Stochastic solutions of kinetic equations and tomographic data analysis

A survey of some mathematical problems related to fusion plasmas

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The notion of stochastic solution

- Linear elliptic and parabolic equations (both with Cauchy and Dirichlet boundary conditions) have a probabilistic interpretation: a classical result and a standard tool in potential theory.
- An example: the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) \qquad \textit{with} \qquad u(0,x) = f(x) \tag{1}$$

- The solution may be written either as

$$u(t,x) = \frac{1}{2\sqrt{\pi}}\int \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\, dy \tag{2}$$

- or as

$$u(t,x) = \mathbb{E}_x f(X_t) \tag{3}$$

$\mathbb{E}_x$ being the expectation value, starting from x, of the Wiener process $dX_t = dW_t$

The notion of stochastic solution

- Eq.(1) is a *specification* of a problem
- (2) and (3) are *solutions* in the sense that they both provide algorithmic means to construct of a function satisfying the specification.
- An important condition for (2) and (3) to be considered as **solutions** is the fact that the algorithm is **independent** of the particular solution,

1. in the first case, an **integration** procedure
2. in the second, a **solution-independent** process.

- This should be contrasted with stochastic processes constructed from a given particular solution, as has been done for example for the Boltzman equation
- New **exact solutions** and also new **numerical algorithms**

Stochastic solutions of nonlinear partial differential equations

- **The basic idea:** In the linear partial differential equation case, once the relevant stochastic process is identified, the process is started from the point x where the solution is to be computed, and the solution is a functional of the exit values of the process (from a space D or a space-time $D \times [0, t]$ domain)
- **Conjecture:** For the nonlinear equations the relevant process has a diffusion, propagation or jump component associated to the linear part of the equation plus a branching mechanism associated to the nonlinear part. The solution will be a functional of the exit measures generated by the process.
- **The construction:** Rewrite the equation as an integral equation. Give a probabilistic interpretation to the integral equation. In the end the stochastic solution is equivalent to the construction of a tree-indexed measure and a sampling evaluation of the Picard series.

Existing results

1. The KPP equation (McKean)
2. Diffusion equation with u^α ($\alpha \in [0,2]$) nonlinearities (Dynkin)
3. The Navier-Stokes equation (LeJan, Schnitzman, Waymire, Ossiander, Batacharia, Orum)
4. The Poisson-Vlasov equation (Cipriano, Floriani, Lima, R. V. M.)
5. The Euler equation (R. V. M.)
6. A fractional version of the KPP equation (Cipriano, Ouerdiane, R. V. M.)
7. Poisson-Vlasov in an external magnetic field (R. V. M.)

Poisson-Vlasov in an external magnetic field

- Poisson-Vlasov

$$
\begin{aligned}
0 &= \frac{\partial f_i}{\partial t} + \left(\overrightarrow{v} \cdot \nabla_x + \frac{e_i}{m_i} \frac{\overrightarrow{v}}{c} \times \overrightarrow{B}(x) \cdot \nabla_v \right) f_i \\
&\quad + \frac{e_i}{m_i} \int d^3x' \sum_j e_j \int d^3u f_j \left(x', u, t \right) \frac{\overrightarrow{x - x'}}{\left| x - x' \right|^3} \cdot \nabla_v f_i (x, v, t)
\end{aligned}
$$

- Fourier transformed version, $F(\xi, t) = \frac{1}{(2\pi)^3} \int d^6\eta f(\eta, t) e^{i\xi\cdot\eta}$, with $\eta = \left(\overrightarrow{x}, \overrightarrow{v} \right)$ and $\xi = \left(\overrightarrow{\xi_1}, \overrightarrow{\xi_2} \right) \mathrel{\hat{=}} (\xi_1, \xi_2)$

$$
\begin{aligned}
\frac{\partial F_i(\xi,t)}{\partial t} &= \left(\overrightarrow{\xi_1} \cdot \nabla_{\xi_2} + \frac{e_i}{cm_i} \nabla_{\xi_2} \times \overrightarrow{B} \left(-i\nabla_{\xi_1} \right) \cdot \overrightarrow{\xi_2} \right) F_i (\xi, t) \\
&\quad - \frac{4\pi e_i}{m_i} \int d^3\xi_1' F_i \left(\xi_1 - \xi_1', \xi_2, t \right) \frac{\overrightarrow{\xi_2} \cdot \overrightarrow{\xi_1'}}{\left| \xi_1' \right|^2} \sum_j e_j F_j \left(\xi_1', 0, t \right)
\end{aligned}
$$

Poisson-Vlasov in an external magnetic field

Linear evolutions

- $$\begin{aligned} \frac{d}{dt}x(t) &= -v(t) \\ \frac{d}{dt}v(t) &= -\frac{e_i}{cm_i}\left(v(t)\times B(x(t))\right) \end{aligned}$$
- $$\begin{aligned} \frac{d}{dt}\xi_1(t) &= -\frac{e_i}{cm_i}\left(\nabla_{\xi_2}(t)\times i\nabla B\left(-i\nabla_{\xi_1}(t)\right)\cdot\xi_2(t)\right) \\ \frac{d}{dt}\xi_2(t) &= \xi_1(t)+\frac{e_i}{cm_i}B\left(-i\nabla_{\xi_1}(t)\right)\times\xi_2(t) \end{aligned}$$
- $$\begin{aligned} \frac{d}{dt}\nabla_{\xi_1}(t) &= -\nabla_{\xi_2}(t) \\ \frac{d}{dt}\nabla_{\xi_2}(t) &= -\frac{e_i}{cm_i}\left(\nabla_{\xi_2}(t)\times B\left(-i\nabla_{\xi_1}(t)\right)\right) \end{aligned}$$
- Write a stochastic solution for

$$\chi_i(\xi_1,\xi_2,t)=e^{-t|\xi_2|}\frac{F_i(\xi_1,\xi_2,t)}{h(\xi_1)}$$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field. Definitions

- Survival probability, up to time t, of an exponential process, $\boxed{e^{-t|\xi_2|}}$ and decaying probability $\boxed{\Pi\left(\xi_1,\xi_2,s\right)=\frac{|\xi_2(s)|e^{(t-s)|\xi_2(s)|-t|\xi_2|}}{N(\xi_1,\xi_2,t)}}$ with normalizing function

$$N\left(\xi_1,\xi_2,t\right)=\frac{1}{1-e^{-t|\xi_2|}}\int_0^t ds\left|\xi_2\left(s\right)\right|e^{(t-s)|\xi_2(s)|-t|\xi_2|}$$

