Some mathematical problems in quantum control

Infinite-dimensional and nonunitary control

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- Wu, Tarn and Li (2005) established controllability criteria on infinite-dimensional manifolds that are generated by non-compact Lie algebras. Left open is the question of when these manifolds are dense on the Hilbert sphere, which would the key requirement for complete controllability in infinite dimensions.
- An alternative approach starts from the study of the infinite dimensional unitary group, $U(\infty)$, which is clearly transitive in the Hilbert sphere, and to study ways of generating it from a finite number of generators.
- The first step is the establishment of the proper mathematical setting for $U(\infty)$
- The most adequate setting for $U(\infty)$ is to consider its action on a Gelfand triplet

$$S^* \supset L^2\left(\mathbb{R}^d\right) \supset S$$

- S is a dense nuclear subspace of $L^2\left(\mathbb{R}^d\right)$, for example the Schwartz space or, alternatively, S is obtained as the limit $S=\bigcap_n S_n$ of a sequence of spaces with increasing Hilbertian norms
$$S^* \supset \cdots \supset S_{-n} \supset \cdots \supset L^2\left(\mathbb{R}^d\right) \supset \cdots \supset S_n \supset \cdots \supset S$$
the Hilbertian norms typically chosen to be
$$\|\xi\|_n = \|A^n \xi\|$$
A being a conveniently chosen unbounded operator of the control algebra.
- Now an element g of $U(\infty)$ is a transformation in S such that
$$\|g\xi\| = \|\xi\|$$
By duality $\langle x, g\xi\rangle = \langle g^* x, \xi\rangle$, $x \in S^*, \xi \in S$, the infinite-dimensional unitary group is also defined on S^*, the two groups being algebraically isomorphic.

Quantum control in infinite dimensions

The infinite dimensional unitary group

- Quantum scattering states are in S^* not in $L^2(\mathbb{R}^d)$.
- For the harmonic analysis on $U(\infty)$ one needs functionals on S^*.
- $U(\infty)$ is a complexificafion of $O(\infty)$, the infinite-diemnsional orthogonal group. A standard result states that if a measure μ is invariant under $O(\infty)$ it must be of the form

$$\mu = a\delta_0 + \int \mu_\sigma dm(\sigma)$$

  a sum of a delta and Gaussian measures μ_σ with variance σ^2
- Hence we are led to consider the (L^2) space of functionals on S^* with a $O(\infty)$ $-$invariant Gaussian measure

$$(L^2) = L^2(S^*, B, \mu)$$

  B generated by cylinder sets in S^* and μ the measure

$$C(f) = \int_{S^*} e^{i\langle x,f\rangle} d\mu(x) = e^{-\frac{1}{2}\|f\|^2}, \qquad x \in S^*, f \in S$$

- For $U(\infty)$ one considers a complexified version (complex white noise space), (S_c^*, B_c, μ_c)

$$S_c = S + iS, \qquad S_c \ni \xi = \xi + i\overline{\xi}$$

$$S_c^* = S^* + iS^*, \qquad S_c^* \ni z_c = z + i\overline{z}$$

- The regular representation of $U(\infty)$

$$U_g \varphi(z) = \varphi(g^* z), \qquad z \in S_c^*, \varphi \in (L_c^2) \cong (L^2) \otimes (L^2)$$

- Decomposes into irreducible representations corresponding to the Fock space (chaos expansion) decomposition of (L_c^2)

$$(L^2) = \oplus_{n=0}^{\infty} \left(\oplus_{k=0}^{n} H_{n-k,k} \right)$$

$H_{n-k,k}$ being a complex Fourier-Hermite polynomial of degree $(n-k)$ in $\langle z, \xi \rangle$ and of degree k in $\left\langle \overline{z}, \overline{\xi} \right\rangle$

- Of particular interest for our purposes is the consideration of subgroups of $U\left(\infty\right)$
- Two classes of subgroups:
  - Subgroups based on coordinate vectors
  - Whiskers
- Examples:

