Network dynamics: tools and examples

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Contents

1 - Introduction

- Networks are everywhere:
  - Extended dynamical systems
  - Metabolic processes of living beings
  - Protein-protein networks
  - Gene expression and regulation
  - Social, economic and political networks
  - The internet

  R. Albert and A.-L. Barabási; Rev. Mod. Phys. 74 (2002) 47-97.
  S. N. Dorogovtsev and J. F. Mendes; Evolution of networks: From biological nets to the internet and WWW, Oxford Univ. Press 2003.
  R. Pastor-Satorras, M. Rubi and A. Diaz-Guilera (Eds.); Statistical mechanics of complex networks, Springer 2003.
- Most studies deal with networks as statistical objects, less attention has been paid to the dynamical phenomena taking place in the networks or to the behavior of the evolving networks as dynamical systems

2 - Differential dynamics tools
2.1 Describing dynamics by global functions

- Cohen - Grossberg theorem
  Symmetric systems $W_{ij}=W_{ji}$

$$\frac{dx_i}{dt} = a_i(x_i)\left\{ b_i(x_i) - \sum_{j=1}^{n} W_{ij} f_j(x_j) \right\}$$

$$V(x_i) = -\sum_{i=1}^{n} \int^{x_i} b_i(\xi_i) f_i^{'}(\xi_i) d\xi_i + \frac{1}{2} \sum_{j,k=1}^{n} W_{jk} f_j(x_j) f_k(x_k)$$

$$\frac{d}{dt} V(x_i) \leq 0 \qquad \text{if} \qquad a_i(x_i) f_i^{'}(x_i) > 0$$

Existence of a Lyapunov function

2.1 Describing dynamics by global functions

- General systems

$$
\begin{aligned}
W_{ij} &= W_{ij}^{(S)} + W_{ij}^{(A)} \\
W_{ij}^{(S)} &= \frac{1}{2}\left(W_{ij} + W_{ji}\right) \\
W_{ij}^{(A)} &= \frac{1}{2}\left(W_{ij} - W_{ji}\right) \\
V^{(S)} &= -\sum_{i=1}^{n} \int^{x_i} b_i(\xi_i) f_i'(\xi_i) d\xi_i \\
&\quad + \frac{1}{2} \sum_{j,k=1}^{n} W_{jk}^{(S)} f_j(x_j) f_k(x_k) \\
H &= \sum_{i=1}^{n} \int^{x_i} \frac{f_i(\xi_i)}{a_i(\xi_i)} d\xi_i
\end{aligned}
$$

2.1 Describing dynamics by global functions

- General systems

Theorem. If $a_i(x_i)/f_i'(x_i) > 0 \; \forall x, i$ and $W_{ij}^{(A)}$ has an inverse, then

$$\dot{x}_i = \dot{x}_i^{(G)} + \dot{x}_i^{(H)}$$

$$\dot{x}_i^{(G)} = -\frac{a_i(x_i)}{f_i'(x_i)} \frac{\partial V^{(S)}}{\partial x_i} = -\sum_j g_{ij}(x) \frac{\partial V^{(S)}}{\partial x_j}$$

$$\dot{x}_i^{(H)} = -\sum_j a_i(x_i) w_{ij}^{(A)}(x) a_j(x_j) \frac{\partial H}{\partial x_j} = \sum_j \Gamma_{ij}(x) \frac{\partial H}{\partial x_j}$$

$$g_{ij}(x) = \frac{a_i(x_i)}{f_i'(x_i)} \delta_{ij}$$

$$\omega_{ij}(x) = -a_i(x_i)^{-1} \left(W^{(A)-1}\right)_{ij} (x) a_j(x_j)^{-1}$$

2.1 Describing dynamics by global functions

- In general

$$\dot{x}_i = -\sum_j g_{ij}(x)\frac{\partial V}{\partial x_j} + \sum_{k=1}^{n-1}\sum_{j=1}^{n}\left(\omega_{(k)}^{-1}(x)\right)_{ij}\frac{\partial H^{(k)}}{\partial x_j}$$

- Applications

  Construction of multistable systems

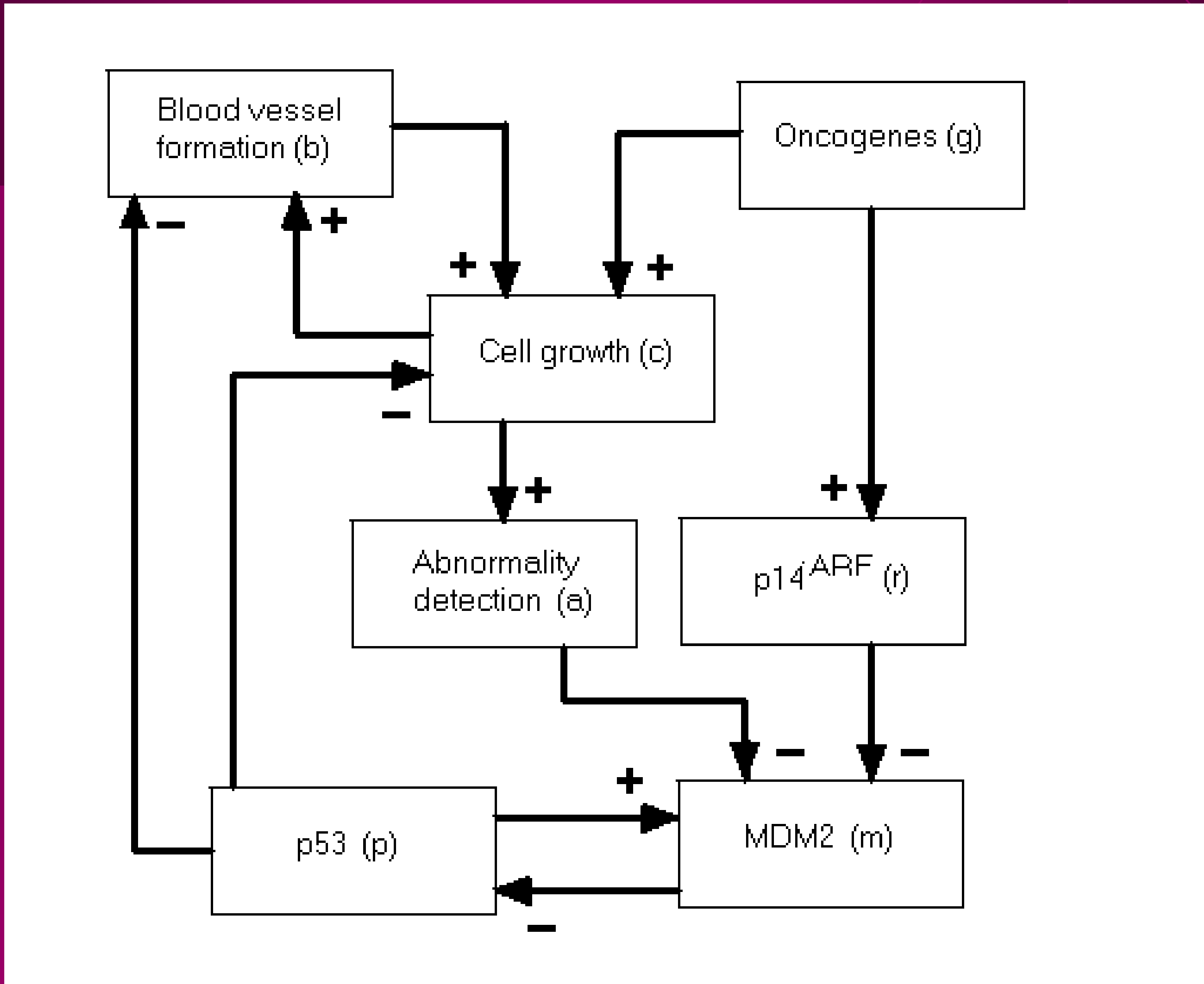
  Construction of invariant measures

2. 1 Cycles (a necessary condition)

$$\oint\left\{\left(\nabla H^{(i)}\cdot\nabla V\right)+\sum_{k\neq i}\omega_{(k)}\left(\nabla H^{(i)}\cdot\nabla H^{(k)}\right)\right\}d\gamma_i = 0$$

