A stochastic approach to the solution and simulation of plasma kinetic equations

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The notion of stochastic solution

- Linear elliptic and parabolic equations (both with Cauchy and Dirichlet boundary conditions) have a probabilistic interpretation: a classical result and a standard tool in potential theory.
- An example: the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad \textit{with} \qquad u(0,x) = f(x) \tag{1}$$

- The solution may be written either as

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\, dy \tag{2}$$

- or as

$$u(t,x) = \mathbb{E}_x f(X_t) \tag{3}$$

$\mathbb{E}_x$ being the expectation value, starting from x, of the Wiener process $dX_t = dW_t$

The notion of stochastic solution

- Eq.(1) is a *specification* of a problem
- (2) and (3) are *solutions* in the sense that they both provide algorithmic means to construct of a function satisfying the specification.
- An important condition for (2) and (3) to be considered as **solutions** is the fact that the algorithm is **independent** of the particular solution,

1. in the first case, an **integration** procedure
2. in the second, a **solution-independent** process.

- This should be contrasted with stochastic processes constructed from a given particular solution, as has been done for example for the Boltzman equation
- New **exact solutions** and also new **numerical algorithms**

Stochastic solutions of nonlinear partial differential equations

- **The basic idea:** In the linear partial differential equation case, once the relevant stochastic process is identified, the process is started from the point x where the solution is to be computed, and the solution is a functional of the exit values of the process (from a space D or a space-time $D \times [0, t]$ domain)
- **Conjecture:** For the nonlinear equations the relevant process has a diffusion, propagation or jump component associated to the linear part of the equation plus a branching mechanism associated to the nonlinear part. The solution will be a functional of the exit measures generated by the process.
- **The construction:** Rewrite the equation as an integral equation. Give a probabilistic interpretation to the integral equation. In the end the stochastic solution is equivalent to the construction of a tree-indexed measure and a sampling evaluation of the Picard series.

Stochastic solutions of nonlinear partial differential equations

Existing results

1. The KPP equation (McKean)
2. Diffusion equation with u^{α} ($\alpha \in [0, 2]$) nonlinearities (Dynkin)
3. The Navier-Stokes equation (LeJan, Schnitzman, Waymire, Ossiander, Batacharia, Orum)
4. The Poisson-Vlasov equation (Cipriano, Floriani, Lima, R. V. M.)
5. The Euler equation (R. V. M.)
6. A fractional version of the KPP equation (Cipriano, Ouerdiane, R. V. M.)
7. Poisson-Vlasov in an external magnetic field (R. V. M.)
8. Magnetohydrodynamics (E. Floriani, RVM)

Stochastic solutions of nonlinear partial differential equations

- What are they good for ?
- New **exact solutions**
- New numerical algorithms? Is a stochastic-based algorithm competitive with other (deterministic) algorithms?
- Deterministic algorithms grow exponentially with the dimension d of the space, roughly N^d ($\frac{L}{N}$ the linear size of the grid). The stochastic process only grows with the dimension d.
- Deterministic algorithms aim at obtaining the solution in the whole domain. Then, even if an efficient deterministic algorithm exists, the stochastic algorithm is competitive if only localized values of the solution are desired. For example by studying only a few high Fourier modes one may obtain information on the small scale fluctuations that only a very fine grid might provide in a deterministic algorithm.

Stochastic solutions of nonlinear partial differential equations. Applications.

- Each sample path is independent. Likewise, paths starting from different points are independent from each other. The stochastic algorithms are a natural choice for parallel and distributed computation.
- Stochastic algorithms handle equally well regular and complex boundary conditions.
- A clever idea (J. Acebron, A. Rodriguez-Rozas, R. Spigler) Domain decomposition using interpolation of localized stochastic solutions and then, in each small domain, use a deterministic code. Fully parallel.

