpha(x_k - x_j)}\right)^{-1}\right) \quad (2)$$

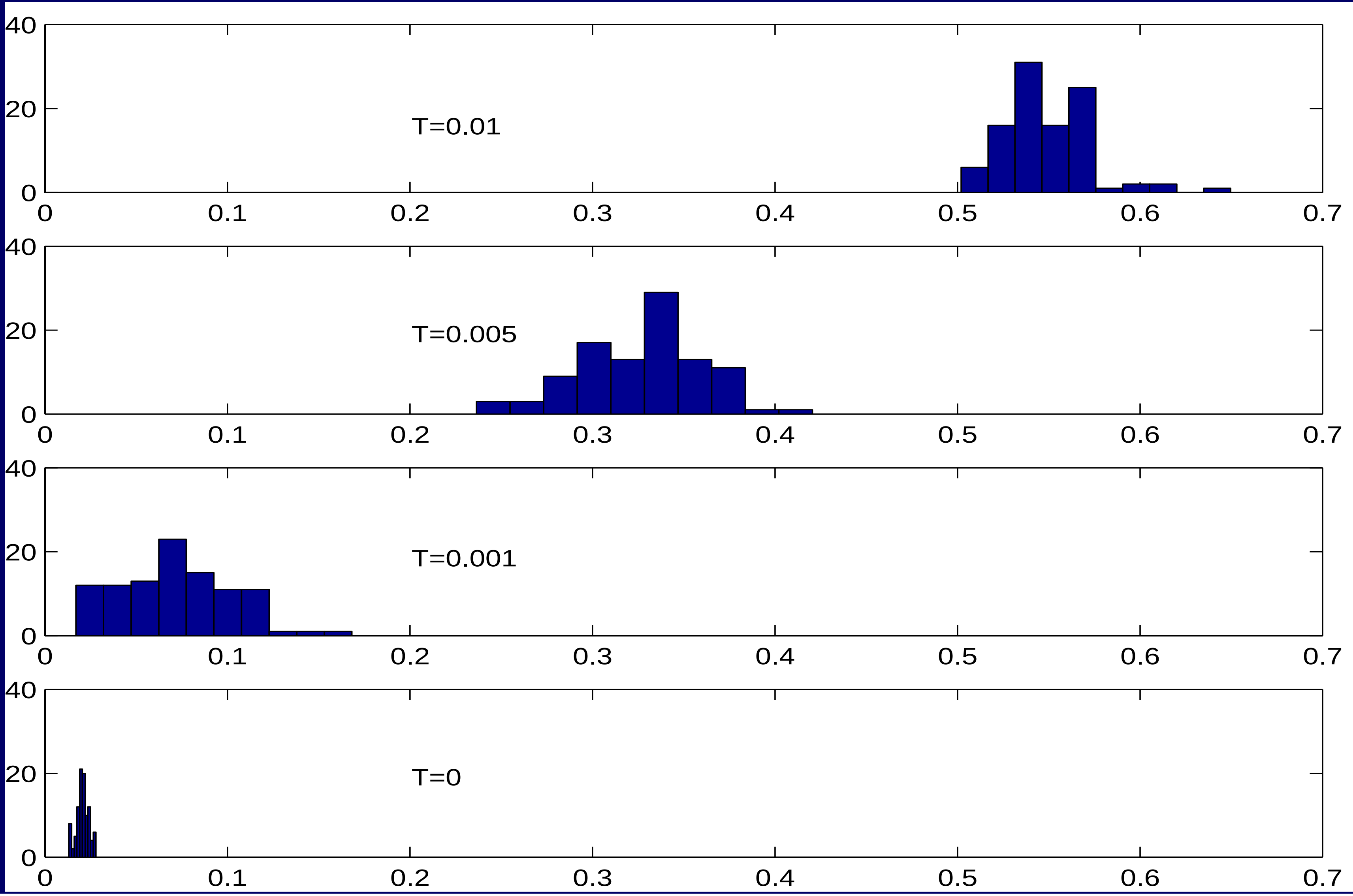
for large α, satisfies the above conditions.

$$\Gamma_i^{(2)}\left(\underset{\sim}{x}\right) = \prod_{j=i-n_V}^{j=i+n_V} \left(1 - \frac{e^{-x_i/T}}{\sum_{j=1}^{N} e^{-x_j/T}}\right) \quad (3)$$

a similar behavior for $T \to 0^+$

A deterministic version of B-S-Head's model

- The absence of power laws for non-zero T is indeed related to the Lyapunov spectrum



A deterministic version of B-S-Head's model

- Notice that at T=0 the Lyapunov spectrum does not reach zero because N=100.