2.3 Network examples

A gene regulation network (simplified p53)



2.3 Network examples

A gene regulation network (simplified p53)

- $$\begin{aligned}\frac{dp}{dt} &= 1 - W_{pm} f_m(m) \\ \frac{dm}{dt} &= f_p(p) - W_{mr} g - W_{ma} a - \gamma_m m \\ \frac{db}{dt} &= f_c(c) - W_{bp} f_p(p) \\ \frac{dc}{dt} &= g + W_{cb} f_b(b) - W_{cp} f_p(p)\end{aligned}$$

Global functions

$$\begin{aligned}V^{(S)} &= -\int^p f_p'(\xi)\, d\xi - \int^c f_c'(\xi)\, d\xi \\ &\quad + \int^m (W_{mr} g + W_{ma} a + \gamma_m \xi) f_m'(\xi)\, d\xi \\ &\quad + \frac{1}{2} \sum_{x_i = p,m,b,c} W_{ij}^{(S)} f_i(x_i) f_j(x_j) \\ H &= \sum_{x_i = p,m,b,c} \int^{x_i} f_i(\xi)\, d\xi\end{aligned}$$

$$g_{ij} = \frac{1}{f_i'(x_i)} \delta_{ij} \qquad \Gamma_{ij} = -W_{ij}^{(A)}$$

2.3 Network examples

A gene regulation network (simplified p53)

- Conclusions :

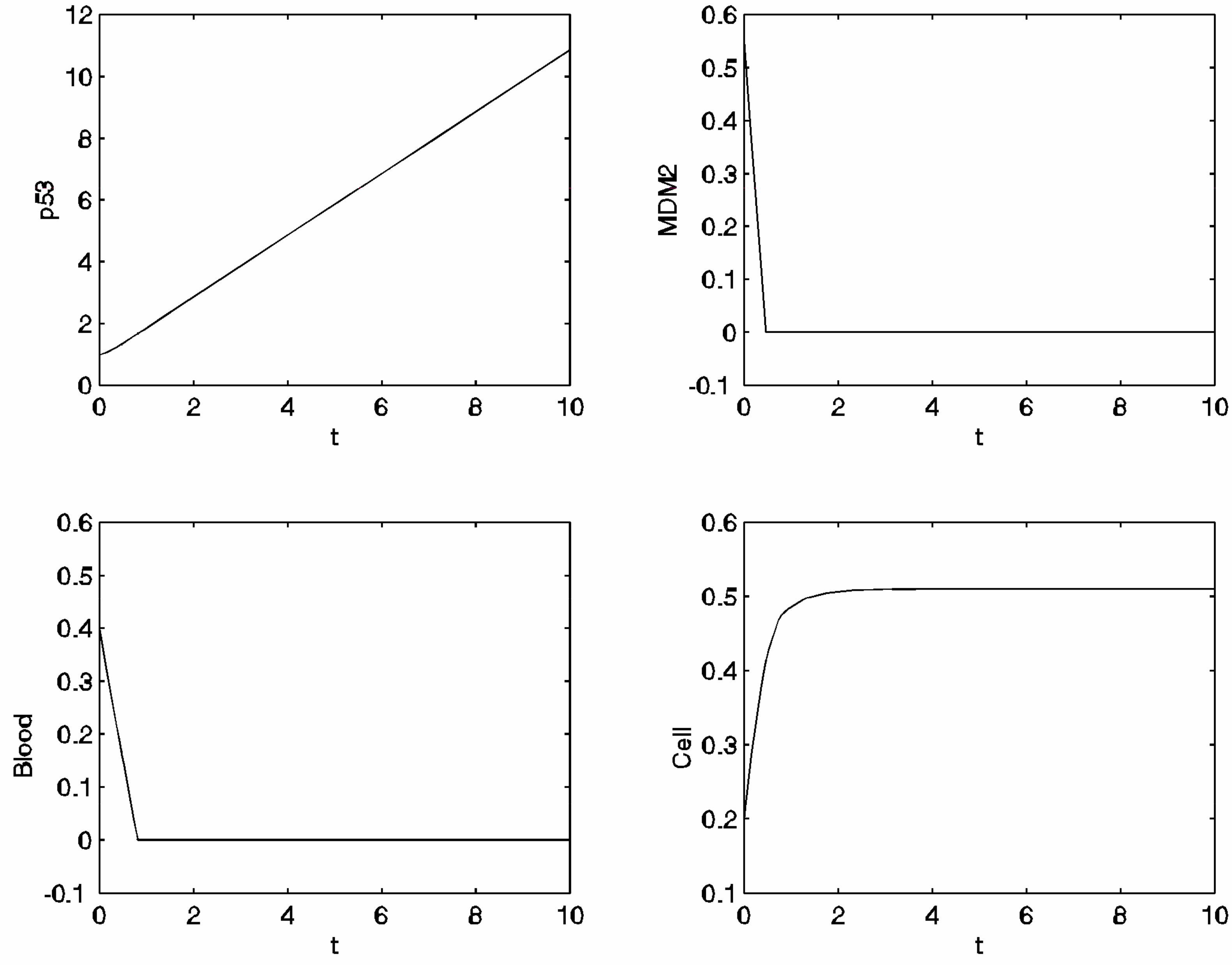
  - Damped Hamiltonian oscillation for the p - m system

  - Runaway behavior of the b-c system arising from its dominantly gradient dynamics

  - Controlling action of p53 is only be effective in particular circumstances. Depends on the initial conditions

2.3 Network examples

A gene regulation network (simplified p53)