Poisson-Vlasov in an external magnetic field

- Poisson-Vlasov

$$0 = \frac{\partial f_i}{\partial t} + \left(\overrightarrow{v} \cdot \nabla_x + \frac{e_i}{m_i} \frac{\overrightarrow{v}}{c} \times \overrightarrow{B}(x) \cdot \nabla_v \right) f_i$$

$$+ \frac{e_i}{m_i} \int d^3 x' \sum_j e_j \int d^3 u f_j \left(x', u, t \right) \frac{\overrightarrow{x - x'}}{\left| x - x' \right|^3} \cdot \nabla_v f_i (x, v, t)$$

- Fourier transformed version, $F(\xi, t) = \frac{1}{(2\pi)^3} \int d^6 \eta f(\eta, t) e^{i \xi \cdot \eta}$, with $\eta = \left(\overrightarrow{x}, \overrightarrow{v} \right)$ and $\xi = \left(\overrightarrow{\xi_1}, \overrightarrow{\xi_2} \right) \hat{=} (\xi_1, \xi_2)$

$$\frac{\partial F_i(\xi, t)}{\partial t} = \left(\overrightarrow{\xi}_1 \cdot \nabla_{\xi_2} + \frac{e_i}{c m_i} \nabla_{\xi_2} \times \overrightarrow{B} \left(-i \nabla_{\xi_1} \right) \cdot \overrightarrow{\xi}_2 \right) F_i(\xi, t)$$

$$- \frac{4\pi e_i}{m_i} \int d^3 \xi_1' F_i \left(\xi_1 - \xi_1', \xi_2, t \right) \frac{\overrightarrow{\xi}_2 \cdot \overrightarrow{\xi}_1'}{\left| \xi_1' \right|^2} \sum_j e_j F_j \left(\xi_1', 0, t \right)$$

Linear evolutions

- $$\begin{aligned} \frac{d}{dt}x(t) &= -v(t) \\ \frac{d}{dt}v(t) &= -\frac{e_i}{cm_i}\left(v(t)\times B(x(t))\right) \end{aligned}$$

- $$\begin{aligned} \frac{d}{dt}\xi_1(t) &= -\frac{e_i}{cm_i}\left(\nabla_{\xi_2}(t)\times i\nabla B\left(-i\nabla_{\xi_1}(t)\right)\cdot\xi_2(t)\right) \\ \frac{d}{dt}\xi_2(t) &= \xi_1(t)+\frac{e_i}{cm_i}B\left(-i\nabla_{\xi_1}(t)\right)\times\xi_2(t) \end{aligned}$$

- $$\begin{aligned} \frac{d}{dt}\nabla_{\xi_1}(t) &= -\nabla_{\xi_2}(t) \\ \frac{d}{dt}\nabla_{\xi_2}(t) &= -\frac{e_i}{cm_i}\left(\nabla_{\xi_2}(t)\times B\left(-i\nabla_{\xi_1}(t)\right)\right) \end{aligned}$$

- Write a stochastic solution for

$$\chi_i(\xi_1,\xi_2,t)=e^{-t\gamma(|\xi_2|)}\frac{F_i(\xi_1,\xi_2,t)}{h(\xi_1)}$$

$\gamma(|\xi_2|)=1$ if $|\xi_2|\leq 1$ and $\gamma(|\xi_2|)=|\xi_2|$ otherwise

Fourier-transformed Poisson-Vlasov. Uniform magnetic field. Definitions

- Survival probability, up to time t, of an exponential process, $\boxed{e^{-t\gamma(|\xi_2|)}}$ and decaying probability $\boxed{\Pi\left(\xi_1, \xi_2, s\right) = \frac{|\xi_2(s)| e^{(t-s)\gamma(|\xi_2(s)|) - t\gamma(|\xi_2|)}}{N(\xi_1, \xi_2, t)}}$ with normalizing function

$$N\left(\xi_1, \xi_2, t\right) = \frac{1}{1 - e^{-t\gamma(|\xi_2|)}} \int_0^t ds \gamma\left(\left|\xi_2\left(s\right)\right|\right) e^{(t-s)\gamma(|\xi_2(s)|) - t\gamma(|\xi_2|)}$$