  1 - G_∞ - Consider a basis $\left\{\xi_i\right\}$, the sequence of subspaces

$$V_n = span\left\{\xi_i, i = 1, \cdots, n\right\}$$

and the sequence of unitary groups

$$G_n = \left\{g \in U\left(\infty\right), \left.g\right|_{V_n} \in U\left(n\right), \left.g\right|_{V_n^\perp} = I\right\}$$

$$G_\infty = \text{proj.}\lim_{n\rightarrow\infty} G_n$$

G_∞ is an infinite-dimensional subgroup but all its transformations may be approximated by finite-dimensional unitary transformations.

- 2 - The Lévy group
  Let π be an automorphism of $\mathbb{Z}_+ = \{1, 2, 3, \cdots\}$. Then

$$g_\pi : \xi = \sum_1^\infty a_n \xi_n \longrightarrow g_\pi \xi = \sum_1^\infty a_n \xi_{\pi(n)}$$

Density of the automorphism

$$d(\pi) = \lim \sup_{N \to \infty} \frac{1}{N} \sharp \{ n \leq N : \pi(n) > N \}$$

The Lévy group

$$G_L = \{ g_\pi : d(\pi) = 0, g_\pi \in U(\infty) \}$$

is a discrete infinite subgroup of $U(\infty)$

- Average power of a transformation of $U(\infty)$

$$a.p(g)(x) = \lim_{N \to \infty} \sup \frac{1}{N} \sum_{1}^{\infty} \langle x, g\xi_n - \xi_n \rangle^2$$

  If $a.p(g)$ is positive $\mu-$almost surely then g is said to be *essentially infinite dimensional.*
  Many elements of the Lévy group are essentially infinite dimensional.
  Example: $\pi(2n-1) = 2n, \pi(2n) = 2n-1$.
  It means that infinitely many coordinates $\langle x, \xi_n \rangle$ change significantly.
- Conclusion: To generate $U(\infty)$ some essentially infinite dimensional elements are needed.
- **The next result shows that one such transformation is enough**.

A control result in the infinite-dimensional Hilbert sphere

(Karwowski, R V M - 2003)

- Consider the space of double-infinite square-integrable sequences

$$a = \{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, \cdots\} \in \ell^2(\mathbb{Z})$$

$$|a| = \left(\sum_{-\infty}^{\infty} |a_k|^2 \right)^{\frac{1}{2}} < \infty$$

with basis $e_k = \{\cdots, 0, 0, 1_k, 0, 0, \cdots\}$

$$a = \sum_{-\infty}^{\infty} a_k e_k$$

- Define:

(i) A linear operator T_+ and its inverse

$$T_+ e_k = e_{k+1}, \qquad k \in \mathbb{Z}$$

$$T_+^{-1} e_k = e_{k-1}, \qquad k \in \mathbb{Z}$$

A control result in the infinite-dimensional Hilbert sphere

- (ii) Another linear operator Π

$$\begin{aligned} \Pi e_0 &= e_1 \\ \Pi e_1 &= e_0 \\ \Pi e_k &= e_k \quad , \quad k \in \mathbb{Z} \setminus \{0,1\} \end{aligned}$$

- Then $\Pi_n = U_+^n \Pi U_+^{-n}$ exchanges a_n with a_{n+1} in $a = \sum_{-\infty}^{\infty} a_k e_k$

$$\begin{aligned} \Pi_n e_n &= e_{n+1} \\ \Pi e_{n+1} &= e_n \\ \Pi e_k &= e_k \qquad , \quad k \neq n, n+1 \end{aligned}$$

Lemma (1)

Given $a \in \ell^2(\mathbb{Z})$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$, the operator

$$\Pi_{k,k+l} a = \Pi_k \Pi_{k+1} \cdots \Pi_{k+l-2} \Pi_{k+l-1} \cdots \Pi_{k+1} \Pi_k a$$

exchanges the coefficients of e_k and e_{k+l} .

