# Stochastic solutions of partial differential equations

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- the solution may be written either as

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- or as

$$u(t,x) = \mathbb{E}_x f(X_t) \tag{3}$$

$\mathbb{E}_x$ being the expectation value, starting from $x$, of the Wiener process $dX_t = dW_t$

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- An important condition for (2) and (3) to be considered as **solutions** is the fact that the algorithmic tools are **independent** of the particular solution,

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- This should be contrasted with stochastic processes constructed from a given particular solution, as has been done for example for the Boltzman equation

# Stochastic solutions of linear equations: Examples

- Let $L$ be the operator

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x) \tag{4}$$

assumed to be strictly elliptic, that is

$$\sum_{i,j=1}^{d} a_{ij}(x)y_i y_j \geq \Lambda(x)\sum_{i=1}^{d} y_i^2 \tag{5}$$

and $X_t$ the solution to the stochastic differential equation

$$dX_t = b(x)dt + \sigma(x)dW_t \tag{6}$$

$\sigma$ being a matrix such that $a = \sigma\sigma^T$ and $W_t$ the Wiener process

$$\begin{aligned} \mathbb{E}(W_t) &= 0 \\ \mathbb{E}(W_t W_s) &= \min(t, s) \\ \mathbb{E}(dW_t dW_t) &= dt \end{aligned}$$

# Stochastic solutions of linear equations: Examples

- *The Poisson equation*

$$Lu_1(x) - \lambda u_1(x) = -f_1(x) \tag{7}$$

where $\lambda > 0$ and $f_1$ is a $C^1$ function with compact support. Then

$$\boxed{u_1(x) = \mathbb{E}^x \int_0^\infty e^{-\lambda t} f_1(X_t)\, dt} \tag{8}$$

$\mathbb{E}^x$ denotes the expectation value for the process that starts from $x$ at time zero.

- *In a bounded domain $D$* with $u_1 = 0$ in $\partial D$ and $f_1 \in C^2$

$$\boxed{u_1(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\lambda t} f_1(X_t) dt} \tag{9}$$

$\tau_D = \inf\{t : X_t \notin D\}$ being the first exit time from $D$.

- *The Dirichlet problem*

$$Lu_2(x) = 0 \qquad \text{with} \qquad u_2 = f_2 \qquad \text{on} \qquad \partial D \tag{10}$$

$$\boxed{u_2(x) = \mathbb{E}^x f_2(X_{\tau_D})} \tag{11}$$

# Stochastic solutions of linear equations: Examples

- *The Cauchy problem of the parabolic equation*

$$\partial_t u_3(x,t) = L u_3(x,t) \qquad \text{with} \qquad u_3(x,0) = f_3(x) \tag{12}$$

$$\boxed{u_3(x,t) = \mathbb{E}^x f_3(X_t)} \tag{13}$$

- The same problem *in a bounded domain $D$* with $u_3(x,0) = f_3(x)$ and $u_3(x,0) = 0$ in $\partial D$

$$\boxed{u_3(x,t) = \mathbb{E}^x \left[ f_3(X_t); t < \tau_D \right]}$$

- For *the Schrödinger operator* $L + v(x)$

$$(L + v(x))\, u_4(x) = 0 \qquad \text{with} \qquad u_4 = f_4 \qquad \text{on} \qquad \partial D \tag{14}$$

$$\boxed{u_4(x) = \mathbb{E}^x \left[ f_4(X_{\tau_D}) e^{\int_0^{\tau_D} v(X_s) ds} \right]} \tag{15}$$

- and

$$(L + v(x))\, u_5(x) = -g(x) \qquad \text{with} \qquad u_5 = 0 \qquad \text{on} \qquad \partial D \quad (16)$$

$$\boxed{u_5(x) = \mathbb{E}^x \left[ \int_0^{\tau_D} g(X_s)\, e^{\int_0^s v(X_r)dr} ds \right]} \quad (17)$$

- *The Cauchy problem for the Schrödinger operator*

$$\partial_t u_6(x,t) = (L + v(x))\, u_6(x,t) \qquad \text{with} \qquad u_6(x,0) = f_6(x) \quad (18)$$

$$\boxed{u_6(x,t) = \mathbb{E}^x \left[ f_6(X_t)\, e^{\int_0^t v(X_s)ds} \right]} \quad (19)$$

# Stochastic solutions and numerical codes

New **exact solutions** and also new **numerical algorithms**

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- Deterministic algorithms grow exponentially with the dimension $d$ of the space, roughly $N^d$ ($\frac{L}{N}$ the linear size of the grid). A stochastic simulation only grows with the dimension of the process, typically of order $d$.
- Deterministic algorithms aim at obtaining the solution in the whole domain. Then, even if an efficient deterministic algorithm exists, the stochastic algorithm is competitive if only localized values of the solution are desired. For example by studying only a few high Fourier modes one may obtain information on the small scale fluctuations that only a very fine grid might provide in a deterministic algorithm.