- Branching probability

$$p\left(\xi_1, \xi_1'\right) = \frac{\left|\xi_1'\right|^{-1} h\left(\xi_1 - \xi_1'\right) h\left(\xi_1'\right)}{\left(\left|\xi_1'\right|^{-1} h * h\right)}$$

$$\left(\left|\xi_1'\right|^{-1} h * h\right)\left(\xi_1\right) = \int d^3\xi_1' \left|\xi_1'\right|^{-1} h\left(\xi_1 - \xi_1'\right) h\left(\xi_1'\right)$$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field

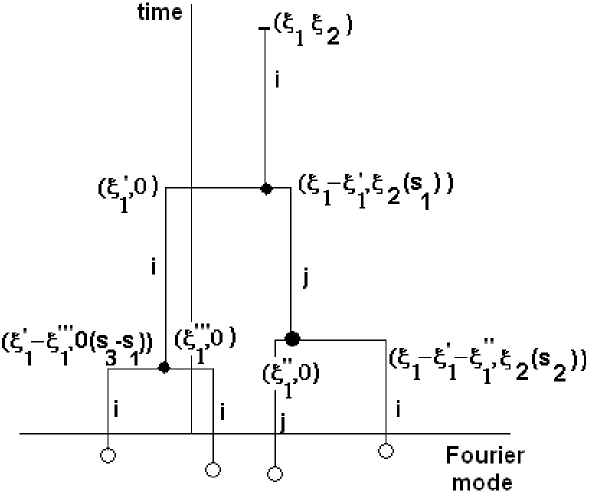
$$\chi_i(\xi_1,\xi_2,t)$$

$$= \boxed{e^{-t\gamma(|\xi_2|)}}\chi_i(\xi_1,\xi_2(t),0) - \frac{8\pi e_i N(\xi_1,\xi_2,t)}{m_i}\frac{\left(\left|\xi_1'\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)}$$

$$\times\int_0^t ds \boxed{\frac{|\xi_2(s)|}{N(\xi_1,\xi_2,t)} e^{(t-s)\gamma(|\xi_2(s)|)-t\gamma(|\xi_2|)}} \int d^3\xi_1' \boxed{p\left(\xi_1,\xi_1'\right)}$$

$$\times\chi_i\left(\xi_1-\xi_1',\xi_2(s),t-s\right)\frac{\overrightarrow{\xi_2}(s)\cdot\widehat{\xi_1'}}{\gamma(|\xi_2(s)|)}\sum_j\frac{e_j}{2}\chi_j\left(\xi_1',0,t-s\right)$$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field



Fourier-transformed Poisson-Vlasov. Uniform magnetic field

- Coupling constants

$$g_{ij}\left(\xi_1,\xi_1^{\prime},s\right) = -\frac{8\pi e_i e_j N(\xi_1,\xi_2,t)}{m_i}\frac{\left(\left|\xi_1^{\prime}\right|^{-1}h*h\right)(\xi_1)}{h(\xi_1)}\frac{\overrightarrow{\xi_2}(s)\cdot\overset{\wedge}{\xi_1^{\prime}}}{\gamma(|\xi_2(s)|)}$$

$$g_{0i}\left(\xi_1,\xi_2\right) = \frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}$$

- Multiplicative functional = product of all the couplings for each realization. The solution $\chi_i\left(\xi_1,\xi_2,t\right)$ is the expectation value
- Existence conditions
  (A) $\left|\frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}\right| \leq 1$
  (B) $\left(\left|\xi_1^{\prime}\right|^{-1}h*h\right)\left(\xi_1\right) \leq h\left(\xi_1\right)$
  (C) $\left|\frac{8\pi e_i e_j N(\xi_1,\xi_2,t)}{\min_i\{m_i\}}\frac{\left(\left|\xi_1^{\prime}\right|^{-1}h*h\right)}{h(\xi_1)}\right| \leq 1$