2.3 Network examples

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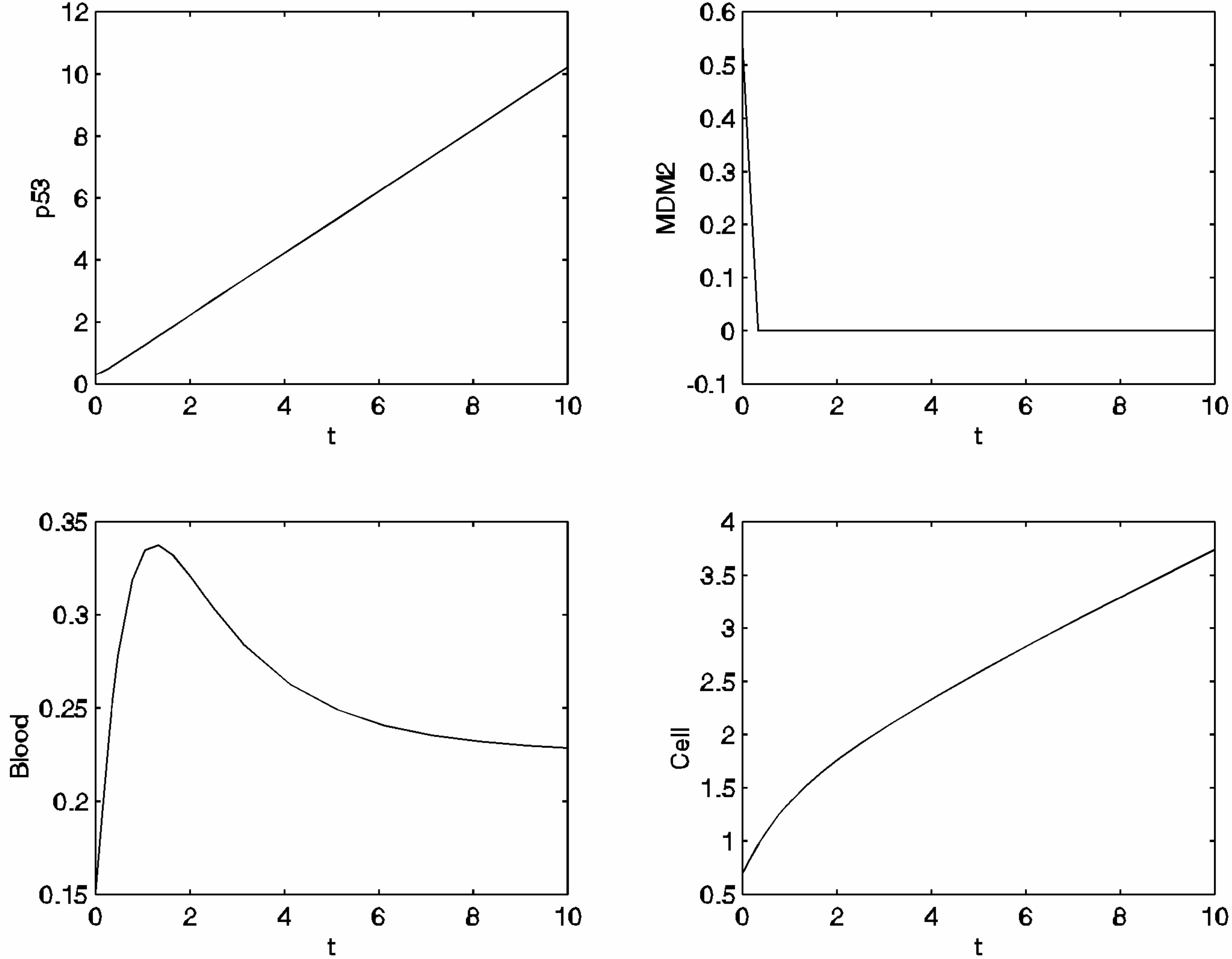


2.3 Network examples

Evolving networks as dynamical systems

- First example :

$$V_1(\{W\}) = \alpha \sum_{i<j} W_{ij}^2 (W_{ij}-1)^2 + \beta \sum_{i\neq j\neq l} (W_{ij}-1)^2 W_{jl}^2$$

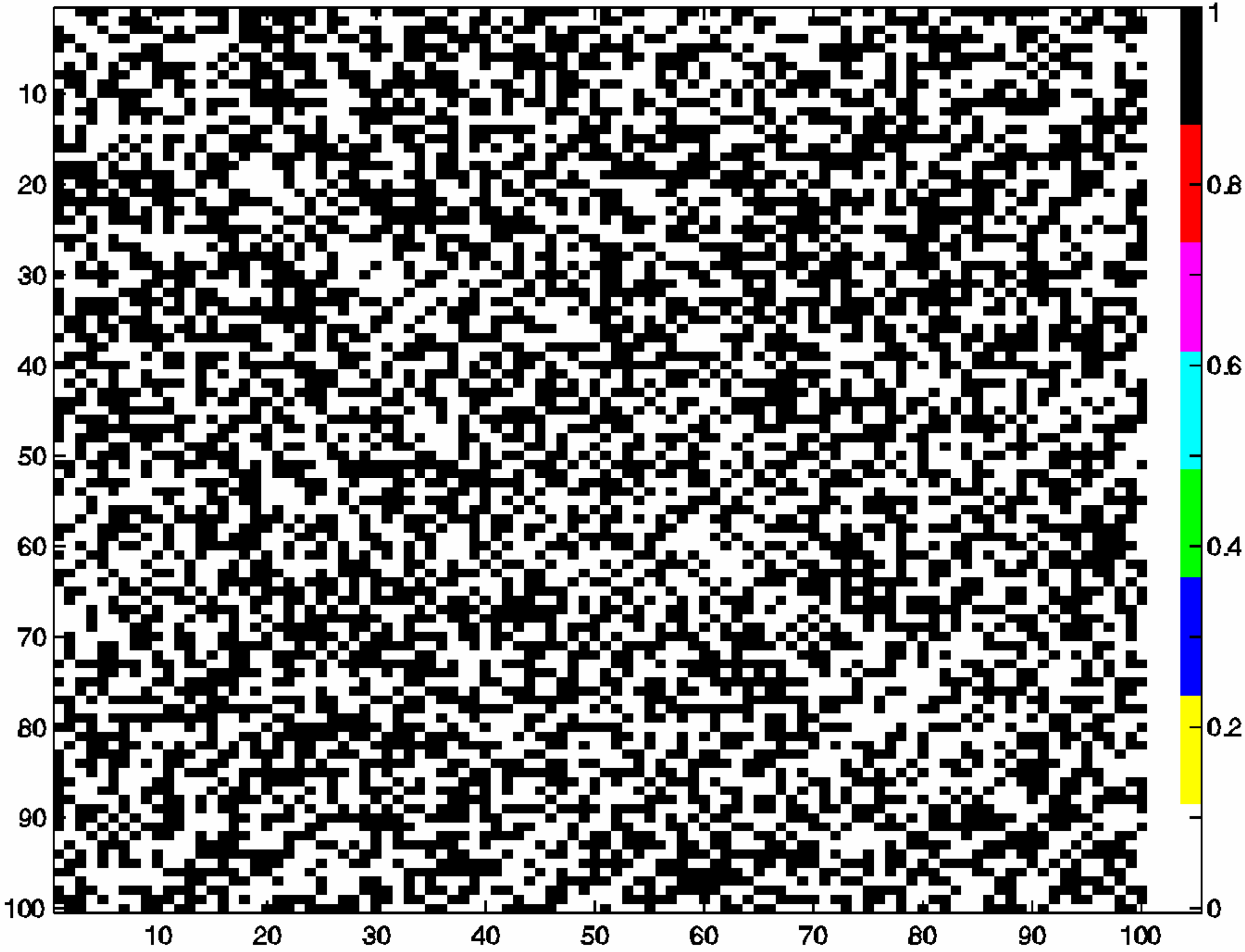
$$\frac{dW_{ij}}{dt} = -\frac{\partial V_1}{\partial W_{ij}}$$