- Branching probability

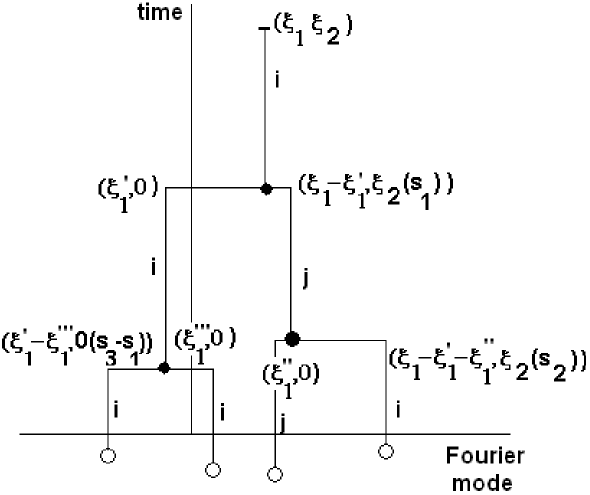
$$p\left(\xi_1,\xi_1^{'}\right)=\frac{\left|\xi_1^{'}\right|^{-1}h\left(\xi_1-\xi_1^{'}\right)h\left(\xi_1^{'}\right)}{\left(\left|\xi_1^{'}\right|^{-1}h*h\right)}$$

$$\left(\left|\xi_1^{'}\right|^{-1}h*h\right)\left(\xi_1\right)=\int d^3\xi_1^{'}\left|\xi_1^{'}\right|^{-1}h\left(\xi_1-\xi_1^{'}\right)h\left(\xi_1^{'}\right)$$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field

$$
\begin{aligned}
&\chi_i(\xi_1, \xi_2, t) \\
= \;& \boxed{e^{-t|\xi_2|}}\chi_i(\xi_1, \xi_2(t), 0) - \frac{8\pi e_i N(\xi_1, \xi_2, t)}{m_i} \frac{\left(\left|\xi_1'\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)} \\
& \times \int_0^t ds \boxed{\frac{|\xi_2(s)|}{N(\xi_1, \xi_2, t)} e^{(t-s)|\xi_2(s)|-t|\xi_2|}} \int d^3\xi_1' \boxed{p\left(\xi_1, \xi_1'\right)} \\
& \times \chi_i\left(\xi_1 - \xi_1', \xi_2(s), t-s\right) \frac{\overrightarrow{\xi}_2(s) \cdot \widehat{\xi_1'}}{|\xi_2(s)|} \sum_j \frac{e_j}{2} \chi_j\left(\xi_1', 0, t-s\right)
\end{aligned}
$$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field



Fourier-transformed Poisson-Vlasov. Uniform magnetic field

- Coupling constants

$$g_{ij}\left(\xi_1,\xi_1',s\right) = -\frac{8\pi e_i e_j N(\xi_1,\xi_2,t)}{m_i}\frac{\left(\left|\xi_1'\right|^{-1}h*h\right)(\xi_1)}{h(\xi_1)}\frac{\vec{\xi}_2(s)\cdot\hat{\xi}_1'}{|\xi_2(s)|}$$

$$g_{0i}\left(\xi_1,\xi_2\right) = \frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}$$

- Multiplicative functional = product of all the couplings for each realization. The solution $\chi_i\left(\xi_1,\xi_2,t\right)$ is the expectation value
- Existence conditions

(A) $\left|\frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}\right| \leq 1$

(B) $\left(\left|\xi_1'\right|^{-1} h * h\right)\left(\xi_1\right) \leq h\left(\xi_1\right)$

(C) $\left|\frac{8\pi e_i e_j N(\xi_1,\xi_2,t)}{\min_i\{m_i\}}\frac{\left(\left|\xi_1'\right|^{-1}h*h\right)}{h(\xi_1)}\right| \leq 1$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field

Theorem: *The stochastic process* $X(\xi_1, \xi_2, t)$*, above described, provides through the multiplicative functional a stochastic solution of the Fourier-transformed Poisson-Vlasov equation in a uniform magnetic field for arbitrary finite values of the arguments, provided the initial conditions at time zero satisfy the boundedness conditions (A).*

Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

$\overrightarrow{B}(\xi_1) = (2\pi)^{3/2}\overrightarrow{B}_0\delta^3(\xi_1) + \overrightarrow{b}(\xi_1)$ and $\chi_i(\xi_1, \xi_2, t) = e^{-t|\xi_2|\frac{F_i(\xi_1,\xi_2,t)}{h(\xi_1)}}$

$$\chi_i(\xi_1, \xi_2, t)$$

$$= \boxed{e^{-t|\xi_2|}}\chi_i(\xi_1, \xi_2(t), 0) - \frac{e_i N(\xi_1, \xi_2, t)}{m_i}\frac{\left(\left|\xi_1'\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)}\int_0^t ds$$

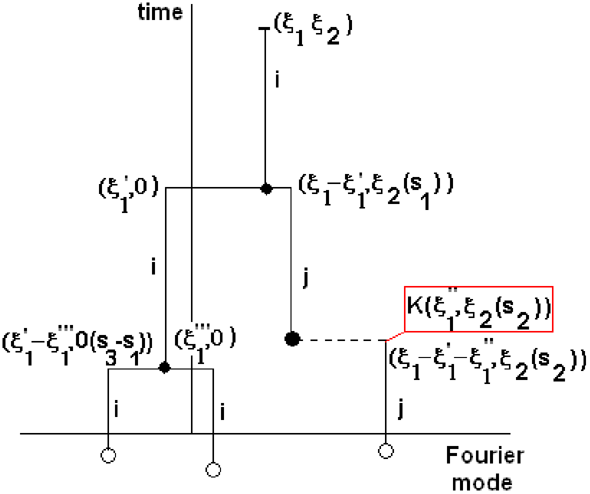
$$\times \boxed{\frac{|\xi_2(s)|}{N(\xi_1,\xi_2,t)} e^{(t-s)|\xi_2(s)|-t|\xi_2|}} \int d^3\xi_1' \boxed{p\left(\xi_1, \xi_1'\right)} \left\{ \frac{1}{2}\frac{16\pi\overrightarrow{\xi}_2(s)\cdot\overset{\wedge}{\xi_1'}}{|\xi_2(s)|} \right.$$

$$\sum_j \frac{1}{2} e_j \chi_j\left(\xi_1', 0, t-s\right) + \frac{1}{2}\frac{2}{(2\pi)^{3/2}}\frac{\overrightarrow{\xi}_2(s)}{|\xi_2(s)|} \cdot \left.\left(\frac{\overrightarrow{b}\left(\xi_1'\right)}{h\left(\xi_1'\right)}\nabla_{\xi_2(s)}\right)\right\}$$