A control result in the infinite-dimensional Hilbert sphere

Theorem (1)

Let $G\left(T_{+},\Pi\right)$ stand for the group generated by T_{+}, T_{+}^{-1} and Π. Then for any $0 \neq a \in \ell^{2}\left(\mathbb{Z}\right)$ the linear span of $G\left(T_{+},\Pi\right) a$ is dense in $\ell^{2}\left(\mathbb{Z}\right)$.

- *Proof*: It is sufficient to show that $b \perp G\left(T_{+},\Pi\right) a$ implies $b = 0$.
  (a) Suppose $b = e_k$ for some k. $a \neq 0 \Rightarrow \exists\ l \in \mathbb{N} \cup \{0\}$ such that at least one of the numbers a_{k+l} or a_{k-l} is $\neq 0$.
  Then $\left(b, \Pi_{k,k+l} a\right) = a_{k+l}$ or $\left(b, \Pi_{k,k-l} a\right) = a_{k-l}$, a contradiction.
  Similarly if both a and b are terminating.
- (b) Suppose b terminating but a not. Then $\exists N$ such that
  $\left(b, a\right) = \sum_{-N}^{N} b_k^{*} a_k = 0$, $b_N^{*} a_N \neq 0$
  and l with $a_{N+l} \neq a_N$ or $a_{-N-l} \neq a_N$.
  Hence $\left(b, \Pi_{N,N+l} a\right) = \sum_{-N}^{N-1} b_k^{*} a_k + b_N^{*} a_{N+l} \neq 0$ or
  $\left(b, \Pi_{N,-N-l} a\right) = \sum_{-N}^{N-1} b_k^{*} a_k + b_N^{*} a_{-N-l} \neq 0$, a contradiction.
  Similarly for a terminating and b nonterminating.

A control result in the infinite-dimensional Hilbert sphere

- (c) If neither a nor b terminates, there are pairs $a_k \neq a_l$ and $b_m \neq b_n$. With appropriate $g, g^{'} \in G\left(T_{+}, \Pi\right)$

$$\left(b, ga\right) = b_m^* a_k + b_n^* a_l + b_k^* a_m + b_l^* a_n + \sum_{r \neq k,l,m,n} b_r^* a_r = 0$$

$$\left(b, ga\right) = b_n^* a_k + b_m^* a_l + b_k^* a_m + b_l^* a_n + \sum_{r \neq k,l,m,n} b_r^* a_r = 0$$

  Hence $b_m^* a_k + b_n^* a_l = b_m^* a_k + b_n^* a_l$, possible only if either $b_m = b_n$ or $a_k = a_l$, a contradiction.

- Instead of the Π operator consider now a $U\left(2\right)$ group operating in $\left\{e_0, e_1\right\}$ and as the identity on $\ell^2\left(\mathbb{Z}\right) \ominus \left\{e_0, e_1\right\}$. In particular $\Pi \in U\left(2\right)$.

Theorem (2)

For any $0 \neq a \in \ell^2\left(\mathbb{Z}\right)$ the set $G\left(T_{+}, U\left(2\right)\right) a$ is dense in $\ell^2\left(\mathbb{Z}\right)$.

Lemma (2)

Suppose $0 \neq a \in \ell^2(\mathbb{Z})$ *is a terminating normalized sequence. Then, there is* $g \in G(T_+, U(2))$ *such that* $ge_0 = a$.

- *Proof*: Let

$$a = a_{-N}e_{-N} + \cdots + a_o e_0 + \cdots + a_N e_N$$

With $U(2)$ in the $\{e_0, e_1\}$ subspace and use of the $\Pi_{k,k+l}$ operators construct the sequence: $(g_i \in G(T_+, U(2)))$

$$\begin{array}{rcl} g_1 e_0 = & x_1 e_0 + a_{-N}e_{-N} & = \alpha_1 \\ g_2\alpha_1 = & x_2 e_0 + a_{-N+1}e_{-N+1} + a_{-N}e_{-N} & = \alpha_2 \\ \cdots & \cdots & \cdots \\ g_{2N}\alpha_{2N-1} = & x_{2N}e_0 + \sum_{-N}^{N} a_k e_k & = \alpha_{2N} \\ g_{2N+1}\alpha_{2N} = & a & \end{array}$$