Fourier-transformed Poisson-Vlasov. Uniform magnetic field

Theorem 1: *The stochastic process* $X(\xi_1, \xi_2, t)$, *above described, provides through the multiplicative functional a stochastic solution of the Fourier-transformed Poisson-Vlasov equation in a uniform magnetic field for arbitrary finite values of the arguments, provided the initial conditions at time zero satisfy the boundedness conditions (A).*

Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

$\overrightarrow{B}(\xi_1) = (2\pi)^{3/2}\overrightarrow{B}_0\delta^3(\xi_1) + \overrightarrow{b}(\xi_1)$ and
$\chi_i(\xi_1, \xi_2, t) = e^{-t\gamma(|\xi_2|)}\frac{F_i(\xi_1,\xi_2,t)}{h(\xi_1)}$

$$\chi_i(\xi_1, \xi_2, t)$$

$$= \boxed{e^{-t\gamma(|\xi_2|)}}\chi_i(\xi_1, \xi_2(t), 0) - \frac{e_i N(\xi_1, \xi_2, t)}{m_i}\frac{\left(\left|\xi_1'\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)}\int_0^t ds$$

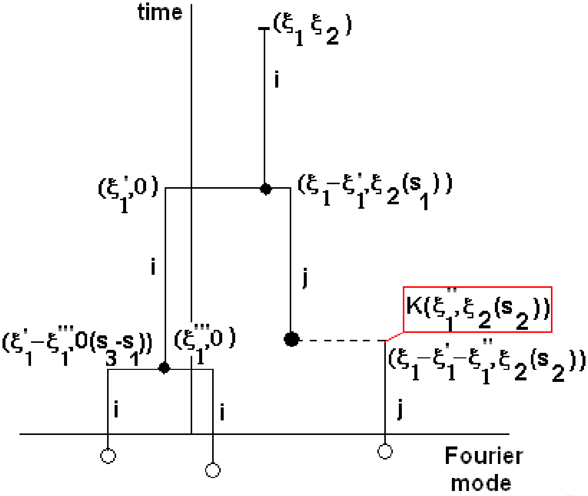
$$\boxed{\frac{|\xi_2(s)|}{N(\xi_1,\xi_2,t)}e^{(t-s)\gamma(|\xi_2(s)|)-t\gamma(|\xi_2|)}}\int d^3\xi_1'\boxed{p\left(\xi_1, \xi_1'\right)}\left\{\frac{1}{2}\frac{16\pi\overrightarrow{\xi}_2(s)\cdot\hat{\xi}_1'}{\gamma(|\xi_2(s)|)}\right.$$

$$\left.\sum_j \frac{1}{2}e_j\chi_j\left(\xi_1', 0, t-s\right) + \frac{1}{2}\frac{2}{(2\pi)^{3/2}}\frac{\overrightarrow{\xi}_2(s)}{\gamma(|\xi_2(s)|)}\cdot\left(\frac{\overrightarrow{b}\left(\xi_1'\right)}{h\left(\xi_1'\right)}\nabla_{\xi_2(s)}\right)\right\}$$

$$\chi_i\left(\xi_1 - \xi_1', \xi_2(s), t-s\right)$$

Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

Two types of vertices



Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

- Operator label

$$K\left(\xi_1^{\prime},\xi_2\left(s\right)\right)=\frac{2}{\left(2\pi\right)^{3/2}}\frac{\overrightarrow{\xi_2}\left(s\right)}{\gamma\left(\left|\xi_2\left(s\right)\right|\right)}\cdot\left(\frac{\overrightarrow{b}\left(\xi_1^{\prime}\right)}{h\left(\xi_1^{\prime}\right)}\times\nabla_{\xi_2(s)}\right)$$