- Two cases :
  (1) $\alpha \neq 0 \quad \beta = 0$
  (2) $\alpha \neq 0 \quad \beta \neq 0$

- Node degree $K_i = \Sigma_j W_{ij}$
- A model of preferential attachment

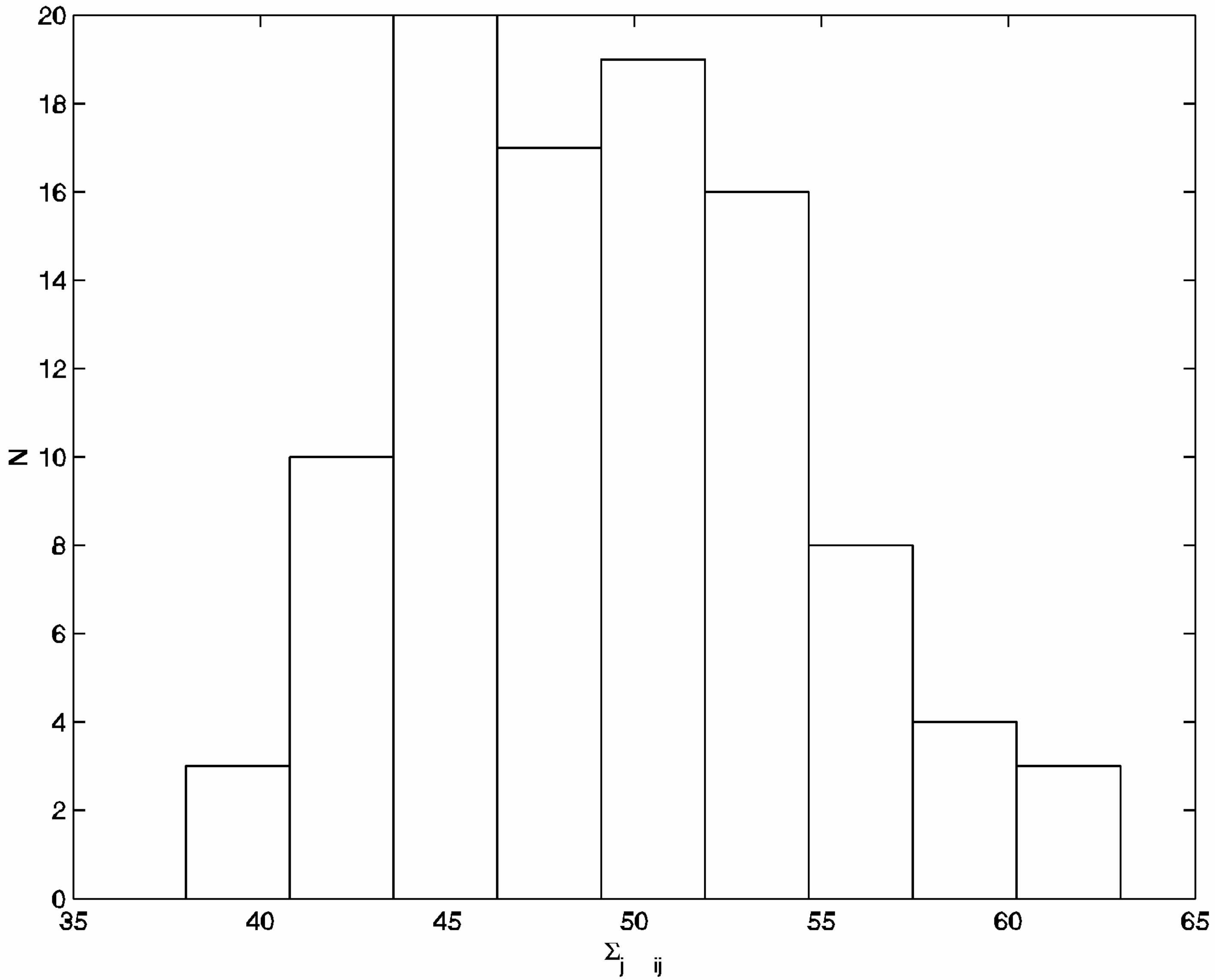
2.3 Network examples

α≠0 β=0



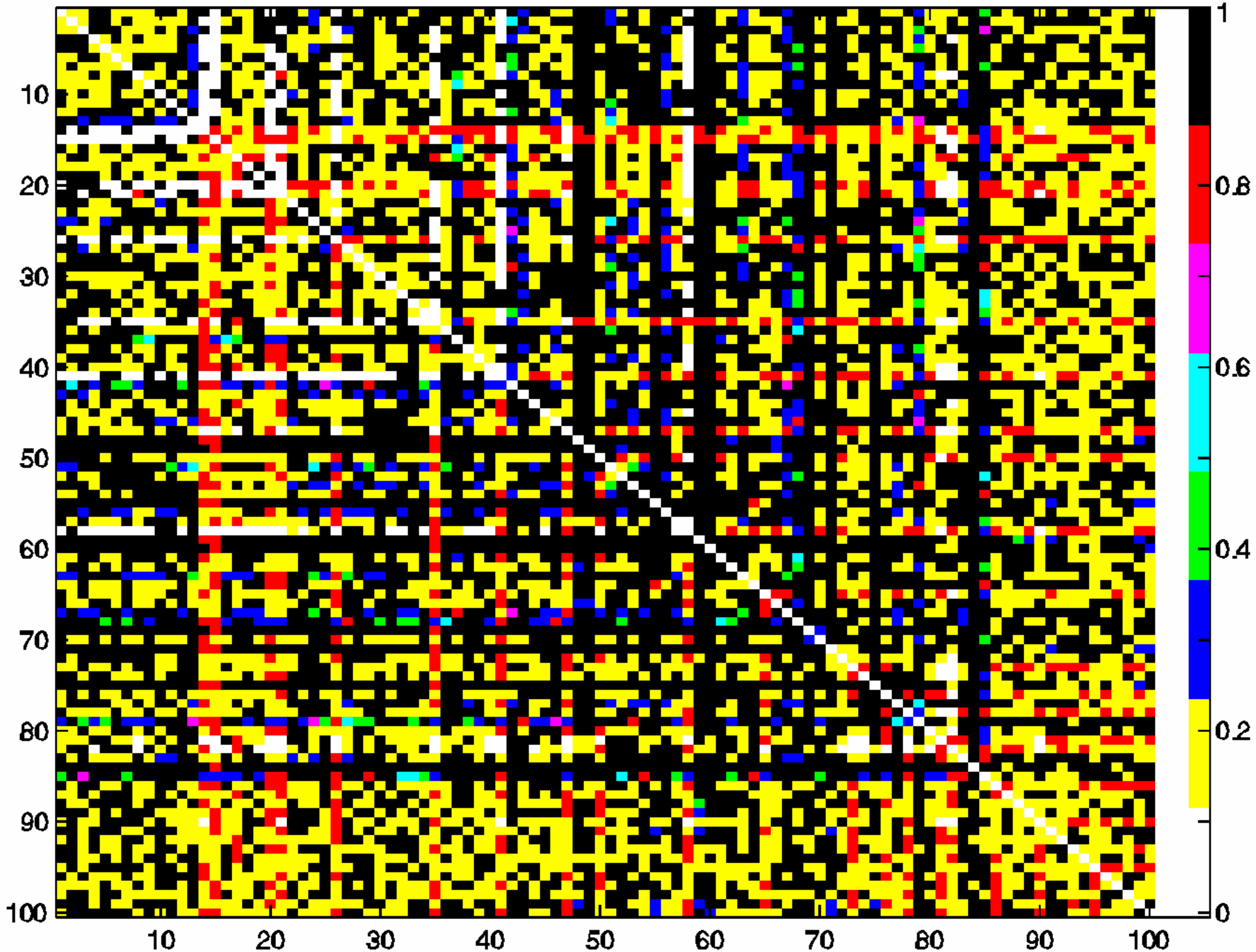
2.3 Network examples

$\alpha \neq 0 \quad \beta = 0$



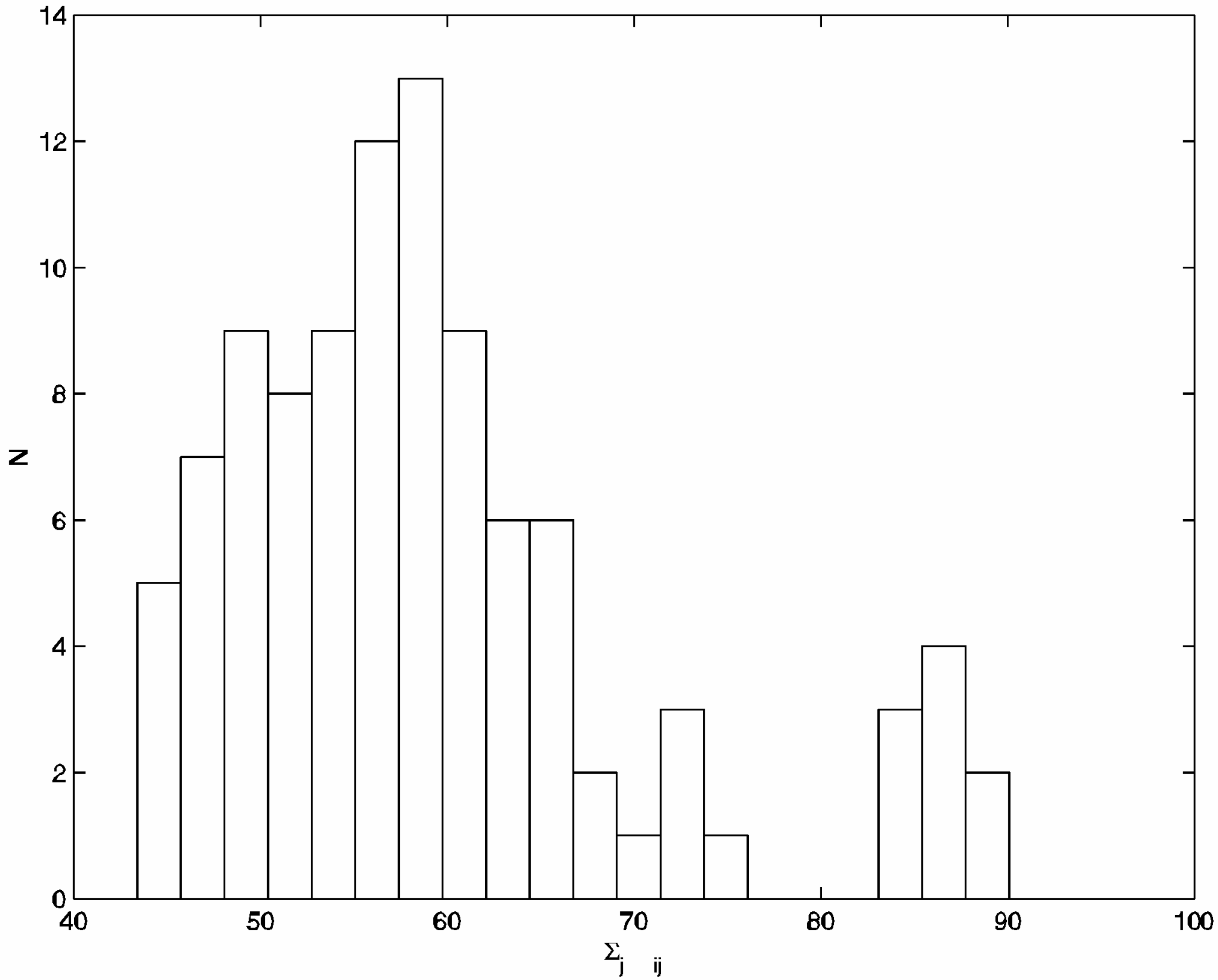
2.3 Network examples