Finally

$$g_{2N+1}g_{2N}\cdots g_2 g_1 e_0 = a$$

A control result in the infinite-dimensional Hilbert sphere

- *Proof of theorem 2*: Consider $a, b \in \ell^2(\mathbb{Z})$ with $|a| = |b| = 1$. Choose ε and N such that

$$\alpha = \left| \sum_{-N}^{N} a_k e_k \right| > 1 - \varepsilon; \qquad \beta = \left| \sum_{-N}^{N} b_k e_k \right| > 1 - \varepsilon$$

By lemma 2 there are $g, g' \in G(T_+, U(2))$ such that

$$g \sum_{-N}^{N} a_k e_k = \alpha e_0; \qquad g'(\alpha e_0) = \frac{\alpha}{\beta} \sum_{-N}^{N} b_k e_k$$

Hence

$$\left| b - g' g a \right| \leq 2\varepsilon + \left| 1 - \frac{\alpha}{\beta} \right| \leq 3\varepsilon$$

- **In conclusion: given any initial state $0 \neq a \in \ell^2(\mathbb{Z})$ it is possible by the unitary action of an element in $G(T_+, U(2))$ to approach as close as desired any other state b in $\ell^2(\mathbb{Z})$.**

Open systems in infinite-dimensions. A universal family for Kraus operators

- Given a topological space X and a family of continuous mappings $T_\alpha : X \to X$ with α belonging to some index set I, an element $x \in X$ is called *universal* if the set

$$\{T_\alpha x : \alpha \in I\}$$

is dense in X. The family $\{T_\alpha : \alpha \in I\}$ will be called universal if there is at least one universal element $x \in X$.

- For open systems consider evolutions by completely positive trace-preserving maps Φ,

$$\Phi(\rho) = \sum K_i \rho K_i^\dagger$$

- The problem of quantum control = search for a universal family of operators acting in the operator algebra of bounded operators $B(H)$
- No countable subset of $B(H)$ can be dense in the operator norm topology. The problem has no practical sense in this topology.

Open systems in infinite-dimensions. A universal family for Kraus operators

- Instead one should discuss density in the strong operator topology, that is, the one with neighborhood basis

$$N\left(x_i, \varepsilon_i; i = 1 \cdots n\right) = \left\{O : \|Ox_i\| < \varepsilon_i\right\}$$

  The $B\left(H\right)$ operator algebra is separable in this topology, meaning that any element may be approximated arbitrarily close by some $n \times n$ matrix.
- Consider a separable Hilbert space isomorphic to $\ell^2\left(\mathbb{Z}\right)$, the shift operator T_+ and its inverse T_+^{-1}, as well as a $U\left(2\right)$ group acting on the subspace $\left\{e_0, e_1\right\}$ and leaving the complementary space unchanged.
- This set of operators, generates all random-unitary transformations (Kraus operators proportional to unitaries) but not all trace-preserving completely positive operations.

Open systems in infinite-dimensions. A universal family for Kraus operators

- A new operator must be added, which may be choosen to be the projection on a basis state, for example $P_0 = |e_0\rangle \langle e_0|$.

Theorem (3)

P_0, T_+, T_+^{-1} and $U(2)$ generate a (strong operator topology-) universal family in the set of all density operators in infinite dimensions, with a dense set of universal elements.

- **Proof:**
  Let ρ be an arbitrary density operator in $n-$dimensional subspace V_n. Using T_+, T_+^{-1} translate the V_n subspace to contain the basis vectors e_0 and e_1. By the construction in Lemma 2, any normalized vector in V_n may be transformed by T_+, T_+^{-1} and $U(2)$ to an arbitrary basis state (say e_0) $\Longrightarrow$ T_+, T_+^{-1} and $U(2)$ generate all $U(n)$. With these transformations ρ may be brought to diagonal form ρ_D.