- All this is expected from the proposition. However the deterministic model also allows to study a few other features :

  - What is the measure of the SOC state ?
  - Is the SOC state an attractor ?
  - Avalanches are return times to the SOC state. What is the prefactor in the return times (avalanches) distribution in the T=0 limit ?

A deterministic version of B-S-Head's model

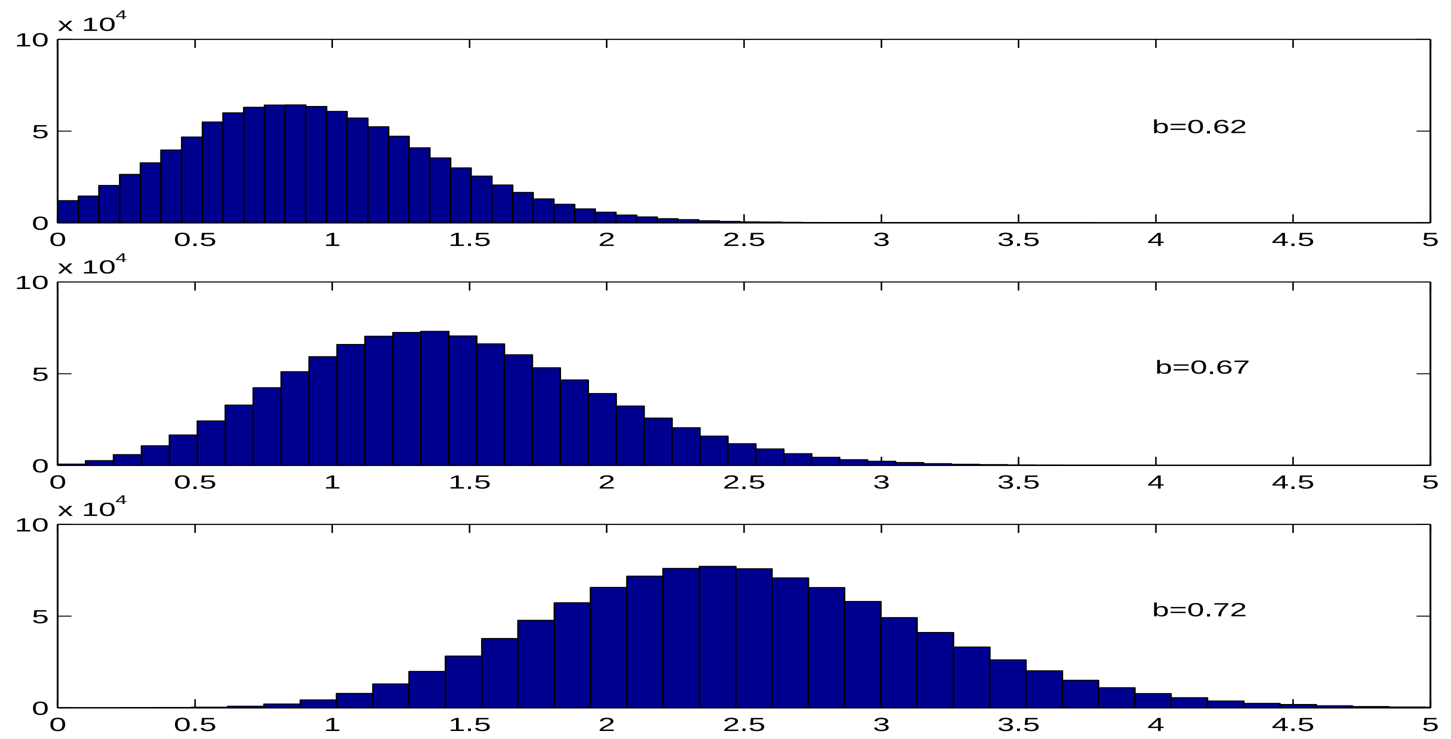
Kac's lemma (for an ergodic invariant measure μ**)**