- Bounds

$$\left|\frac{16\pi e_i e_j N\left(\xi_1,\xi_2,t\right)}{\min_i\left\{m_i\right\}}\frac{\left(\left|\xi_1^{\prime}\right|^{-1}h*h\right)}{h\left(\xi_1\right)}\right| \leq 1$$

$$\left|K\left(\xi_1^{\prime},\xi_2\left(s_1\right)\right)\cdots K\left(\xi_1^{(n)},\xi_2\left(s_n\right)\right)\frac{F_i\left(\xi_1,\xi_2,0\right)}{h\left(\xi_1\right)}\right| \leq 1$$

Fourier-transformed Poisson-Vlasov. Non-uniform magnetic field

Theorem 2: *The stochastic process* $Y(\xi_1, \xi_2, t)$*, above described, provides a stochastic solution to the Fourier-transformed Poisson-Vlasov equation in a static non-uniform magnetic field, provided the initial conditions at time zero and the non-uniform part of the field satisfy the boundedness conditions*

Poisson-Vlasov in an external magnetic field. Configuration space

$$0 = \frac{\partial f_i}{\partial t} + \left(\vec{v} \cdot \nabla_x + \frac{e_i}{m_i} \frac{\vec{v}}{c} \times \vec{B} \cdot \nabla_v \right) f_i$$

$$+\frac{e_i}{m_i} \int d^3 x' \sum_j e_j \int d^3 v f \left(x', v, t \right) \frac{\overrightarrow{x - x'}}{\left| x - x' \right|^3} \cdot \nabla_v f_i \left(x, v, t \right)$$

$$\begin{aligned} \tfrac{d}{dt} x(t) &= -v(t) \\ \tfrac{d}{dt} v(t) &= -\tfrac{e_i}{c m_i} \left(v(t) \times B(x(t)) \right) \end{aligned}$$

$$f_i(x, v, t) = f_i(x(t), v(t), 0) - \frac{e_i}{m_i} \int_0^t ds \int d^3 x' \sum_j e_j \int d^3 u$$

$$\times f \left(x', u, t - s \right) \frac{\overrightarrow{x(s)} - \overrightarrow{x'}}{\left| x(s) - x' \right|^3} \cdot \nabla_v f_i \left(x(s), v(s), t - s \right)$$

Poisson-Vlasov in an external magnetic field. Configuration space

$$G_i\left(\overrightarrow{x},\overrightarrow{v},t\right)=\frac{f_i\left(\overrightarrow{x},\overrightarrow{v},t\right)}{\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)}$$

$$\begin{aligned}
& G_i\left(\overrightarrow{x},\overrightarrow{v},t\right) \\
= \quad & G_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right),0\right)-2\sum_j\frac{1}{2}\frac{e_ie_j}{m_i}\int_0^t dsA_{x,v,t}^{(j)} \\
& \times\int d^3x'd^3u\boxed{p_{x,v,t}^{(j)}\left(\overrightarrow{x'},\overrightarrow{u},s\right)}G_j\left(\overrightarrow{x'},\overrightarrow{u},t-s\right)\left(\widehat{\overrightarrow{x}\left(s\right)-\overrightarrow{x'}}\right) \\
& \bullet\frac{1}{\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)}\nabla_{v(s)}\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)G_i\left(\overrightarrow{x}\left(s\right),\overrightarrow{v}\left(s\right),t-s\right)
\end{aligned}$$

Poisson-Vlasov in an external magnetic field. Configuration space

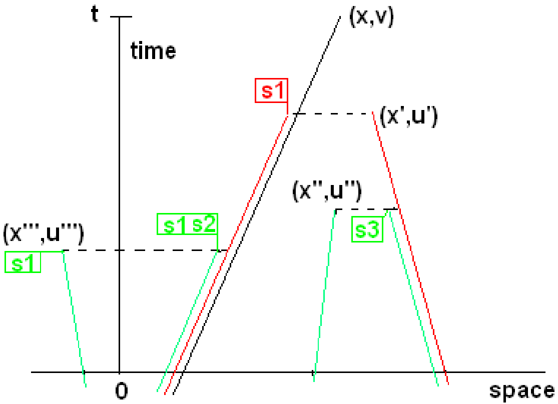
$$p_{x,v,t}^{(j)}\left(\overrightarrow{x'},\overrightarrow{u},s\right)=\frac{1}{A_{x,v,t}^{(j)}}\frac{\varphi_j\left(\overrightarrow{x'}\left(t-s\right),\overrightarrow{u}\left(t-s\right)\right)}{\left|\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right|^2}$$