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- Each sample path is independent. Likewise, paths starting from different points are independent from each other. The stochastic algorithms are a natural choice for parallel and distributed computation.
- Stochastic algorithms are appropriate for domain decomposition and handle equally well regular and complex boundary conditions.

# Stochastic solutions of nonlinear partial differential equations

- **The basic idea:** One notices that in the linear partial differential equation case, once the relevant stochastic process is identified, the process is started from the point $x$ where the solution is to be computed, and the solution is a functional of the exit values of the process (from a space $D$ or a space-time $D \times [0, t]$ domain)

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- **Conjecture:** For the nonlinear equations the relevant process has a diffusion, propagation or jump component associated to the linear part of the equation plus a branching mechanism associated to the nonlinear part. The solution will be a functional of the exit measures generated by the process.

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- **Conjecture:** For the nonlinear equations the relevant process has a diffusion, propagation or jump component associated to the linear part of the equation plus a branching mechanism associated to the nonlinear part. The solution will be a functional of the exit measures generated by the process.
- **The construction:** Rewrite the equation as an integral equation. Give a probabilistic interpretation to the integral equation. In the end the stochastic solution is equivalent to a sampling evaluation of the Picard series.

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4. The Poisson-Vlasov equation (Cipriano, Floriani, Lima, R. V. M.)
5. The Euler equation (Cipriano, R. V. M.)
6. A fractional version of the KPP equation (Cipriano, Ouerdiane, R. V. M.)

# The Poisson-Vlasov equation

$$\boxed{\frac{\partial f_i}{\partial t} + \overrightarrow{v} \cdot \nabla_x f_i - \frac{e_i}{m_i} \nabla_x \Phi \cdot \nabla_v f_i = 0} \tag{20}$$

$$\Delta_x \Phi = -4\pi \left\{ \sum_i e_i \int f_i \left( \overrightarrow{x}, \overrightarrow{v}, t \right) d^3 v \right\} \tag{21}$$

Fourier transforming Eq.(20) and (21), with

$$F_i(\xi, t) = \frac{1}{(2\pi)^3} \int d^6 \eta f_i(\eta, t) e^{i\xi \cdot \eta} \tag{22}$$

$\eta = \left( \overrightarrow{x}, \overrightarrow{v} \right)$ and $\xi = \left( \overrightarrow{\xi_1}, \overrightarrow{\xi_2} \right) \triangleq (\xi_1, \xi_2)$, one obtains

$$\begin{aligned} 0 \quad = \quad & \frac{\partial F_i(\xi, t)}{\partial t} - \overrightarrow{\xi_1} \cdot \nabla_{\xi_2} F_i(\xi, t) \\ & + \frac{4\pi e_i}{m_i} \int d^3 \xi_1' F_i \left( \xi_1 - \xi_1', \xi_2, t \right) \frac{\overrightarrow{\xi_2} \cdot \overrightarrow{\xi_1'}}{\left| \xi_1' \right|^2} \sum_j e_j F_j \left( \xi_1', 0, t \right) \end{aligned} \tag{23}$$

Changing variables to

$$\tau = \gamma\left(\left|\xi_2\right|\right) t \tag{24}$$

$\gamma\left(\left|\xi_2\right|\right)$ is a positive continuous function satisfying

$$\begin{array}{lll} \gamma\left(\left|\xi_2\right|\right) = 1 & if & \left|\xi_2\right| < 1 \\ \gamma\left(\left|\xi_2\right|\right) \geq \left|\xi_2\right| & if & \left|\xi_2\right| \geq 1 \end{array}$$

$$\begin{aligned} \frac{\partial F_i\left(\xi,\tau\right)}{\partial \tau} &= \frac{\overrightarrow{\xi_1}}{\gamma\left(\left|\xi_2\right|\right)} \cdot \nabla_{\xi_2} F_i\left(\xi,\tau\right) - \frac{4\pi e_i}{m_i} \int d^3\xi_1^{'} F_i\left(\xi_1 - \xi_1^{'}, \xi_2, \tau\right) \\ & \times \frac{\overrightarrow{\xi_2} \cdot \hat{\xi_1^{'}}}{\gamma\left(\left|\xi_2\right|\right)\left|\xi_1^{'}\right|} \sum_j e_j F_j\left(\xi_1^{'}, 0, \tau\right) \end{aligned} \tag{25}$$

with $\hat{\xi_1} = \frac{\overrightarrow{\xi_1}}{\left|\xi_1\right|}$.