$$\chi_i\left(\xi_1 - \xi_1', \xi_2(s), t-s\right)$$

Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

Two types of vertices



Poisson-Vlasov in an external magnetic field. Configuration space

$$
\begin{aligned}
0 &= \frac{\partial f_i}{\partial t} + \left(\vec{v} \cdot \nabla_x + \frac{e_i}{m_i} \frac{\vec{v}}{c} \times \vec{B} \cdot \nabla_v \right) f_i \\
&+ \frac{e_i}{m_i} \int d^3 x' \sum_j e_j \int d^3 v f \left(x', v, t \right) \frac{\overrightarrow{x - x'}}{\left| x - x' \right|^3} \cdot \nabla_v f_i \left(x, v, t \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt} x(t) &= -v(t) \\
\frac{d}{dt} v(t) &= -\frac{e_i}{c m_i} \left(v(t) \times B(x(t)) \right)
\end{aligned}
$$

$$
\begin{aligned}
f_i(x, v, t) &= f_i(x(t), v(t), 0) - \frac{e_i}{m_i} \int_0^t ds \int d^3 x' \sum_j e_j \int d^3 u \\
&\times f\left(x', u, t - s \right) \frac{\overrightarrow{x(s) - x'}}{\left| x(s) - x' \right|^3} \cdot \nabla_v f_i \left(x(s), v(s), t - s \right)
\end{aligned}
$$

Poisson-Vlasov in an external magnetic field. Configuration space

$$G_i\left(\vec{x},\vec{v},t\right)=\frac{f_i\left(\vec{x},\vec{v},t\right)}{\varphi_i\left(\vec{x}(t),\vec{v}(t)\right)}$$

$$\begin{aligned}
& G_i\left(\vec{x},\vec{v},t\right) \\
= \quad & G_i\left(\vec{x}(t),\vec{v}(t),0\right)-\sum_j\frac{e_ie_j}{m_i}\int_0^t dsA_{x,v,t}^{(j)} \\
& \times\int d^3x'd^3u\boxed{p_{x,v,t}^{(j)}\left(\vec{x'},\vec{u},s\right)}G_j\left(\vec{x'},\vec{u},t-s\right)\left(\widehat{\vec{x}(s)-\vec{x'}}\right) \\
& \bullet\frac{1}{\varphi_i\left(\vec{x}(t),\vec{v}(t)\right)}\nabla_{v(s)}\varphi_i\left(\vec{x}(t),\vec{v}(t)\right)G_i\left(\vec{x}(s),\vec{v}(s),t-s\right)
\end{aligned}$$

Poisson-Vlasov in an external magnetic field. Configuration space

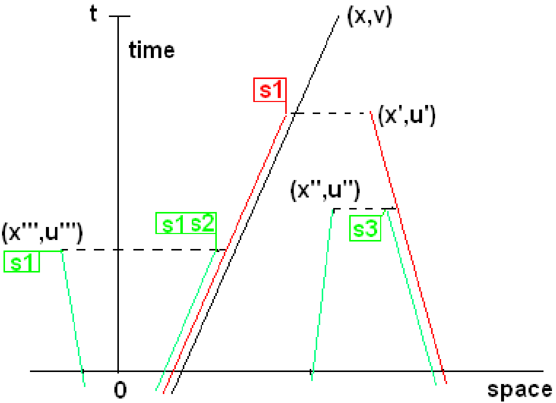
$$p_{x,v,t}^{(j)}\left(\overrightarrow{x'},\overrightarrow{u},s\right)=\frac{1}{A_{x,v,t}^{(j)}}\frac{\varphi_j\left(\overrightarrow{x'}\left(t-s\right),\overrightarrow{u}\left(t-s\right)\right)}{\left|\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right|^2}$$

a probability in the space $[0,t]\times\mathbb{R}^3\times\mathbb{R}^3$

$$A_{x,v,t}^{(j)}=\int_0^t ds\int\int d^3x'd^3u\frac{\varphi_j\left(\overrightarrow{x'},\overrightarrow{u}\right)}{\left|\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right|^2}$$

$$K\left(s\right)=\widehat{\left(\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right)}\bullet\frac{1}{\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)}\nabla_{v(s)}\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)$$

Poisson-Vlasov in an external magnetic field. Configuration space



Poisson-Vlasov in an external magnetic field. Configuration space

$$\breve{G}_i\left(\overrightarrow{x},\overrightarrow{v},t\right) = G_i\left(\overrightarrow{x}(t),\overrightarrow{v}(t),0\right) - \frac{e_i e_j}{m_i} A^{(j)}_{x,v,t} \breve{G}_j\left(\overrightarrow{x'},\overrightarrow{u},t-s\right)$$
$$\times K(s)\,\breve{G}_i\left(\overrightarrow{x}(s),\overrightarrow{v}(s),t-s\right)$$

Let

$$\left|G_i\left(\overrightarrow{x},\overrightarrow{v},0\right)\right| \leq M \tag{4}$$

$$\left|K(s_1)\,K(s_2)\cdots K(s_n)\,G_i\left(\overrightarrow{x},\overrightarrow{v},0\right)\right| \leq M \tag{5}$$

for all n. Then the iteration has a stable fixed point if

$$8\max\left(\frac{A^{(j)}_{x,v,t}}{m_i}\right) M < 1 \tag{6}$$

Poisson-Vlasov in an external magnetic field. Configuration space

$$G_i\left(\overrightarrow{x},\overrightarrow{v},t\right)=\mathbb{E}\left\{\overset{\smile}{G}_i\left(\overrightarrow{x},\overrightarrow{v},t\right)\right\}$$

Theorem. *There is a tree-labelled stochastic process which, if conditions (4-6) are satisfied, provides a stochastic solution to the configuration space Poisson-Vlasov equation in an external magnetic field.*

Stochastic solutions of nonlinear partial differential equations. Applications.

- What are they good for ?
- New **exact solutions**
- New numerical algorithms? Is a stochastic-based algorithm competitive with other (deterministic) algorithms?
- Deterministic algorithms grow exponentially with the dimension d of the space, roughly N^d ($\frac{L}{N}$ the linear size of the grid). The stochastic process only grows with the dimension d.
- Deterministic algorithms aim at obtaining the solution in the whole domain. Then, even if an efficient deterministic algorithm exists, the stochastic algorithm is competitive if only localized values of the solution are desired. For example by studying only a few high Fourier modes one may obtain information on the small scale fluctuations that only a very fine grid might provide in a deterministic algorithm.

Stochastic solutions of nonlinear partial differential equations. Applications.