Average return time to a set A **of measure** $\mu(A)$ **is** $1/\mu(A)$**.**

For a scaling law $\rho(\tau) \sim 1/\tau^{\alpha}$**,** $\alpha \leq 2$ **implies** $\mu(A) = 0$**.**

The distance process d

$$d = \sum_i \max(b - x_i, 0) \quad (4)$$



A deterministic version of B-S-Head’s model

- The SOC state has zero measure, but its finite-dimensional projections have full measure.
- It is not an attractor, nor a repeller (not invariant)
- “Ghost weak repeller”
- The invariant measure is like a cloud around the SOC state.

A deterministic version of B-S-Head's model

The zero measure of this "repeller" makes the direct measurement of the distribution law of avalanches a delicate matter.

A more robust method

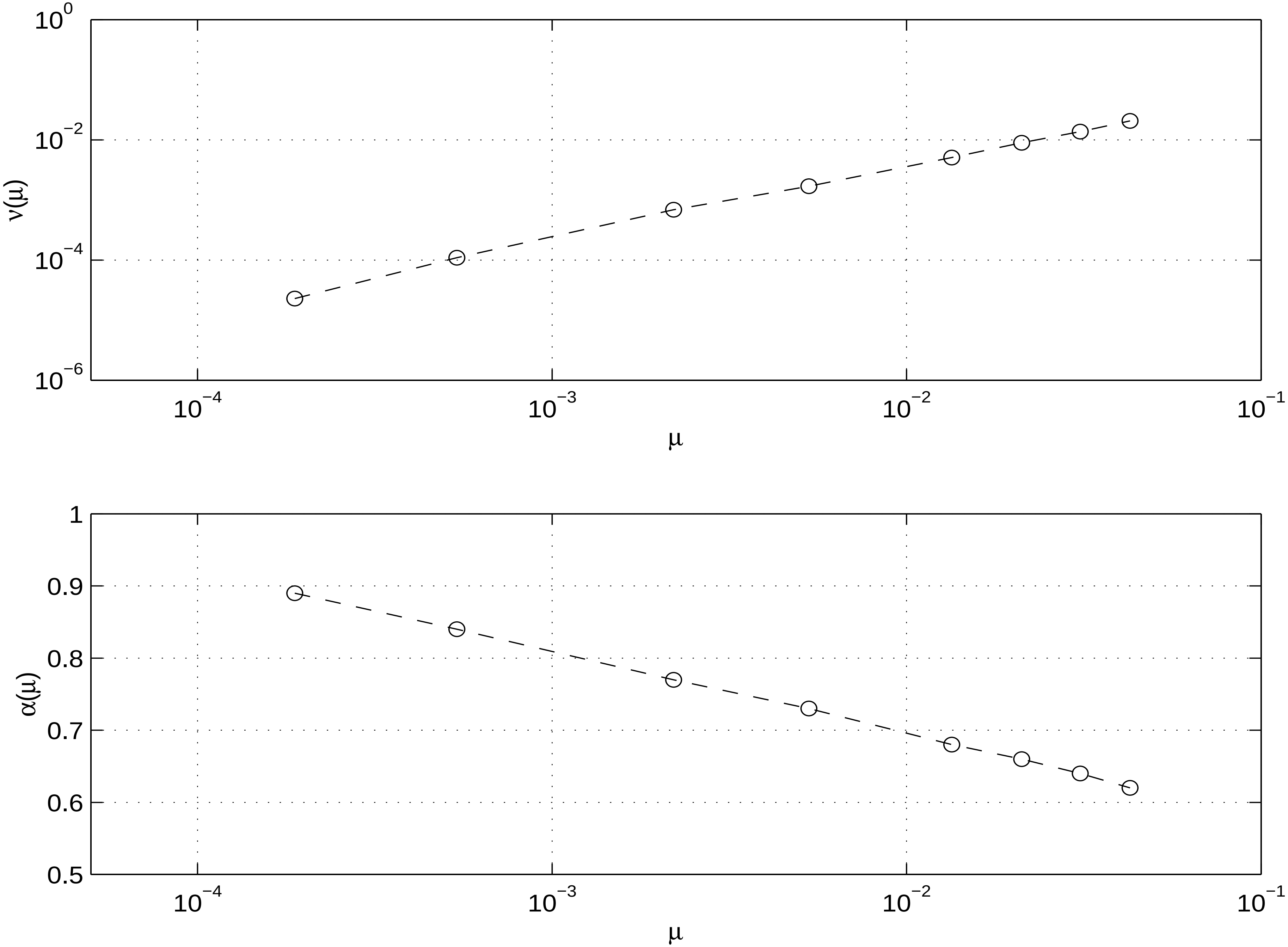
$$C\left(x\right)=\left\langle e^{ikx}\right\rangle \tag{5}$$