$\alpha \neq 0$ $\beta \neq 0$



2.3 Network examples

$\alpha \neq 0 \quad \beta \neq 0$



2.3 Network examples

Evolving networks as dynamical systems

- Second example :

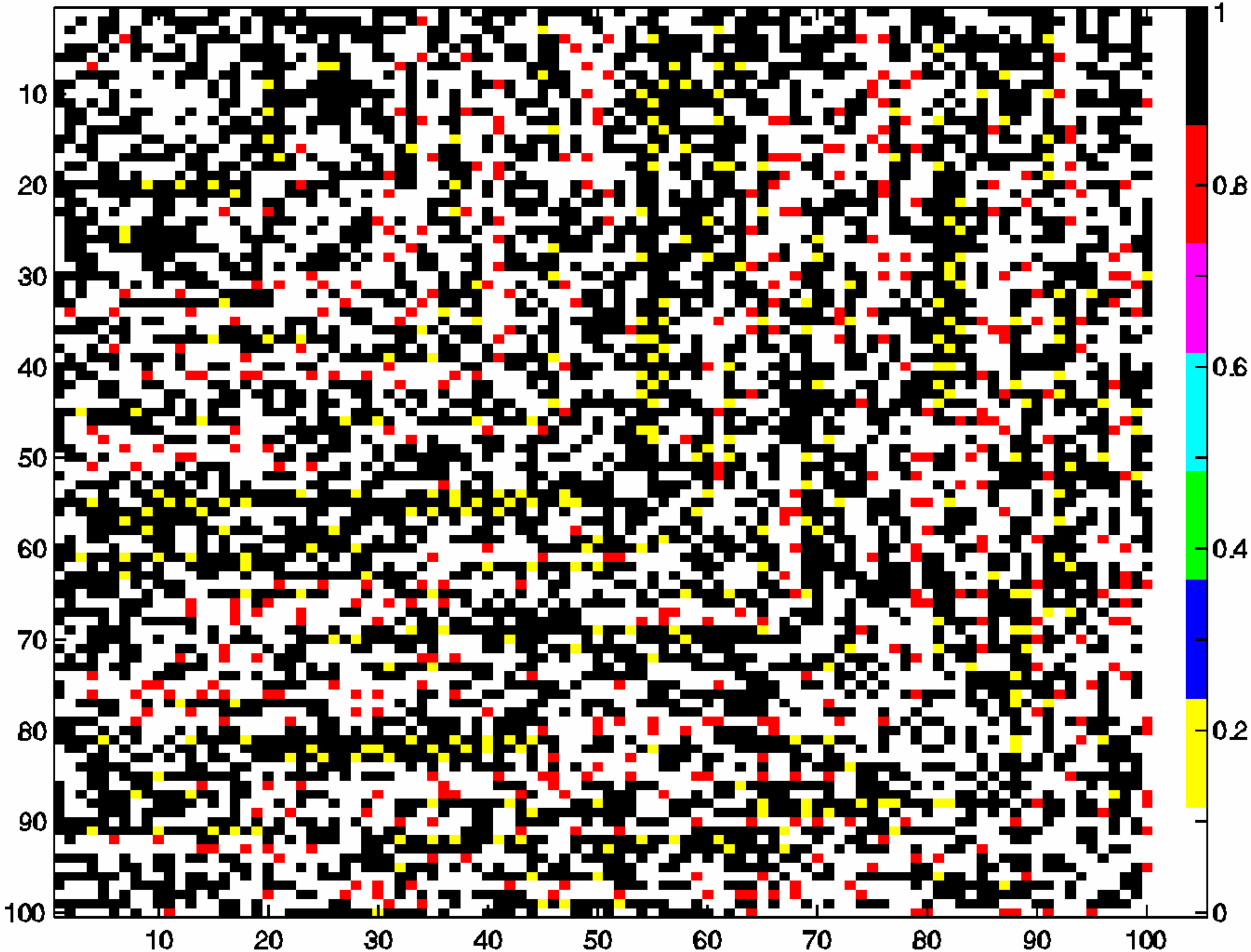
$$V_2(\{W\}) = \alpha \sum_{i<j} W_{ij}^2 (W_{ij}-1)^2 + \beta \sum_{i<j} \sum_{k\neq i,j} \frac{1}{|i-j|} (W_{ik}^2 + W_{jk}^2) \left((W_{ik}-1)^2 + (W_{jk}-1)^2\right)$$