Stochastic representation written for the following functions

$$\chi_i(\xi_1,\xi_2,\tau)=e^{-\lambda\tau}\frac{F_i(\xi_1,\xi_2,\tau)}{h(\xi_1)} \tag{26}$$

with $\lambda$ a constant and $h(\xi_1)$ a positive function to be specified later. Define

$$\left(\left|\xi_1'\right|^{-1}h*h\right)=\int d^3\xi_1'\left|\xi_1'\right|^{-1}h\left(\xi_1-\xi_1'\right)h\left(\xi_1'\right) \tag{27}$$

$$p\left(\xi_1,\xi_1'\right)=\frac{\left|\xi_1'\right|^{-1}h\left(\xi_1-\xi_1'\right)h\left(\xi_1'\right)}{\left(\left|\xi_1'\right|^{-1}h*h\right)} \tag{28}$$

# The Poisson-Vlasov equation

## The Fourier transformed equation

$$
\begin{aligned}
&\chi_i(\xi_1,\xi_2,\tau) \\
=\ & \boxed{e^{-\lambda\tau}}\chi_i\left(\xi_1,\xi_2+\tau\frac{\xi_1}{\gamma(|\xi_2|)},0\right)-\frac{8\pi e_i}{m_i\lambda}\frac{\left(|\xi_1|^{-1}h*h\right)(\xi_1)}{h(\xi_1)} \\
&\times\int_0^\tau ds\boxed{\lambda e^{-\lambda s}}\int d^3\xi_1'\boxed{p\left(\xi_1,{\xi_1'}'\right)}\chi_i\left(\xi_1-{\xi_1'}',\xi_2+s\frac{\xi_1}{\gamma(|\xi_2|)},\tau-s\right) \\
&\times\frac{\left(\xi_2+s\frac{\xi_1}{\gamma(|\xi_2|)}\right)\cdot\hat{\xi_1'}}{\gamma\left(\left|\xi_2+s\frac{\xi_1}{\gamma(|\xi_2|)}\right|\right)}\sum_j\boxed{\frac{1}{2}}e_je^{\lambda(\tau-s)}\chi_j\left({\xi_1'}',0,\tau-s\right)
\end{aligned}
\tag{29}
$$

Eq.(29) has a stochastic interpretation *(an* **exponential process** *plus* **branching** *and* **Bernoulli** *processes)*.

$e^{-\lambda\tau} =$ survival probability during time $\tau$ of the exponential process

$\lambda e^{-\lambda s} ds =$ the decay probability

$p\left(\xi_1, \xi_1^{\prime\prime}\right) d^3\xi_1 =$ branching probability of $\xi_1$ mode into $\left(\xi_1 - \xi_1^{\prime\prime}, \xi_1^{\prime\prime}\right)$

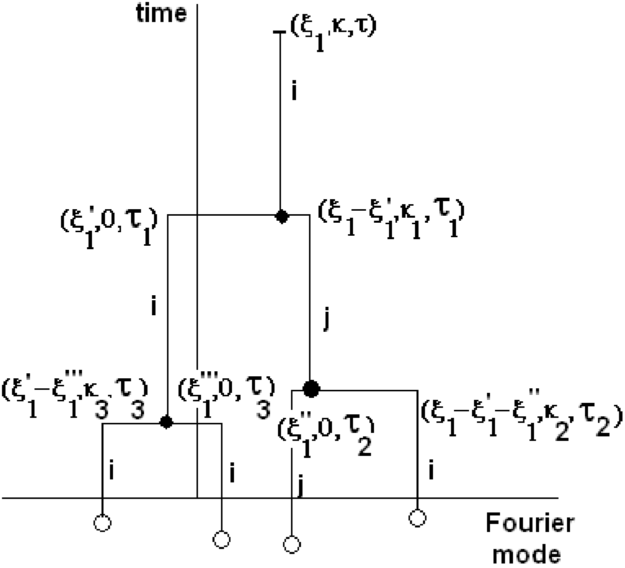
$\chi\left(\xi_1, \xi_2, \tau\right)$ computed from the expectation value of a multiplicative functional