a probability in the space $[0,t]\times\mathbb{R}^3\times\mathbb{R}^3$

$$A_{x,v,t}^{(j)}=\int_0^t ds\int\int d^3x'd^3u\frac{\varphi_j\left(\overrightarrow{x'},\overrightarrow{u}\right)}{\left|\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right|^2}$$

$$K\left(s\right)=\widehat{\left(\overrightarrow{x}\left(s\right)-\overrightarrow{x'}\right)}\bullet\frac{1}{\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)}\nabla_{v(s)}\varphi_i\left(\overrightarrow{x}\left(t\right),\overrightarrow{v}\left(t\right)\right)$$

Poisson-Vlasov in an external magnetic field. Configuration space



Poisson-Vlasov in an external magnetic field. Configuration space

$$\breve{G}_i\left(\overrightarrow{x},\overrightarrow{v},t\right) = G_i\left(\overrightarrow{x}(t),\overrightarrow{v}(t),0\right) - 2\frac{e_i e_j}{m_i} A^{(j)}_{x,v,t} \breve{G}_j\left(\overrightarrow{x'},\overrightarrow{u},t-s\right)$$
$$\times K(s)\,\breve{G}_i\left(\overrightarrow{x}(s),\overrightarrow{v}(s),t-s\right)$$

Let

$$\left|G_i\left(\overrightarrow{x},\overrightarrow{v},0\right)\right| \leq M \tag{4}$$

$$\left|K(s_1)\,K(s_2)\cdots K(s_n)\,G_i\left(\overrightarrow{x},\overrightarrow{v},0\right)\right| \leq M \tag{5}$$

for all n. Then the iteration has a stable fixed point if

$$8\max\left(\frac{A^{(j)}_{x,v,t}}{m_i}\right) M < 1 \tag{6}$$

Poisson-Vlasov in an external magnetic field. Configuration space

$$G_i\left(\overrightarrow{x},\overrightarrow{v},t\right)=\mathbb{E}\left\{\overset{\backsim}{G}_i\left(\overrightarrow{x},\overrightarrow{v},t\right)\right\}$$

Theorem 3: *There is a tree-labelled stochastic process which, if conditions (4-6) are satisfied, provides a stochastic solution to the configuration space Poisson-Vlasov equation in an external magnetic field.*

Magnetohydrodynamics

Non-relativistic approximation, 3 dimensions, non-zero fluid viscosity and electric resistivity. Incompressible fluid with density $\rho(x,t) = \rho_0$ constant and uniform. The equations for the velocity $V(x,t)$ of the fluid and the magnetic field $B(x,t)$:

$$\frac{\partial V}{\partial t} = -(V\cdot\nabla)V + \frac{1}{\rho_0\mu_0}(B\cdot\nabla)B - \frac{1}{2\rho_0\mu_0}\nabla B^2 - \frac{1}{\rho_0}\nabla P + \nu\nabla^2 V$$

$$\frac{\partial B}{\partial t} = -(V\cdot\nabla)B + (B\cdot\nabla)V + \frac{\eta}{\mu_0}\nabla^2 B$$

ν is the kinematic viscosity and η the electric resistivity.