- Each sample path is independent. Likewise, paths starting from different points are independent from each other. The stochastic algorithms are a natural choice for parallel and distributed computation.
- Stochastic algorithms handle equally well regular and complex boundary conditions.
- A very clever idea (J. Acebron, A. Rodriguez-Rozas, R. Spigler) Domain decomposition using interpolation of localized stochastic solutions and then, in each small domain, use a deterministic code. Fully parallel.
- *And now "something completely different", but also related to fusion plasmas*

Tomographic data analysis. General setting

- **Integral transforms**
- *Linear transforms*: Fourier, Wavelets, Hilbert, ...
- *Bilinear transforms*: Wigner-Ville, Bertrand, Tomograms
- **General setting**
  Consider signals $f(t)$ as vectors $\mid f\rangle \in$ dense nuclear subspace $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ with dual space $\mathcal{N}^*$
- $\{U(\alpha) : \alpha \in I\}$ a family of operators defined on $\mathcal{N}^*$. (In many cases $U(\alpha)$ generates a unitary group $U(\alpha) = e^{iB(\alpha)}$)
- **Three types of transforms**
  Let $h \in \mathcal{N}^*$ be a reference vector such that the linear span of $\{U(\alpha)h \in \mathcal{N}^* : \alpha \in I\}$ is dense in $\mathcal{N}^*$. In the set $\{U(\alpha)h\}$, a complete set of vectors can be chosen to serve as a basis
- **1 - Wavelet-type transform**

$$W_f^{(h)}(\alpha) = \langle U(\alpha)\, h \mid f\rangle,$$

- **2 - Quasi-distribution**

$$Q_f(\alpha) = \langle U(\alpha) f \mid f \rangle.$$

- If $U(\alpha)$ is a unitary operator there is a self-adjoint operator $B(\alpha)$

$$W_f^{(h)}(\alpha) = \langle h \mid e^{iB(\alpha)} \mid f \rangle$$

$$Q_f^{(B)}(\alpha) = \langle f \mid e^{iB(\alpha)} \mid f \rangle$$

- **3 - Tomographic transform or tomogram**

$$M_f^{(B)}(X) = \langle f \mid \delta(B(\alpha) - X) \mid f \rangle$$

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator
- **Ambiguity function:** $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$
- **Wigner–Ville transform**: $Q_f(\alpha)$ for the same $B_F\left(\overrightarrow{\alpha}\right)$ plus the parity operator

$$B^{(WV)}(\alpha_1, \alpha_2) = -i2\alpha_1 \frac{d}{dt} - 2\alpha_2 t + \frac{\pi\left(t^2 - \frac{d^2}{dt^2} - 1\right)}{2}.$$

- **Wavelet transform**: $W_f^{(h)}(\alpha)$ for $B_W\left(\overrightarrow{\alpha}\right) = \alpha_1 D + i\alpha_2 \frac{d}{dt}$, D being the dilation operator $D = -\frac{1}{2}\left(it\frac{d}{dt} + i\frac{d}{dt}t\right)$
- **Bertrand transform:** $Q_f(\alpha)$ for B_W

The tomographic transform (tomogram)

- $$M_f^{(B)}(X) = \langle f \mid \delta\left(B\left(\alpha\right) - X\right) \mid f \rangle$$
- $M_f^{(B)}(\alpha)$ is positive and may be interpreted as a probability distribution. Benefits from the properties of the bilinear transforms, without interpretation ambiguities
- For normalized $\mid f \rangle$,
  $$\langle f \mid f \rangle = 1$$
  the tomogram is normalized
  $$\int M_f^{(B)}\left(X\right)\, dX = 1$$
  It is a probability distribution for the random variable X corresponding to the observable defined by the operator $B\left(\alpha\right)$
- The tomogram is a homogeneous function
  $$M_f^{(B/p)}(X) = |p| M_f^{(B)}(pX)$$

Relations between the three types of transforms

- $$M_f^{(B)}(X) = \frac{1}{2\pi} \int Q_f^{(kB)}(\alpha)\, e^{-ikX}\, dk$$
- $$Q_f^{(B)}(\alpha) = \int M_f^{(B/p)}(X)\, e^{ipX}\, dX.$$
- $$Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha),$$
- $$W_f^{(h)}(\alpha) = \frac{1}{4} \int e^{iX} \begin{bmatrix} M_{f_1}^{(B)}(X) - iM_{f_2}^{(B)}(X) \\ -M_{f_3}^{(B)}(X) + iM_{f_4}^{(B)}(X) \end{bmatrix} dX,$$

  with

$$| \quad f_1\rangle =| h\rangle + | f\rangle; \qquad | f_3\rangle =| h\rangle - | f\rangle;$$
$$| \quad f_2\rangle =| h\rangle + i \,| f\rangle; \qquad | f_4\rangle =| h\rangle - i \,| f\rangle.$$

Husimi–Kano type quasi-distribution

- Other type of operator

$$U(\alpha) = e^{iB(\alpha)} P_h e^{-iB(\alpha)},$$

P_h = projector on a reference vector $| h\rangle$

- Quasidistribution of the Husimi–Kano type

$$H_f^{(b)}(\alpha) = \langle f \mid U(\alpha) \mid f \rangle.$$

The conformal group

- The generators of the conformal group

in $\mathbb{R}^d$

$$\begin{aligned} \omega_k &= i\frac{\partial}{\partial t_k} \\ D &= i\left(t \bullet \nabla + \frac{d}{2}\right) \\ R_{j,k} &= i\left(t_j\frac{\partial}{\partial t_k} - t_k\frac{\partial}{\partial t_j}\right) \\ K_j &= i\left(t_j^2\frac{\partial}{\partial t_j} + t_j\right) \end{aligned}$$

- For $d = 1$

in $\mathbb{R}$

$$\begin{aligned} \omega &= i\frac{d}{dt} \\ D &= i\left(t\frac{d}{dt} + \frac{1}{2}\right) \\ K &= i\left(t^2\frac{d}{dt} + t\right) \end{aligned}$$

Tomograms associated to the conformal group

- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

- Time-scale

$$B_2 = \mu t + i\nu \left(t \frac{d}{dt} + \frac{1}{2} \right)$$

- Frequency-scale

$$B_3 = i\mu \frac{d}{dt} + i\nu \left(t \frac{d}{dt} + \frac{1}{2} \right)$$

- Time-conformal

$$B_4 = \mu t + i\nu \left(t^2 \frac{d}{dt} + t \right)$$

Tomograms associated to the conformal group

- General construction of the tomograms: Let

$$\int dY \left|Y\right\rangle \left\langle Y\right| = 1$$

be a decomposition of the unit, with generalized eigenvectors of the operator B. Then

$$M(\alpha, X) = \int dY \langle f \mid \delta \left(B\left(\alpha\right) - X\right) \left|Y\right\rangle \left\langle Y\right| \mid f \rangle = \left|\langle X \mid f \rangle\right|^2$$

- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each B operator
- $B_1 \psi_1 \left(\mu, \nu, t, X\right) = X \psi_1 \left(\mu, \nu, t, X\right)$

$$\psi_1 \left(\mu, \nu, t, X\right) = \exp i \left(\frac{\mu t^2}{2\nu} - \frac{tX}{\nu}\right)$$

$$\int dt \psi_1^* \left(\mu, \nu, t, X\right) \psi_1 \left(\mu, \nu, t, X'\right) = 2\pi\nu\delta \left(X - X'\right)$$

Tomograms associated to the conformal group

- $B_2\psi_2(\mu,\nu,t,X) = X\psi_2(\mu,\nu,t,X)$

$$\psi_2(\mu,\nu,t,X) = \frac{1}{\sqrt{|t|}}\exp i\left(\frac{\mu t}{\nu} - \frac{X}{\nu}\log|t|\right)$$

$$\int dt\psi_2^*(\mu,\nu,t,X)\,\psi_2(\mu,\nu,t,X') = 4\pi\nu\delta(X-X')$$

- $B_3\psi_3(\mu,\nu,\omega,X) = X\psi_3(\mu,\nu,\omega,X)$

$$\psi_3(\mu,\nu,t,X) = \exp(-i)\left(\frac{\mu}{\nu}\omega - \frac{X}{\nu}\log|\omega|\right)$$

$$\int d\omega\psi_1^*(\mu,\nu,\omega,X)\,\psi_1(\mu,\nu,\omega,X') = 2\pi\nu\delta(X-X')$$

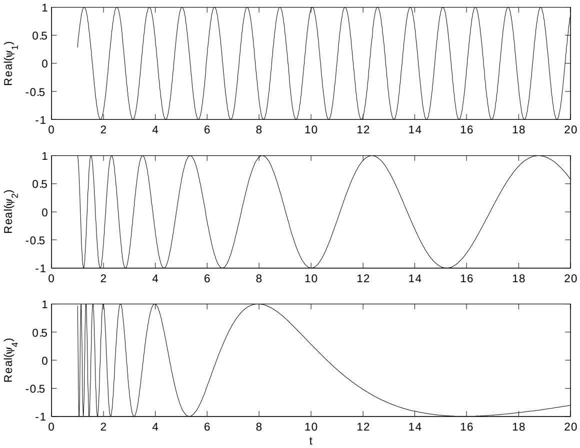
- $B_4\psi_4(\mu,\nu,t,X) = X\psi_4(\mu,\nu,t,X)$

$$\psi_4(\mu,\nu,t,X) = \frac{1}{|t|}\exp i\left(\frac{X}{\nu t} + \frac{\mu}{\nu}\log|t|\right)$$

$$\int dt\psi_4^*(\mu,\nu,t,s)\,\psi_4(\mu,\nu,t,s') = 2\pi\nu\delta(s-s')$$

Tomograms associated to the conformal group

$\mu = 0$



Tomograms associated to the conformal group

- Time-frequency tomogram

$$M_1\left(\mu,\nu,X\right)=\frac{1}{2\,\pi|\nu|}\left|\int \exp\left[\frac{i\mu t^2}{2\,\nu}-\frac{itX}{\nu}\right]f\left(t\right)dt\right|^2$$

- Time-scale tomogram

$$M_2(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f\left(t\right)}{\sqrt{|t|}}e^{\left[i\left(\frac{\mu}{\nu}t-\frac{X}{\nu}\log|t|\right)\right]}\right|^2$$

- Frequency-scale tomogram

$$M_3(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int d\omega\,\frac{f\left(\omega\right)}{\sqrt{|\omega|}}e^{\left[-i\left(\frac{\mu}{\nu}\omega-\frac{X}{\nu}\log|\omega|\right)\right]}\right|^2$$

$f\left(\omega\right)=$ Fourier transform of $f\left(t\right)$

- Time-conformal tomogram

$$M_4(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f\left(t\right)}{|t|}e^{\left[i\left(\frac{X}{\nu t}+\frac{\mu}{\nu}\log|t|\right)\right]}\right|^2$$

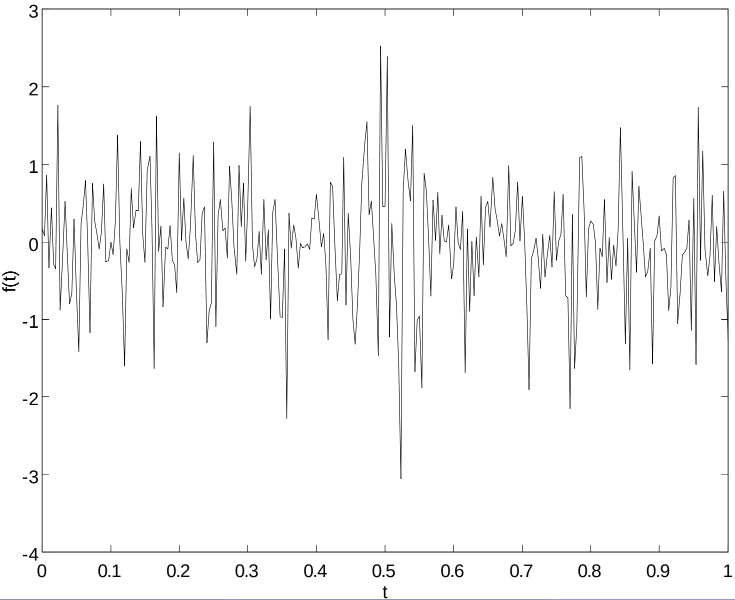
Applications

- 1 - **Detection of small signals in noise**
- Let in $M_1(\mu, \nu; X)$

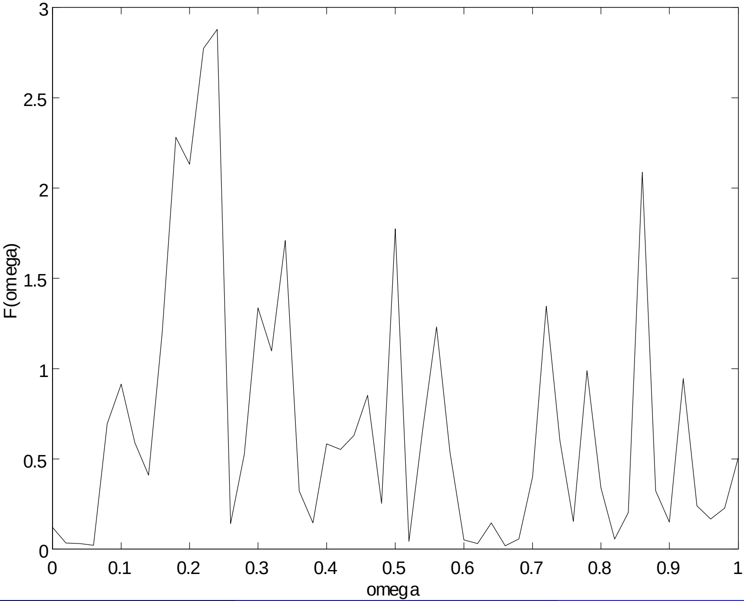
$$\mu = \frac{\cos\,\theta}{T},\ \nu = \frac{\sin\,\theta}{\Omega}$$