Two cases :
Sum of correlations between node connections
(1) For $\alpha=1$ $\beta=0$ is 4.5
(2) For $\alpha=1$ $\beta=0.005$ is 20

- A model of node duplication

2.3 Network examples

Evolving networks as dynamical systems



3 - Ergodic tools

- Invariant measures and ergodic parameters

$$I_F(\mu) = \lim_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} F\left(f^n x_0\right)$$

- 3.1 Lyapunov and conditional exponents

From the k x k and (n-k)x(n-k) blocks of the Jacobian

Eigenvalues of the limits

$$\lim_{n\to\infty} \left(D_k f^{n*}(x) D_k f^n(x)\right)^{\frac{1}{2n}}$$

$$\lim_{n\to\infty} \left(D_{m-k} f^{n*}(x) D_k f^n(x)\right)^{\frac{1}{2n}}$$

Or

$$\lim_{n\to\infty} \frac{1}{n} \log \left\| D_k f^n(x) u \right\| = \xi_i^{(k)}$$

$$0 \neq u \in E_x^i / E_x^{i+1}$$

E_x^i is the subspace spanned by the eigenstates corresponding to eigenvalues $\leq \exp(\xi_i^{(k)})$

3 - Ergodic tools

- 3.2 Structure index

$$S = \frac{1}{N} \sum_{i=1}^{N_+} \left(\frac{\lambda_0}{\lambda_i} - 1 \right)$$

diverges whenever a Lyapunov exponent approaches zero from above (points where long time correlations develop)

- Self-organization (partitions $\Sigma_k = R^k \times R^{m-k}$)

$$I_\Sigma(\mu) = \sum_{k=1}^{N} \{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\}$$

$$h_k(\mu) = \sum_{\xi_i^{(k)}>0} \xi_i^{(k)}; h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)}>0} \xi_i^{(m-k)}; h(\mu) = \sum_{\lambda_i>0} \lambda_i$$

3 - Ergodic tools

- Self-organization concerns the dynamical relation of the whole to its parts. Therefore, $I_\Sigma(\mu)$ is a measure of dynamical self-organization
- Also, it is a measure of apparent dynamical freedom (or apparent rate of information production).
- These parameters characterize the dynamics of multiagent networks.
- Also, provide insight on the relation between dynamics and the topology of the network (the small world phase, for example)

3 - Ergodic tools

- 3.3 Construction of invariant measures by small random perturbations
  Physical (BRS) measures
  Let $dx_i = X_i(x)dt + \varepsilon\sigma(X)dW(t)$

  If
$$X(x) = -\nabla_{(g)} V(x)$$
$$\nabla_{(g)} = \text{gradient in the metric}$$
$$ds^2 = \sum a_{ij}(x)\, dx_i dx_j$$
$$a_{ij}(x) = \left(\sigma(x)\,\sigma^*(x)\right)^{-1}_{ij} = g_{ij}(x)$$
$$\rho^{\varepsilon}(x) = C_{\varepsilon} \exp\left(-2\varepsilon^{-2} V(x)\right)$$

  $\rho^{\varepsilon}(x)$ is the density of the invariant measure
  Also possible in more general cases

3 - Ergodic tools

- 3.4 Synchronization and dynamical correlations (Classical example: the Kuramoto model)
  A similar, discrete-time oscillators model :

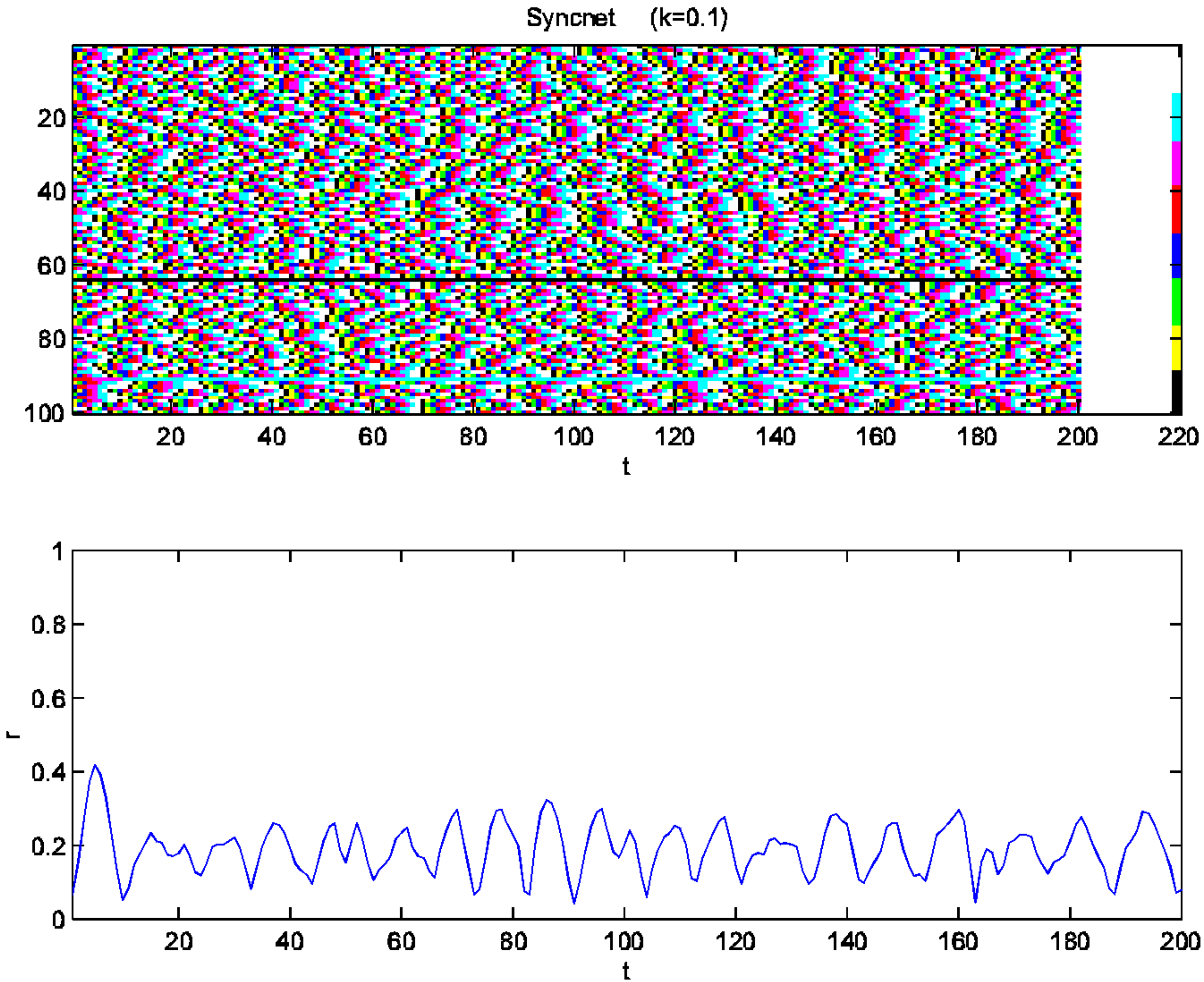
$$x_i(t+1) = x_i(t) + \omega_i + \frac{k}{N-1}\sum_{j=1}^{N} f_\alpha(x_j - x_i)$$