Passing to the Fourier transform

$$\begin{aligned} v(k,t) &= (2\pi)^{-3/2}\int d^3x\, V(x,t)\, e^{ik\cdot x} \\ b(k,t) &= (2\pi)^{-3/2}\int d^3x\, B(x,t)\, e^{ik\cdot x} \end{aligned}$$

$$\begin{aligned}\frac{\partial v(k,t)}{\partial t} = \; & -\nu k^2 v(k,t) + \frac{i\,k}{\rho_0}\, p(k,t) + \\ & +(2\pi)^{3/2} \int d^3q \; \{ i\,[k \cdot v(q,t)]\, v(k-q,t) - \\ & \frac{i}{\rho_0 \mu_0}\,[k \cdot b(q,t)]\, b(k-q,t) + \frac{1}{2\rho_0\mu_0}\,[B(q,t) \cdot B(k-q,t)]\, k \Big\}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial b(k,t)}{\partial t} = \; & -\frac{\eta}{\mu_0}\, k^2 b(k,t) + \\ & +(2\pi)^{3/2} \int d^3q \, \{ i\,[k \cdot v(q,t)]\, b(k-q,t) \\ & -i\,[k \cdot b(q,t)]\, v(k-q,t) \}\end{aligned}$$

$p(k,t)$ is the Fourier transform of the pressure $P(x,t)$ and we have used the fact that the divergences of $V(x,t)$, $B(x,t)$ vanish.

Project on the plane orthogonal to the vector k. Since $k \cdot v(k,t) = k \cdot b(k,t) = 0$ no information is lost on the velocity and magnetic fields.
The projection operator $\pi_{(k)}(\xi) = \xi - (\xi \cdot e_k)\, e_k \;, \quad e_k = \frac{k}{|k|}$ leads to

$$
\begin{aligned}
\frac{\partial v(k,t)}{\partial t} &= -\nu k^2 v(k,t) + (2\pi)^{3/2} |k| \int d^3q \, \Big\{ v(q,t) \times_{(k)} v(k-q,t) \\
&\quad - \frac{1}{\rho_0 \mu_0}\, b(q,t) \times_{(k)} b(k-q,t) \Big\} \\
\frac{\partial b(k,t)}{\partial t} &= -\frac{\eta}{\mu_0} k^2 b(k,t) + (2\pi)^{3/2} |k| \int d^3q \, \Big\{ v(q,t) \times_{(k)} b(k-q,t) \\
&\quad - b(q,t) \times_{(k)} v(k-q,t) \Big\}
\end{aligned}
$$

with the product $\times_{(k)}$ between two vectors ξ, ω defined by

$$
\xi \times_{(k)} \omega = i(e_k \cdot \xi)\, \pi_{(k)} \omega
$$

Magnetohydrodynamics

Now write the equations in integral form:

$$v(k,t) = v(k,0)\, e^{-\nu k^2 t} + (2\pi)^{3/2}|k| \int ds\, e^{-\nu k^2 s} \int d^3 q$$
$$\left\{ v(q,t-s) \times_{(k)} v(k-q,t-s) - \frac{1}{\rho_0 \mu_0} b(q,t-s) \times_{(k)} b(k-q,t-s) \right\}$$

$$b(k,t) = b(k,0)\, e^{-\frac{\eta}{\mu_0} k^2 t} + (2\pi)^{3/2}|k| \int ds\, e^{-\frac{\eta}{\mu_0} k^2 s} \int d^3 q$$
$$\left\{ v(q,t-s) \times_{(k)} b(k-q,t-s) - b(q,t-s) \times_{(k)} v(k-q,t-s) \right\}$$

To give the equations a probabilistic interpretation, rescale the fields by a positive function $h(k)$:

$$v(k,t) = h(k)\, \chi_v(k,t)\ , \qquad b(k,t) = \sqrt{\rho_0 \mu_0}\, h(k)\, \chi_b(k,t)$$