$$\rho\left(k\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}C\left(x\right)e^{-ikx}dx \tag{6}$$

Alternatively

$$p\left(k,\mu\right)=ck^{-\alpha(\mu)}e^{-\nu(\mu,\alpha)} \tag{7}$$

with c **and** $\nu\left(\mu,\alpha\right)$ **obtained from normalization and Kac's lemma,** $\langle k\rangle=\frac{1}{\mu}$ **.**



10^0
10^−2
10^−4
10^−6
ν(μ)
10^−4
10^−3
10^−2
10^−1
μ
1
0.9
0.8
0.7
0.6
0.5
α(μ)
10^−4
10^−3
10^−2
10^−1
μ

A deterministic version of B-S-Head's model

The discussion above refers to the problem of direct determination of the scaling exponents.

With the additional assumption of a scaling form for $p(k)$ **near the critical barrier, further results may be obtained. Assuming that close to** $\mu = 0$

$$p(k, \mu) = k^{-\tau} f(k^s \mu) \tag{8}$$

$$\begin{aligned} \langle k \rangle &\sim \mu^{\frac{\tau-2}{s}} \\ \langle k^2 \rangle &\sim \mu^{\frac{\tau-3}{s}} \end{aligned} \tag{9}$$

From Kac's lemma

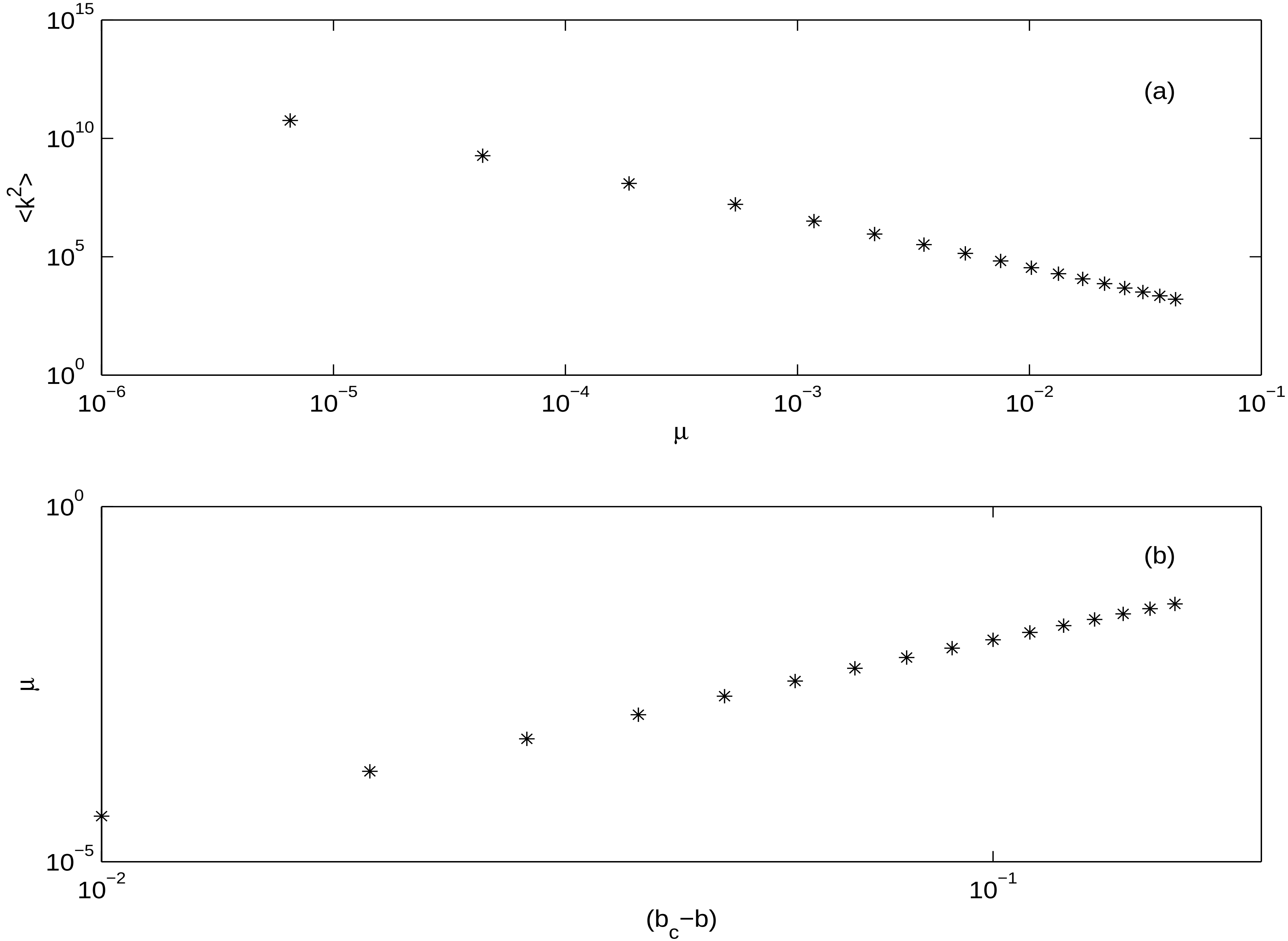
$$s = 2 - \tau$$

and from the numerical data (Fig. 4a), $\frac{\tau-3}{s} \simeq 2.07$**, leading to** $\tau \simeq 1.067$**,** $s \simeq 0.93$**.**

Another exponent

$$\mu \sim (b_c - b)^{\eta}$$

with (Fig. 4b) $\eta \simeq 2.55$**.**


(a)
10^{15}
10^{10}
10^{5}
10^{0}
$<k^2>$