  (Radon transform)
- A signal generated as a superposition of a normally distributed random amplitude - random phase noise with a sinusoidal signal of same average amplitude, operating only during the time $0.45 - 0.55$. The signal to noise power ratio is $1/10$.
- The. following figures show the signal, its Fourier transform and the tomogram $M_f^{(S)}(s, \mu, \nu)$ ($T = 1$ and $\Omega = 1000$)

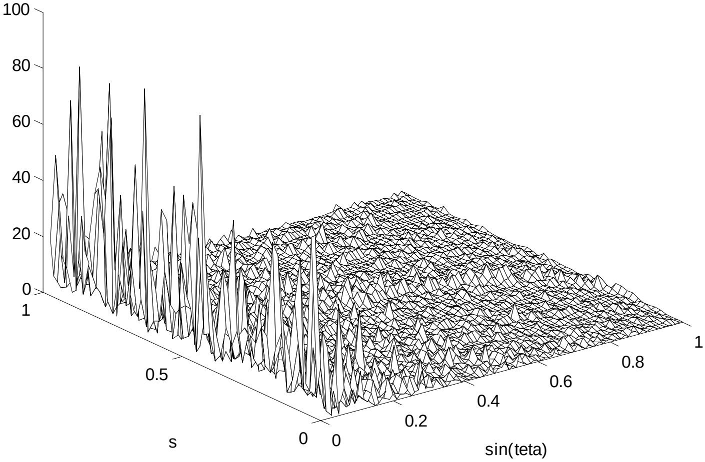
Detection of signals in noise



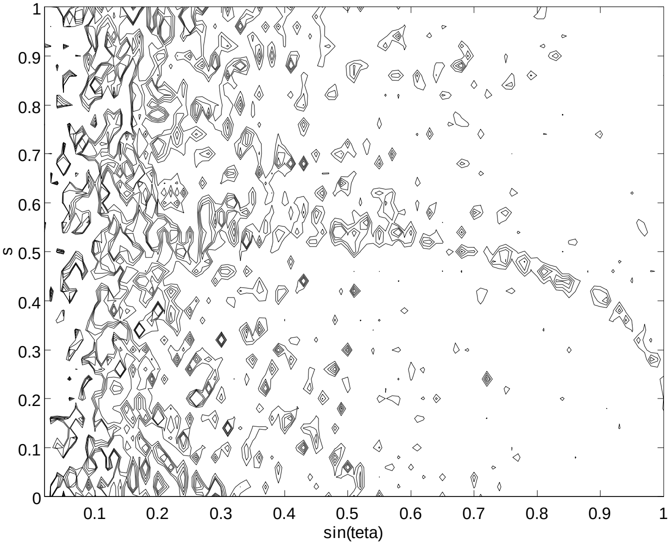
Detection of signals in noise



Detection of signals in noise



Detection of signals in noise



Detection of signals in noise

- One clearly sees a sequence of small peaks connecting a time around 0.5 to a frequency around 200.
- The signature that the signal leaves on the tomogram is a manifestation of the fact that, despite its low SNR, there is a certain number of directions in the (t, ω) plane along which detection happens to be more favorable. For different trials the coherent peaks appear at different locations, but the overall geometry of the ridge is the same.
- A ridge of small peaks is reliable because the rigorous probability interpretation of $M(\theta, X)$ renders the method immune to spurious effects.

Component decomposition

- Most natural and man-made signals are nonstationary and have a multicomponent structure.
  Examples: Bat echolocation, whale sounds, radar, sonar, etc.
- The concept of signal component is not uniquely defined. The notion of *component* depends as much on the observer as on the observed object. When we speak about a component of a signal we are in fact referring to a particular feature of the signal that we want to emphasize.
- One possibility: Separation of components using its behavior in the time-frequency plane. Consider the finite-time tomogram

$$M(\theta, X) = \left| \int f(t) \psi_{\theta,X}(t)\, dt \right|^2 = |< f, \psi >|^2$$

with

$$\psi_{\theta,X}(t) = \frac{1}{\sqrt{T}} \exp\left(\frac{-i\cos\theta}{2\sin\theta} t^2 + \frac{iX}{\sin\theta} t \right)$$

$\mu = \cos\theta, \nu = \sin\theta$.

Component decomposition

- θ is a parameter that interpolates between the time and the frequency operators, running from 0 to $\pi/2$ whereas X is allowed to be any real number.
- For all different θ's the $U(\theta)$ are unitarily equivalent operators, hence all the tomograms share the same information. The component separation technique is based on the search for an intermediate value of θ where a good compromise might be found between time localization and frequency information.
- First select a subset X_n in such a way that the corresponding family $\left\{\psi_{\theta,X_n}(t)\right\}$ is orthogonal and normalized,

$$< \psi_{\theta,X_n}\psi_{\theta,X_m} >= \delta_{m,n}$$

This is possible by taking the sequence

$$X_n = X_0 + \frac{2n\pi}{T}\sin\theta$$

where X_0 is freely chosen (in general we take $X_0 = 0$)

Component decomposition

- We then consider the projections of the signal $f(t)$

$$c_{X_n}^{\theta}(f) = < f, \psi_{\theta,X_n} >$$

which are used for the signal processing.

- Denoising consists in eliminating the $c_{X_n}^{\theta}(f)$ such that

$$\left| c_{X_n}^{\theta}(f) \right|^2 \leq \epsilon$$

for some threshold ϵ

- Multi-component analysis is done by selecting subsets $\mathcal{F}_k$ of the X_n and reconstructing partial signals (k-components) by restricting the sum to

$$f_k(t) = \sum_{n \in \mathcal{F}_k} c_{X_n}^{\theta}(f) \psi_{\theta,X_n}(t)$$

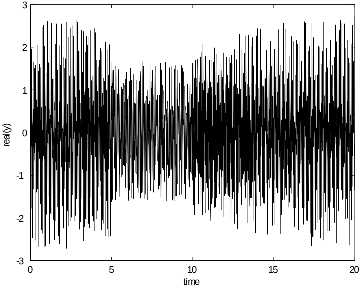
for each k.