**Convergence of the multiplicative functional:**

(A) $\left|\frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}\right| \leq 1$

(B) $\left(\left|\xi_1^{\prime}\right|^{-1} h * h\right)\left(\xi_1\right) \leq h\left(\xi_1\right)$, satisfied, for example,

for $h\left(\xi_1\right) = \frac{c}{\left(1+\left|\xi_1\right|^2\right)^2}$ and $c \leq \frac{1}{3\pi}$


time
(ξ1,κ,τ)
i
(ξ1',0,τ1)
(ξ1−ξ1',κ1,τ1)
i
j
(ξ1'−ξ1''',κ3,τ3)
(ξ1''',0,τ3)
(ξ1'',0,τ2)
(ξ1−ξ1'−ξ1'',κ2,τ2)
i
i
j
i
Fourier mode

The process $X(\xi_1, \xi_2, \tau)$ is the limit of the following iteration

$$
\begin{aligned}
& X_i^{(k+1)}(\xi_1, \xi_2, \tau) \\
= \; & \chi_i\left(\xi_1, \xi_2 + \tau \frac{\xi_1}{\gamma(|\xi_2|)}, 0\right) \mathbf{1}_{[s>\tau]} + g_{ii}\left(\xi_1, \xi_1', s\right) \\
& \times X_i^{(k)}\left(\xi_1 - \xi_1', \xi_2 + \frac{s\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) X_i^{(k)}\left(\xi_1', 0, \tau - s\right) \mathbf{1}_{[s<\tau]} \mathbf{1}_{[l_s=0]} \\
& + g_{ij}\left(\xi_1, \xi_1'\right) X_i^{(k)}\left(\xi_1 - \xi_1', \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) X_j^{(k)}\left(\xi_1', 0, \tau - s\right) \\
& \times \mathbf{1}_{[s<\tau]} \mathbf{1}_{[l_s=1]}
\end{aligned}
$$

The multiplicative functional of the process $X(\xi_1, \xi_2, \tau)$ is the product of:
- **At each branching point where 2 particles are born**

$$g_{ij}\left(\xi_1, \xi_1', s\right) = -e^{\lambda(\tau-s)}\frac{8\pi e_i e_j}{m_i\lambda}\frac{\left(\left|\xi_1'\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)}\frac{\left(\xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right)\cdot \hat{\xi_1'}}{\gamma\left(\left|\xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right|\right)}$$

- **When one particle reaches time zero and samples the initial condition**

$$g_{0i}(\xi_1, \xi_2) = \frac{F_i(\xi_1, \xi_2, 0)}{h(\xi_1)}$$

$$\boxed{\chi_i(\xi_1, \xi_2, \tau) = \mathbb{E}\left\{\Pi\left(g_0 g_0' \cdots\right)\left(g_{ii} g_{ii}' \cdots\right)\left(g_{ij} g_{ij}' \cdots\right)\right\}}$$

- Choose $\lambda \geq \left| \frac{8\pi e_i e_j}{\min_i\{m_i\}} \right|$ and $c \leq e^{-\lambda\tau} \frac{1}{3\pi} \Longrightarrow$ the absolute value of all coupling constants is bounded by one.

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- The branching process, identical to a Galton-Watson process, terminates with probability one $\implies$ number of inputs to the functional is finite (with probability one).

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- The branching process, identical to a Galton-Watson process, terminates with probability one $\implies$ number of inputs to the functional is finite (with probability one).
- With the bounds on the coupling constants, the multiplicative functional is bounded by one in absolute value almost surely.
- **Theorem 1.** *The stochastic process* $X(\xi_1, \xi_2, \tau)$*, above described, provides a stochastic solution for the Fourier-transformed Poisson-Vlasov equation* $F_i(\xi_1, \xi_2, t)$ *for any arbitrary finite value of the arguments, provided the initial conditions at time zero satisfy the boundedness conditions (A).*

Instead of renormalizing the time (Eq.(24)) one may write

$$\Theta_i\left(\xi_1, \xi_2, t\right) = e^{-t|\xi_2|} \frac{F_i\left(\xi_1, \xi_2, t\right)}{h\left(\xi_1\right)}$$