Magnetohydrodynamics

One obtains

$$\chi_v(k,t) = \chi_v(k,0)\boxed{e^{-\nu k^2 t}} + \int ds \boxed{\nu k^2 \, e^{-\nu k^2 s}} \int d^3q \boxed{\frac{h(q)h(k-q)}{(h*h)(k)}}$$
$$\left\{ \frac{1}{2} g_{v\to vv}(k)\, \chi_v(q,t-s) \times_{(k)} \chi_v(k-q,t-s) + \right.$$
$$\left. \frac{1}{2} g_{v\to bb}(k)\, \chi_b(q,t-s) \times_{(k)} \chi_b(k-q,t-s) \right\}$$

and

$$\chi_b(k,t) = \chi_b(k,0)\boxed{e^{-\frac{\eta}{\mu_0} k^2 t}} + \int ds \boxed{\frac{\eta}{\mu_0} k^2 \, e^{-\frac{\eta}{\mu_0} k^2 s}} \int d^3q \boxed{\frac{h(q)h(k-q)}{(h*h)(k)}}$$
$$\left\{ \frac{1}{2} g_{b\to vb}(k)\, \chi_v(q,t-s) \times_{(k)} \chi_b(k-q,t-s) + \right.$$
$$\left. \frac{1}{2} g_{b\to bv}(k)\, \chi_b(q,t-s) \times_{(k)} \chi_v(k-q,t-s) \right\}$$

Magnetohydrodynamics

$*$ denotes convolution and the functions $g_{\bullet\to\bullet\bullet}$ are:

$$
\begin{aligned}
g_{v\to vv}(k) &= -g_{v\to bb}(k) = \frac{2(2\pi)^{3/2}(h*h)(k)}{\nu\,|k|\,h(k)} \\
g_{b\to vb}(k) &= -g_{b\to bv}(k) = \frac{2(2\pi)^{3/2}\mu_0\,(h*h)(k)}{\eta\,|k|\,h(k)}
\end{aligned}
$$

Stochastic interpretation: a combination of exponential with branching processes.
- $e^{-\nu k^2 t}$, $e^{-(\eta/\mu_0)k^2 t}$ survival probabilities up to time t,
- $\nu k^2\, e^{-\nu k^2 s}\, ds$, $(\eta/\mu_0)k^2\, e^{-(\eta/\mu_0)k^2 s}\, ds$ decay probabilities in the interval $(s, s+ds)$
- $h(q)h(k-q)/(h*h)(k)\, d^3q$ the probability that, given a k mode, one obtains a branching to modes $q, k-q$.
- The functions $g_{\bullet\to\bullet\bullet}$ play the role of coupling constants.

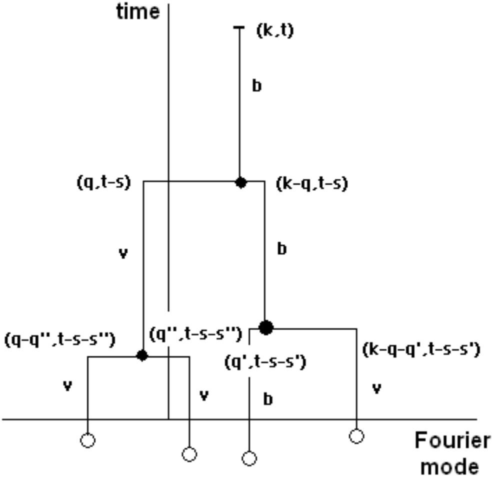
$\chi_v(k,t)$, $\chi_b(k,t)$ are then the expectation values of multiplicative functionals associated to the processes, whose convergence is assured by the following conditions:
on coupling constants (conditions on the function $h(k)$):

$$\begin{aligned} g_{v\to vv}(k) &\leq 1; g_{b\to vb}(k) \leq 1 \\ &\Longleftrightarrow 2(2\pi)^{3/2}(h*h)(k) \leq \min\left(\nu, \frac{\eta}{\mu_0}\right)|k|\,h(k) \end{aligned}$$

on initial conditions:

$$|\chi_v(k,0)| \leq 1;\ |\chi_v(k,0)| \leq 1 \Leftrightarrow |v(k,0)| \leq h(k);\ |b(k,0)| \leq \sqrt{\rho_0\mu_0}\,h(k)$$

Magnetohydrodynamics



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