Open systems in infinite-dimensions. A universal family for Kraus operators

- To ρ_D apply the Kraus transformation

$$\sum_{i=1}^{n} K_i \rho_D K_i^{\dagger}$$

with $K_i = P_0 \Pi_{0,i}$ $(i = 0, \cdots, n-1)$ ($\Pi_{0,0}$ is the identity, an element of the $U(2)$ group). This transforms ρ_D into the single projector $P_0 = |e_0\rangle \langle e_0|$.

- Conversely by applying the Kraus operators $K_i = \sqrt{\rho_{D,i}} \Pi_{0,i}$ to P_0 and reversing the operations of the unitary group and the shift, P_0 may be transformed into any density operator of any other $m-$dimensional subspace.
- The fact that the density operators in finite-dimensional subspaces are dense (in the strong operator topology) on the set of all the density operators in infinite dimensions, completes the proof.

Smaller sets ?

- *Hypercyclic operators* - Universal family generated by a single operator and its powers.
- If it is

$$\{\lambda T^n x\}$$

  with λ a scalar, that is dense in X, the operator is called *supercyclic*.
- These notions being related to the density of a set, they depend on the topology of X.
- Hypercyclicity is a purely infinite-dimensional phenomenon.
- T_+,$U(2)$ (and P_0 for open systems) are already relatively small sets of generators, but an interesting question is whether a smaller set may be found, namely whether there are unitary hypercyclic or supercyclic operators.
- The answer depends both on the space topology and on the nature of the measure μ used in the $L^2(\mu)$ space. With norm topology in X, the answer is negative because no hyponormal operator $(\|Tx\| \geq \|T^*x\|; x \in X)$ can be hypercyclic or supercyclic.

Smaller sets ?

- The situation is different if density in the space X is relative to the weak topology, with neighborhood basis

$$N\left(\psi_1 \cdots \psi_n, \varepsilon_1 \cdots \varepsilon_n\right) = \left\{\phi : \left|\left\langle \psi_i \middle| \phi \right\rangle\right| < \varepsilon_i\right\}$$

- Then there are weakly supercyclic normal operators which are necessarily multiples of unitary operators. An example of unitary hypercyclic operator was constructed in a $L^2(\mu)$ space (Bayart and Matheron 2006). The construction is somewhat particular in that μ is a singular continuous measure in a thin Kronecker set.
- For measures that are absolutely continuous with respect to the Lebesgue measure one has no weakly supercyclic operator.
- Nevertheless a set is usually considered as "large" if it carries a probability measure μ for which the Fourier coefficients $\hat{\mu}(n)$ vanish at infinity. It has recently been proved that there is such a probability measure for which the corresponding $L^2(\mu)$ space has a weakly supercyclic operator (Shkarin 2007).

Smaller sets ?

- These results raise the interesting possibility that in some quantum spaces associated to singular continuous measures (hierarchical systems, for example), complete infinite-dimensional quantum controllability might be implemented with a single operator and its powers.

Non-unitary control

Classical control and differential forms

- A classical control system is a dynamical system $\frac{dx}{dt} = X$

Theorem (4)

(J. Math. Phys. 22, 1420, 1981); Let X be a vector field on a Riemannian manifold M_g. Then for each $x \in M_g$, there is a neighborhood Ω of x and a sympletic form ω on Ω such that the X is the sum of a gradient and an Hamiltonian vector field

$$\frac{dx}{dt} = \omega^{-1}(dH) + g(dS)$$

- Classical techniques of control (bang-bang, sliding mode, etc.) use both types of dynamics
- To compare with the situation in quantum control, the Strocchi map formulation is useful

- F. Strocchi; Rev. Mod. Phys. 38 (1966) 36

Kibble, Heslot, Anadan, Aharonov, Cirelli, Manià, Pizzocchero, Ashtekar, Schilling

- *Identifying real and imaginary parts of the wave function with coordinates and momenta, quantum evolution may be mapped onto a classical Hamiltonian system*
- With a basis $\{|k\rangle\}$ (of finite or infinite cardinality n) in a separable complex Hilbert space $\mathcal{H}^*$, a general quantum state $|\psi\rangle$ is

$$|\psi\rangle = \sum_k \psi_k |k\rangle$$

Define

$$\psi_k = \frac{1}{\sqrt{2}} (q_k + ip_k)$$

$\{q_k, p_k\}$ is a countable set of real phase-space coordinates.