Component decomposition. Examples

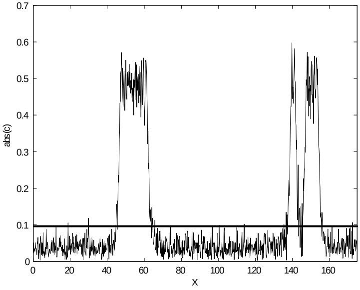
- $$y(t) = y_1(t) + y_2(t) + y_3(t) + b(t)$$

$$\begin{aligned} y_1(t) &= \exp(i25t), t \in [0, 20] \\ y_2(t) &= \exp(i75t), t \in [0, 5] \\ y_3(t) &= \exp(i75t), t \in [10, 20] \end{aligned}$$

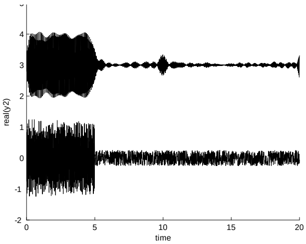
- Real part of the time signal



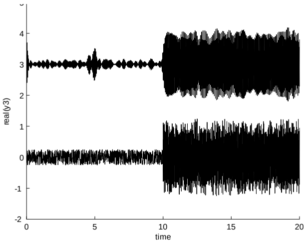
Component decomposition. Examples

- Separation at $\theta = \frac{\pi}{5}$
- 

- Reconstrution of the $y_2\left(t\right)$



- and $y_3\left(t\right)$ components



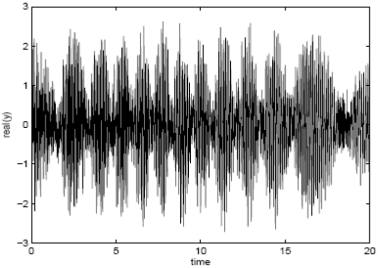
Component decomposition. Examples

- Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of an "incident" $y_0(t)$ and a "deformed reflected" chirp $y_R(t)$ delayed by $3s$ with white noise added.

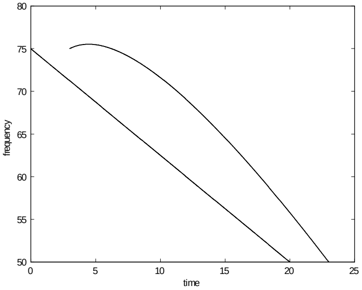
$$y_0(t) = e^{i\Phi_0(t)} \qquad\qquad y_R(t) = e^{i\Phi_R(t)}$$

  $\Phi_0(t) = a_0 t^2 + b_0 t$ and

  $\Phi_R(t) = a_R(t - t_R)^2 + b_R(t - t_R) + 10(t - t_R)^{\frac{3}{2}}$.

- 

- Comparison of the phase derivatives $\frac{d}{dt}\Phi_0(t)$ and $\frac{d}{dt}\Phi_R(t)$. Except for the three first seconds, the spectrum of the signals $y_0(t)$ and $y_R(t)$ is almost the same



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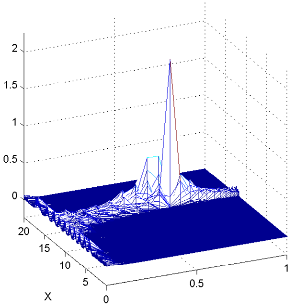
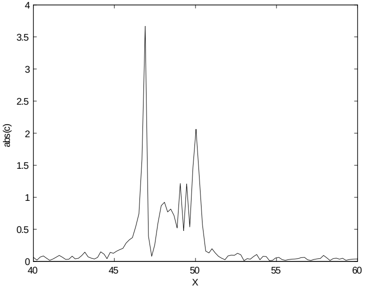
Component decomposition. Examples