$$p(\omega) = \frac{\gamma}{\pi\left[\gamma^2 + (\omega - \omega_0)^2\right]}$$

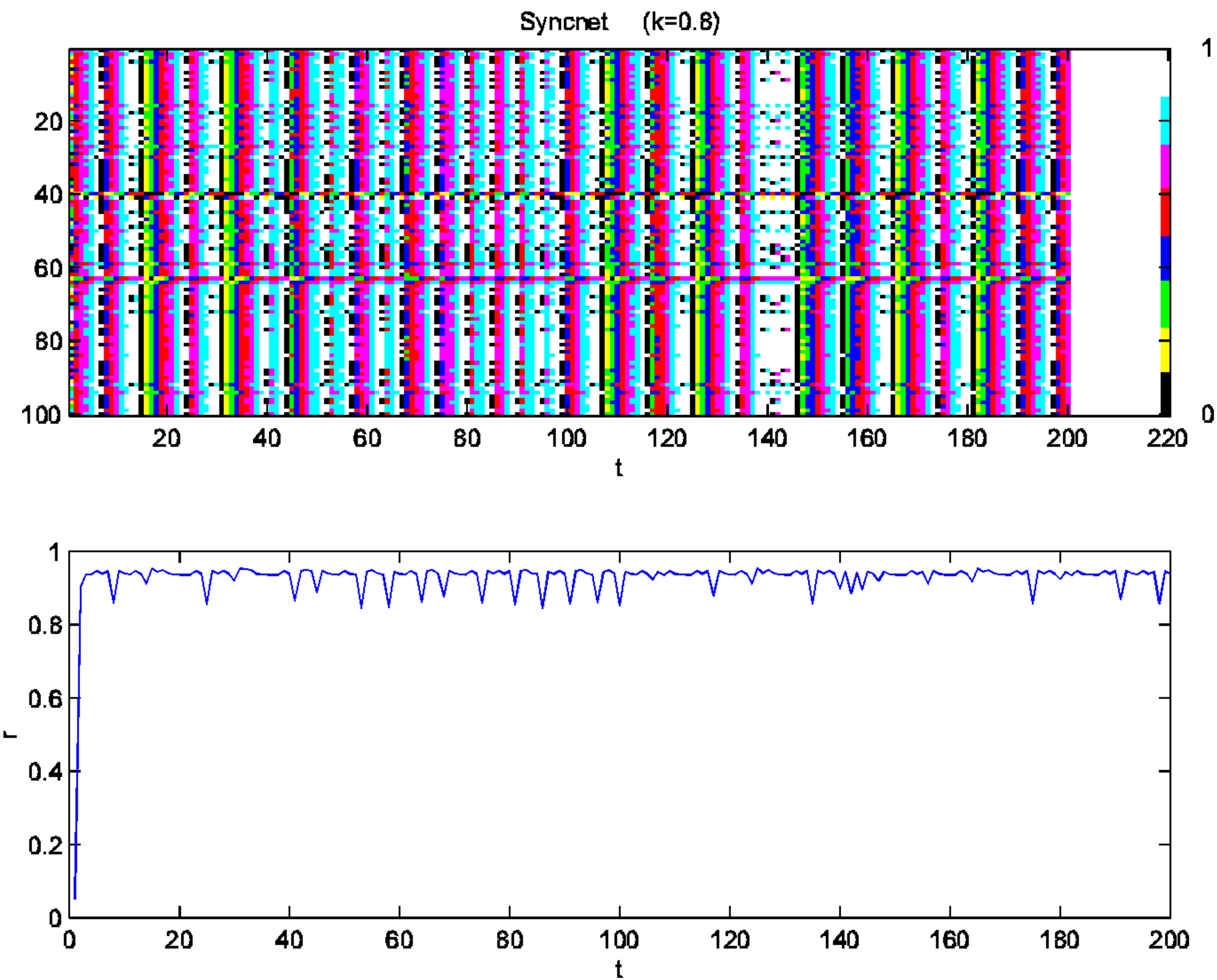
$$f_\alpha(x_j - x_i) = \alpha(x_j - x_i)\,(mod 1)$$