Non-unitary control

The Strocchi map

- The scalar product in the complex Hilbert space $\mathcal{H}^*$

$$\left\langle \psi^{'} | \psi \right\rangle = \frac{1}{2} \sum_k \left(q_k^{'} q_k + p_k^{'} p_k \right) + i \left(q_k^{'} p_k - p_k^{'} q_k \right)$$

decomposes into a positive real inner product

$$G\left(\psi^{'}, \psi\right) = \frac{1}{2} \sum_k \left(q_k^{'} q_k + p_k^{'} p_k \right)$$

and a symplectic form

$$\Omega\left(\psi^{'}, \psi\right) = \frac{1}{2} \sum_k \left(q_k^{'} p_k - p_k^{'} q_k \right)$$

- Considering $\mathcal{H}^* = (\mathcal{H}, J)$ as a real Hilbert space $\mathcal{H}$ with a complex structure J, the triple (J, G, Ω) equips $\mathcal{H}$ with the structure of a Kähler space because

$$G\left(\psi^{'}, \psi\right) = \Omega\left(\psi^{'}, J\psi\right)$$

- The Schrödinger equation $i\frac{\partial}{\partial t}\left|\psi\right\rangle = H\left|\psi\right\rangle$ becomes the set of Hamilton's equations

$$\begin{aligned} \frac{d}{dt}q_k &= \frac{\partial}{\partial p_k}\mathbb{H} \\ \frac{d}{dt}p_k &= -\frac{\partial}{\partial q_k}\mathbb{H} \end{aligned}$$

associated to the symplectic form $\Omega\left(\psi^{'},\psi\right)$ and the "classical" Hamiltonian

$$\mathbb{H} = \frac{1}{2}\sum_{k,j}\left\{\left(q_k q_j + p_k p_j\right) ReH_{kj} + \left(p_k q_j - q_k p_j\right) ImH_{kj}\right\}$$

with $H_{kj} = \langle k|H|j\rangle$.

- *The time evolution of quantum mechanics is equivalent to the classical dynamics of a countable set of coupled oscillators* (the role of the symplectic form Ω)

- **What is the role of the metric G?**
Let $\mathcal{S}$ be the Hilbert sphere ($\|\psi\| = 1$). $G\left(\psi', \psi\right)$ is a metric in $\mathcal{S}$.
Measurement of an observable A. Let a be an eigenvalue of A and P_a the projector on the subspace V_a of $\mathcal{S}$ associated to this eigenvalue.
When the result of the measurement is a , the quantum state changes from $\psi \in \mathcal{S}$ to $\psi_a = \frac{P_a\psi}{\|P_a\psi\|} \in \mathcal{S}$ with probability $\|P_a\psi\|^2$. Given $\psi \in \mathcal{S}$ and $\phi \in V_a \subset \mathcal{S}$
$$(\psi - \phi, \psi - \phi)$$
is minimal when $\phi = \psi_a$.
- Because $(\psi - \phi, \psi - \phi) = G(\psi - \phi, \psi - \phi)$ one concludes that the **measurement projects ψ on the element of V_a that is closest to ψ in the $G-$metric**. The probability for this projection is
$$p_a = \|P_a\psi\|^2 = \left(1 - \frac{1}{2}G\left(\psi - \frac{P_a\psi}{\|P_a\psi\|}, \psi - \frac{P_a\psi}{\|P_a\psi\|}\right)\right)^2$$