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
μ
(b)
10^{0}
10^{-5}
μ
10^{-2}
10^{-1}
(b_c-b)

10 - Beyond the classical ergodic parameters

- Lyapunov and conditional exponents and derived quantities depend on the actual (or expected) ***average*** rates of expansion
- ***Fluctuations*** of the expansion rates along the trajectories
Generalized Lyapunov exponents

$$\Lambda(\beta) = \lim_{N\to\infty} \frac{1}{\beta N} \log \int d\mu(x_0) \exp\left[\beta \sum_{n=0}^{N-1} \log\left| f^{'}(x_n) \right| \right]$$

Dynamical Rényi entropies

$$K(\alpha) = \lim_{N\to\infty} \frac{1}{1-\alpha} \frac{1}{N} \log \sum_{i_0 \dots i_{N-1}} (p(i_0 \dots i_{N-1}))^{\alpha} \qquad \Lambda(\beta) = K(1-\beta)$$

Cumulants of the Lyapunov spectrum

$$K(\alpha) \cong \sum_{s=1}^{\infty} c_s \frac{(1-\alpha)^{s-1}}{s!}$$

Traces of Hessian powers

$$\frac{1}{2} H_N = \delta_{\alpha,\beta} \delta_{j,k} - (1-\delta_{k,N}) \delta_{k,j-1} \frac{\partial f^{\alpha}(x_k)}{\partial x_k^{\beta}} - (1-\delta_{j,N}) \delta_{j,k-1} \frac{\partial f^{\beta}(x_j)}{\partial x_j^{\alpha}} + (1-\delta_{j,N}) \delta_{j,k} \frac{\partial f^{\gamma}(x_j)}{\partial x_j^{\beta}} \frac{\partial f^{\gamma}(x_j)}{\partial x_j^{\alpha}}$$

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