- Order parameter

$$r(t) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi x_j(t)}\right|$$



Syncnet (k=0.1)
20
40
60
80
100
t
220
1
0
r
0.2
0.4
0.6
0.8
120
140
160
180
200



Syncnet (k=0.8)
t
r
t

3 - Ergodic tools

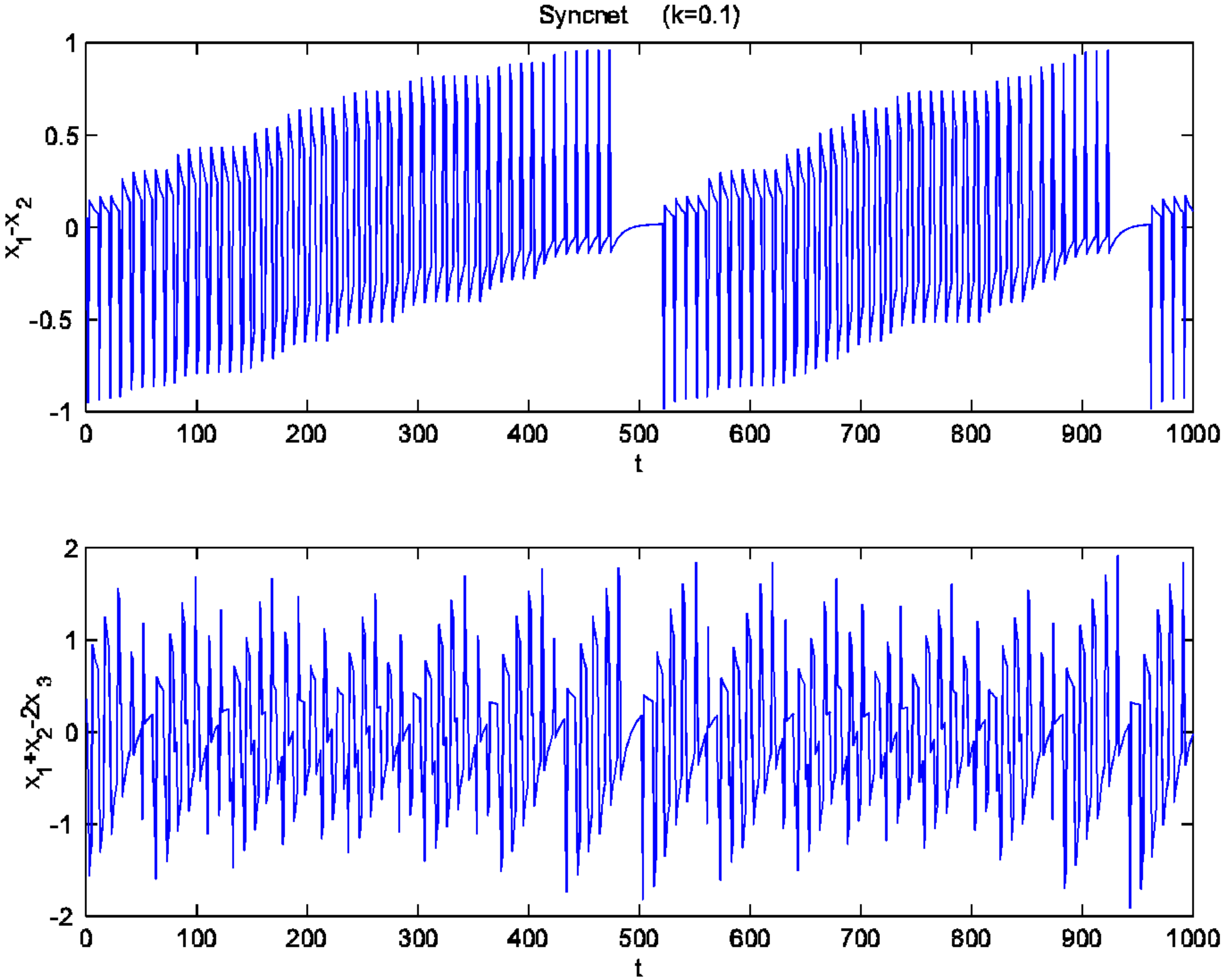
- The Lyapunov spectrum controls the dynamical self-organization of the system.
- In this case

  $\lambda_1=0$ and

  $\lambda_i=\log(1-\alpha\lambda k(N/N-1))$ (N-1) times

  N-1 contracting directions for $k\neq 0$

- $\Rightarrow$ strong dynamical correlations even before synchronization


Syncnet (k=0.1)
1
0.5
0
-0.5
-1
x1-x2
0
100
200
300
400
500
600
700
800
900
1000
t
2
1
0
-1
-2
x1+x2-2x3
0
100
200
300
400
500
600
700
800
900
1000
t

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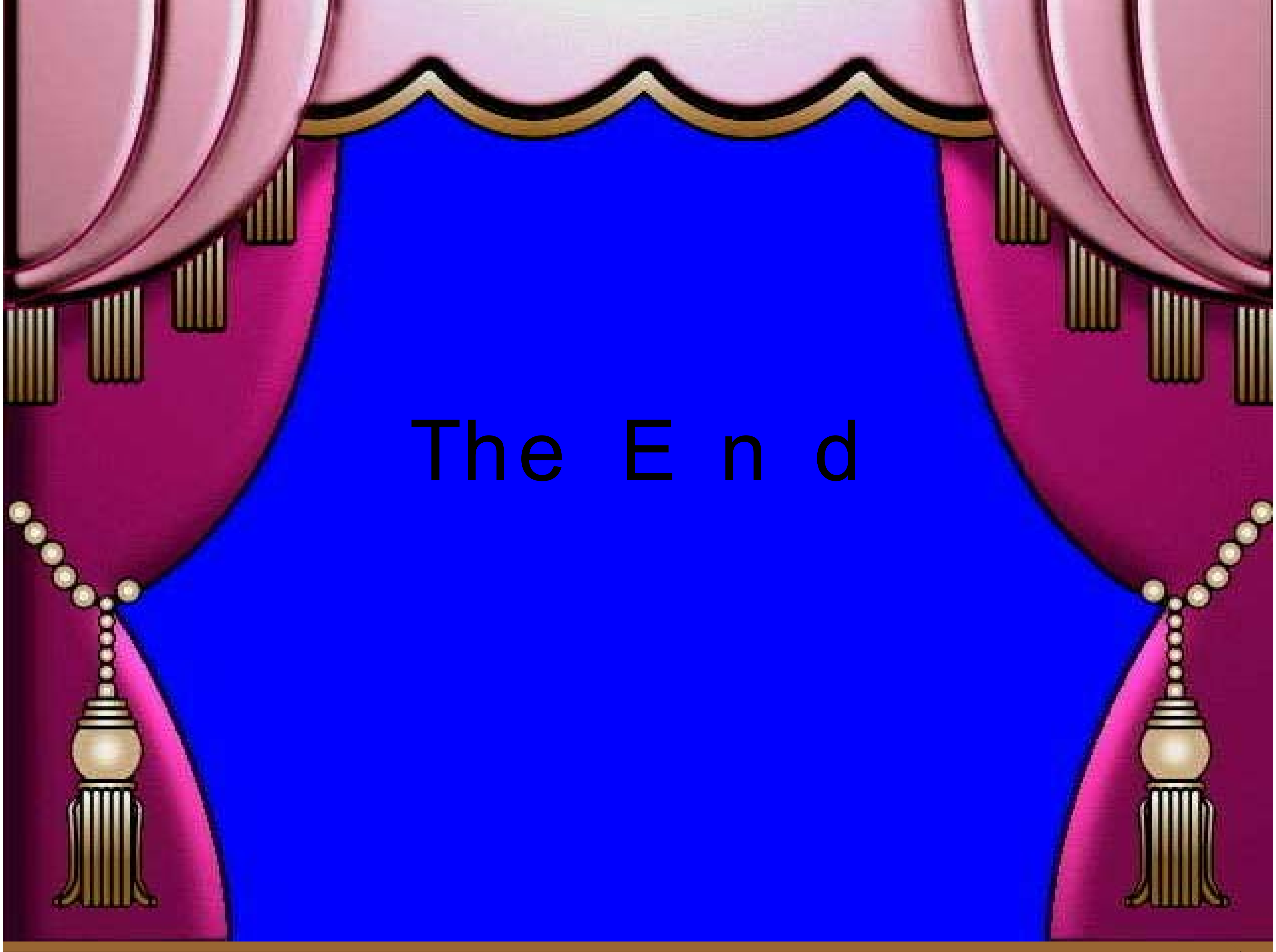
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