- *Therefore, whereas the symplectic form Ω determines time-evolution, the $G-$metric controls the measurement process. It is the special role played by the metric that, in this framework, sets apart quantum from classical mechanics.*
- **Pure states** are represented by **points** $(\overrightarrow{q}, \overrightarrow{p})$ in a "phase-space" of dimension 2χ.
  Mixed states by **densities**: For a density matrix $\rho(t) = \sum_n \rho_n |\psi_n(t)\rangle \langle\psi_n(t)|$ $(\sum_n \rho_n = 1)$

$$\rho(t) = \int d\overrightarrow{q}\, d\overrightarrow{p}\, \rho(\overrightarrow{q}, \overrightarrow{p}) \sum_{k,k'} (q_k + ip_k)(q_{k'} - ip_{k'}) |k\rangle \left\langle k' \right|$$

with equation of motion

$$\frac{d}{dt}\rho(\overrightarrow{q}, \overrightarrow{p}) = -\frac{\partial\rho}{\partial\overrightarrow{q}} \cdot \frac{\partial\mathbb{H}}{\partial\overrightarrow{p}} + \frac{\partial\rho}{\partial\overrightarrow{p}} \cdot \frac{\partial\mathbb{H}}{\partial\overrightarrow{q}} = -\{\rho, \mathbb{H}\}$$

- **Measurements**

  The basis $\{|k\rangle\}$ being arbitrary, suppose it to be a basis of eigenstates of measured set A of observables. Before the measurement

$$\rho\left(\overrightarrow{\mu},\overrightarrow{\nu}\right)=\delta^{n}\left(\overrightarrow{\mu}-\overrightarrow{q}\right)\delta^{n}\left(\overrightarrow{\nu}-\overrightarrow{p}\right)$$

  After the measurement is performed and the result registered is k, the state becomes

$$\rho\left(\overrightarrow{\mu},\overrightarrow{\nu}\right)=\delta\left(\overrightarrow{\mu}-\frac{q_k}{\sqrt{q^2+p^2}}\overrightarrow{e_k}\right)\delta\left(\overrightarrow{\nu}-\frac{p_k}{\sqrt{q^2+p^2}}\overrightarrow{e_k^{'}}\right)$$

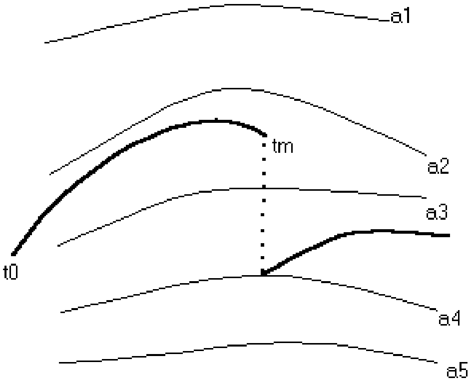
  ($\overrightarrow{e_k}$ and $\overrightarrow{e_k^{'}}$ are unit vectors along the $k-$coordinate and the $k-$momentum)

- For non-selective measurements one obtains a mixed state

$$\rho\left(\overrightarrow{\mu},\overrightarrow{\nu}\right)=\sum_k\left(q_k^2+p_k^2\right)\delta\left(\overrightarrow{\mu}-\frac{q_k\overrightarrow{e_k}}{\sqrt{q^2+p^2}}\right)\delta\left(\overrightarrow{\nu}-\frac{p_k\overrightarrow{e_k^{'}}}{\sqrt{q^2+p^2}}\right)$$

Non-unitary control

The Strocchi map



Non-unitary control

The Strocchi map. Using the metric structure for control.

- **Summarizing:**
  In the SM formulation:
  1) The (unobserved) dynamics of quantum states is a continuous symplectic evolution in a phase space.
  2) Quantum measurements are (minimal distance) jumps in a phase space
  3) Decoherence corresponds to splittings of the densities
- Measurements are the natural extension for quantum control techniques.