- Tomogram of the chirps signal



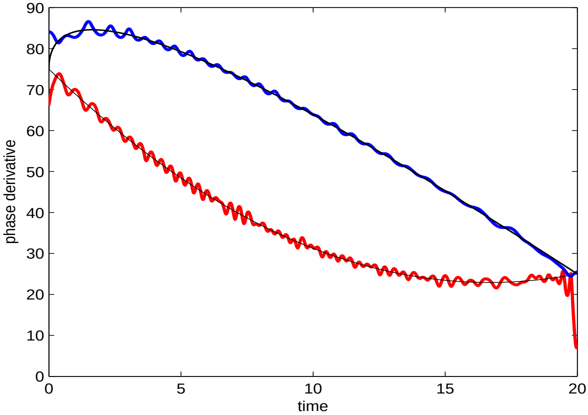
Component decomposition. Examples

- Separable spectrum at $\theta = \frac{\pi}{5}$



SD # 2 : Component decomposition and phase derivative

- Phase derivative $\theta = atan(\frac{\Delta T}{\Delta F})$



Reflectometry data : choc 42824

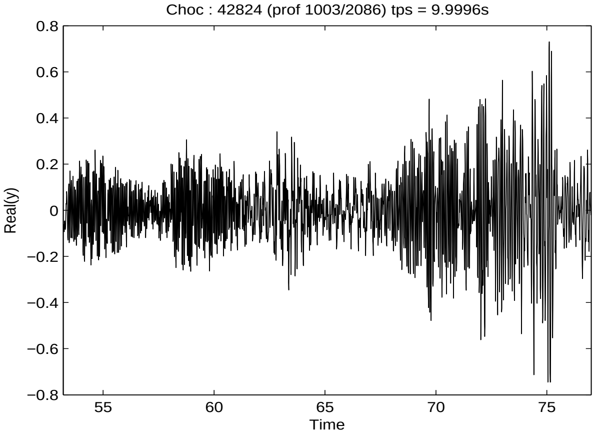


Figure: Time representation

Reflectometry data : choc 42824

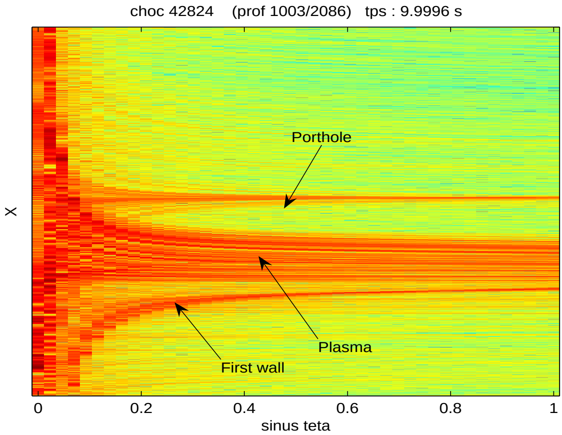
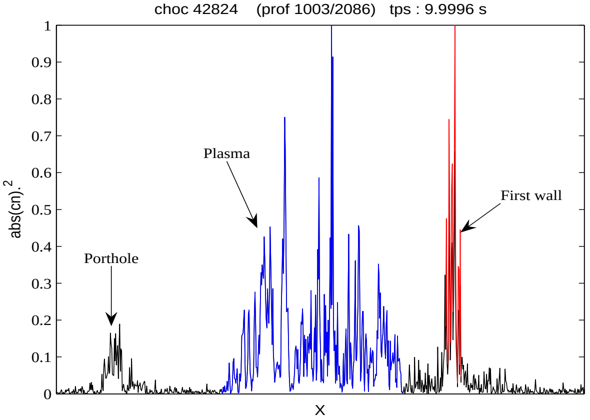


Figure: Tomogram of the reflectometry signal.

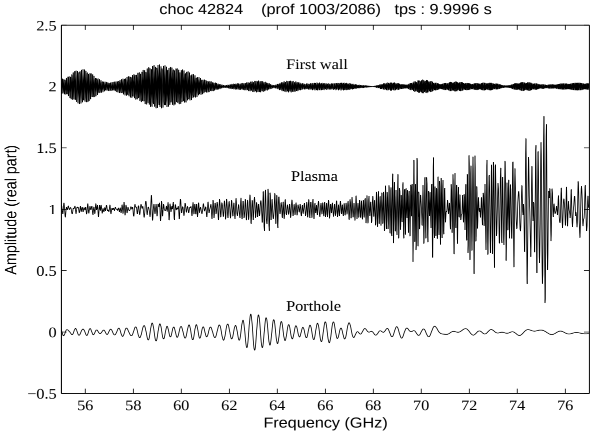
Reflectometry data : choc 42824

- Spectrum $c_{x_n}^{\theta}$ of the reflectometry signal $y(t)$ for $\theta = pi - \frac{\pi}{5}$



Reflectometry data : choc 42824

- Components of the reflectometry signal : $\theta = pi - \frac{\pi}{5}$



Reflectometry data : choc 42824

- Separation of the components in the reflectometry signal

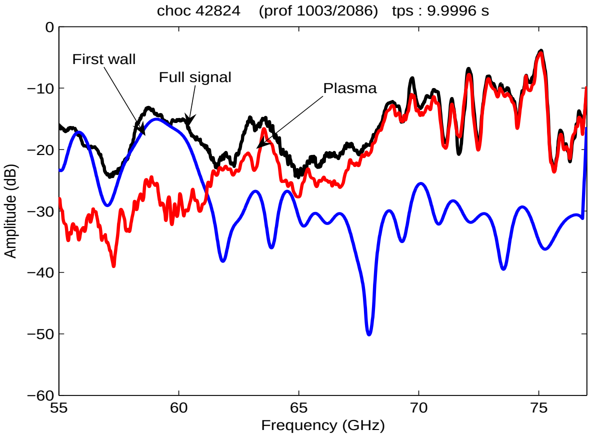
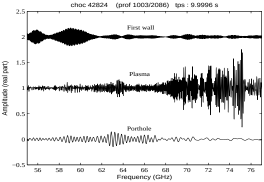


Figure: Full signal and reflexions on the wall and on the plasma (dB)

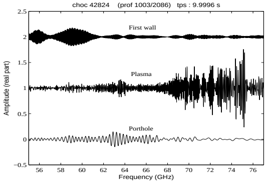
Reflectometry data : choc 42824

- Components of the reflectometry signal : $\theta = pi - \frac{\pi}{5}$

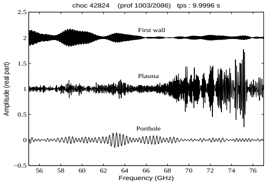


Reflectometry data : choc 42824

- Components of the reflectometry signal : $\theta = pi - \frac{\pi}{5}$

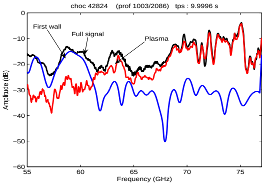


- Components of the reflectometry signal : $\theta = \frac{\pi}{2}$



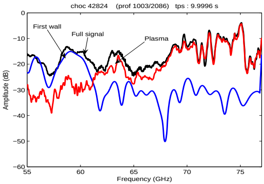
Reflectometry data : choc 42824

- Components in the reflectometry signal : $\theta = pi - \frac{\pi}{5}$

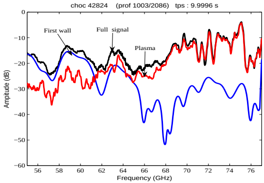


Reflectometry data : choc 42824

- Components in the reflectometry signal : $\theta = pi - \frac{\pi}{5}$

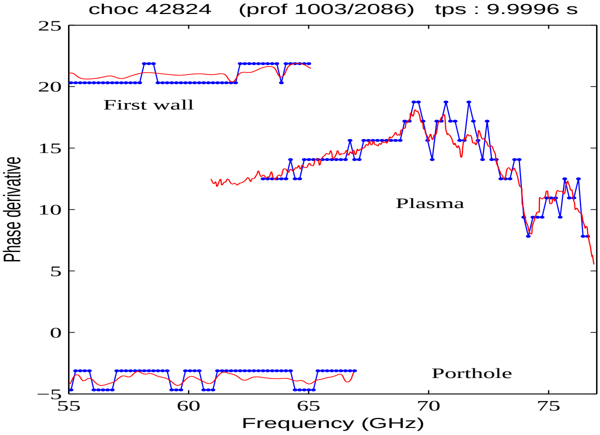


- Components in the reflectometry signal : $\theta = \frac{\pi}{2}$



Reflectometry data : choc 42824

- Phase derivative of the three components $(\theta = \pi - \frac{\pi}{5})$



Random sampling. Going beyond the limitations of Shannon's theorem

Theorem 4. *Let* $x_n = n\lambda + X_n$ *with* X_n *being a sequence of i.i.d. random variables uniformly distributed in* $[0,\lambda]$*. Then, almost every configuration* $\{x_n\}$ *of the point process has the property that if* f *is a function in the linear chirp space* $\mathcal{LC}$ *satisfying* $f(x_n) = 0 \quad \forall n \in \mathbb{Z}$, *then* $f \equiv 0$.

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