Control by measurement plus unitary evolution

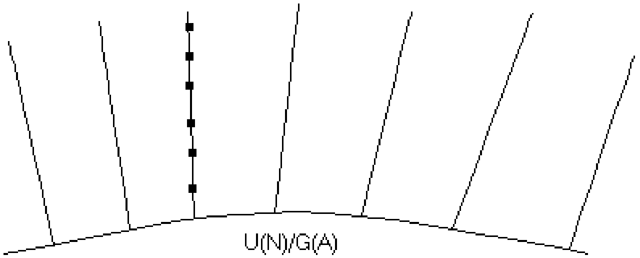
- Unitary controllability for Hamiltonians

$$H(t) = H_0 + \sum_{j=1}^{r} u_j(t) H_j$$

General result by Huang, Tarn and Clark. For systems with a finite number N of allowed states a necessary and sufficient condition for pure-state controllability is that the Lie algebra generated by $\{H_0, H_1, \cdots, H_r\}$ contains $su(N)$ or $sp(N/2)$ (if N is even) because these subgroups act transitively on the sphere S^{2N-1}.

- Suppose that $\mathcal{A} = \{H_0, H_1, \cdots, H_r\}_{LA}$ is a proper subalgebra of $u(N)$.
  Each orbit of the subgroup $G(\mathcal{A}) \subset U(N)$ may not cover $S_{\mathbb{C}}^{N-1}$.
  $S_{\mathbb{C}}^{N-1}$ would be a fiber space with the orbits of $G(\mathcal{A})$ as fibers and base $U(N)/G(\mathcal{A})$. A goal state ψ_f can only be reached from ψ_0 if ψ_0 and ψ_f belong to the same fiber.

Control by measurement plus unitary evolution



Control by measurement plus unitary evolution

- Control by the joint action of *measurement plus evolution:*

Theorem (5)

Given any goal state ψ_f , there is a family of observables $M\left(\psi_f\right)$ such that measurement of one of these observables on any ψ_0 plus unitary evolution leads to ψ_f if $G(\mathcal{A})$ is either $O\left(N\right)$ or $Sp\left(\frac{1}{2}N\right)$.

- *Proof:* If $G(\mathcal{A}) = O\left(N\right)$ or $Sp\left(\frac{1}{2}N\right)$ we may choose an orthonormal basis $\{\phi_i\}$ for S^{N-1} in the orbit $G(\mathcal{A})\psi_f$. Construct an observable $M = \sum_i a_i P_{\phi_i}$, P_{ϕ_i} being the projector on ϕ_i. Measuring this observable on any state ψ_0 and recording the measured value a_k the state becomes ϕ_k and then, by unitary evolution, ψ_f may be reached.
- *Remarks:*
  (i) A large family of observables for this type of control.

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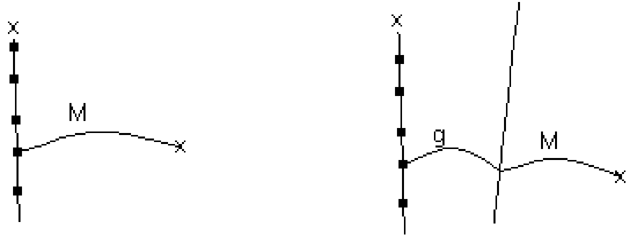
- (ii) If both ψ_0 and ψ_f are fixed a simpler set of controls H_j may be sufficient.
  Construct the M observable by $N-1$ vectors in the $N-1$-dimensional subspace orthogonal to ψ_0 plus a single vector in the orbit $G(\mathcal{A})\psi_f$, non-orthogonal to ψ_0.
  (iii) In case $G(\mathcal{A}) = Sp\,(N/2)$, the system is already pure-state controllable but, even in this case, it might be more efficient to use the measurement-plus-evolution scheme.
- General case



Control by measurement plus unitary evolution

Further developments in non-unitary control by
Mandilara and Clark;
Pechen, Il'in, Shuang and Rabitz;
Shuang, Zhou, Pechen, Wu, Shir and Rabitz

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