$p\left(\xi_1, \xi_1^{'}\right)$ and the conditions on $h\left(\xi_1\right)$ are the same as before. The main difference is the survival probability, namely $e^{-t|\xi_2|}$ and $ds\Pi\left(\xi_1, \xi_2, s\right)$ the dying probability in time $ds$

$$\Pi\left(\xi_1, \xi_2, s\right) = \frac{\left|\xi_2 + s\xi_1\right| e^{(t-s)|\xi_2+s\xi_1|-t|\xi_2|}}{N\left(\xi_1, \xi_2, t\right)}$$

$$N\left(\xi_1, \xi_2, t\right) = \frac{1}{1-e^{-t|\xi_2|}} \int_0^t ds \left|\xi_2 + s\xi_1\right| e^{(t-s)|\xi_2+s\xi_1|-t|\xi_2|}$$

# The Poisson-Vlasov equation

## The configuration space equation

$$\boxed{\frac{\partial f_i}{\partial t} + \overrightarrow{v} \cdot \nabla_x f_i - \frac{e_i}{m_i} \nabla_x \Phi \cdot \nabla_v f_i = 0}$$

$$G_i\left(\overrightarrow{x}, \overrightarrow{v}, t\right) = e^{-\lambda t} \frac{f_i\left(\overrightarrow{x}, \overrightarrow{v}, t\right)}{\varphi_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}\right)}$$

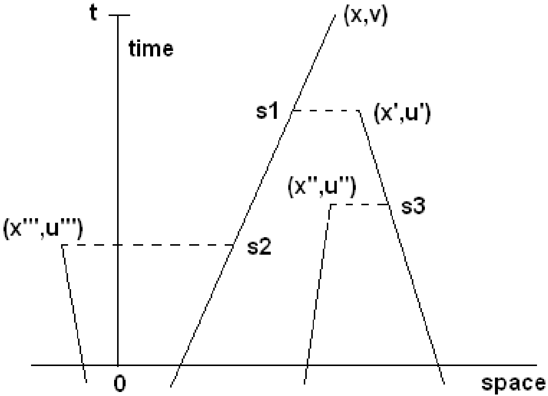
$$\begin{aligned} & G_i\left(\overrightarrow{x}, \overrightarrow{v}, t\right) \\ = \; & \boxed{e^{-\lambda t}} G_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}, 0\right) - 2\sum_j \boxed{\frac{1}{2}} \frac{e_i e_j}{m_i \lambda} \int_0^t ds \boxed{\lambda e^{-\lambda s}} A_{x,v,t,s}^{(j)} e^{\lambda(t-s)} \\ & \times \int d^3x' d^3u \boxed{p_{x,v,t,s}^{(j)}\left(\overrightarrow{x'}, \overrightarrow{u}\right)} G_j\left(\overrightarrow{x'}, \overrightarrow{u}, t-s\right) \left(\widehat{\overrightarrow{x} - s\overrightarrow{v} - \overrightarrow{x'}}\right) \\ & \bullet \frac{1}{\varphi_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}\right)} \left(\nabla_v + s\nabla_x\right) \varphi_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}\right) G_i\left(\overrightarrow{x} - s\overrightarrow{v}, \overrightarrow{v}, t-s\right) \end{aligned}$$

$$p_{x,v,t,s}^{(j)}\left(\overrightarrow{x'},\overrightarrow{u}\right)=\frac{1}{A_{x,v,t,s}^{(j)}}\frac{\varphi_j\left(\overrightarrow{x'}-u\left(t-s\right),\overrightarrow{u}\right)}{\left|\overrightarrow{x}-s\overrightarrow{v}-\overrightarrow{x'}\right|^2}$$

$$A_{x,v,t,s}^{(j)}=\int d^3x'd^3u\frac{\varphi_j\left(\overrightarrow{x'}-u\left(t-s\right),\overrightarrow{u}\right)}{\left|\overrightarrow{x}-s\overrightarrow{v}-\overrightarrow{x'}\right|^2}$$

The simplest choice for $\varphi_i\left(\overrightarrow{x},\overrightarrow{v}\right)$ is $\varphi_i\left(\overrightarrow{x},\overrightarrow{v}\right)=f_i\left(\overrightarrow{x},\overrightarrow{v},0\right)$
The probabilistic interpretation requires finiteness

of $\quad A_{x,v,t,s}^{(j)}=\int d^3x'd^3u\frac{f_j\left(\overrightarrow{x'}-u(t-s),\overrightarrow{u},0\right)}{\left|\overrightarrow{x}-s\overrightarrow{v}-\overrightarrow{x'}\right|^2}$



t
time
(x,v)
s1
(x',u')
(x'',u'')
s3
(x''',u''')
s2
0
space

Contributions to the multiplicative functional:

- The coupling constants at the creation of each new particle is

$$g_{ij}\left(x,v,t,s\right)=\frac{2e_ie_j}{m_i\lambda}A_{x,v,t,s}^{(j)}e^{\lambda(t-s)}$$

- The terminal contribution of a particle that in the course of its evolution has received the labels $s_1, s_2, \cdots s_n$ is

$$\frac{1}{f_i\left(\overrightarrow{x}-t\overrightarrow{v},\overrightarrow{v},0\right)}\left(\nabla_v+s_1\nabla_x\right)\cdots\left(\nabla_v+s_n\nabla_x\right)f_i\left(\overrightarrow{x}-t\overrightarrow{v},\overrightarrow{v},0\right)$$

- Convergence of the multiplicative functional in the stochastic solution requires

$$\left| \frac{2e_i e_j}{\min(m_i)\lambda} \max_s \left( A^{(j)}_{x,v,t,s} \right) e^{\lambda(t-s)} \right| \leq 1 \tag{30}$$

$$\left| \frac{1}{f_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}, 0\right)} \left(\nabla_v + s_1 \nabla_x\right) \cdots \left(\nabla_v + s_n \nabla_x\right) f_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}, 0\right) \right| \leq 1 \tag{31}$$

- Convergence of the multiplicative functional in the stochastic solution requires

$$\left| \frac{2e_i e_j}{\min(m_i)\,\lambda} \max_s \left( A^{(j)}_{x,v,t,s} \right) e^{\lambda(t-s)} \right| \leq 1 \tag{30}$$

$$\left| \frac{1}{f_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}, 0\right)} \left(\nabla_v + s_1 \nabla_x\right) \cdots \left(\nabla_v + s_n \nabla_x\right) f_i\left(\overrightarrow{x} - t\overrightarrow{v}, \overrightarrow{v}, 0\right) \right| \leq 1 \tag{31}$$

- **Theorem 2.** *The stochastic process $Y\left(\overrightarrow{x}, \overrightarrow{u}, t\right)$, above described, provides a stochastic solution for the configuration space Poisson-Vlasov equation provided the initial conditions satisfy the constraints (30) and (31).*

# A fractional nonlinear equation

A fractional version of the KPP equation, studied by McKean

$$\boxed{{}_tD_*^\alpha u(t,x) = \frac{1}{2}\,{}_xD_\theta^\beta u(t,x) + u^2(t,x) - u(t,x)} \tag{32}$$

${}_tD_*^\alpha$ is a Caputo derivative of order $\alpha$

$$ {}_tD_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t) & \alpha = m \end{cases} \tag{33}$$

${}_xD_\theta^\beta$ is a Riesz-Feller derivative defined through its Fourier symbol

$$\mathcal{F}\left\{{}_xD_\theta^\beta f(x)\right\}(k) = -\psi_\beta^\theta(k)\,\mathcal{F}\{f(x)\}(k) \tag{34}$$

with $\psi_\beta^\theta(k) = |k|^\beta e^{i(\text{sign}k)\theta\pi/2}$.
Physically it describes a nonlinear diffusion with growing mass and in our fractional generalization it would represent the same phenomenon taking into account memory effects in time and long range correlations in space.

# A fractional nonlinear equation

The first step towards a probabilistic formulation is the rewriting of Eq.(32) as an integral equation.Take the Fourier transform $(\mathcal{F})$ in space and the Laplace transform $(\mathcal{L})$ in time

$$s^{\alpha}\tilde{\hat{u}}(s,k) = s^{\alpha-1}\hat{u}(0^{+},k) - \frac{1}{2}\psi_{\beta}^{\theta}(k)\tilde{\hat{u}}(s,k) - \tilde{\hat{u}}(s,k) + \int_{0}^{\infty} dt e^{-st}\mathcal{F}(u^{2})$$

where

$$\hat{u}(t,k) = \mathcal{F}(u(t,x)) = \int_{-\infty}^{\infty} e^{ikx}u(t,x)$$
$$\tilde{u}(s,x) = \mathcal{L}(u(t,x)) = \int_{0}^{\infty} e^{-st}u(t,x)$$

This equation holds for $0 < \alpha \leq 1$ or for $0 < \alpha \leq 2$ with $\frac{\partial}{dt}u(0^{+},x) = 0$.

Solving for $\tilde{\hat{u}}(s,k)$ one obtains an integral equation

$$\tilde{\hat{u}}(s,k) = \frac{s^{\alpha-1}}{s^{\alpha} + \frac{1}{2}\psi_{\beta}^{\theta}(k)}\hat{u}(0^{+},k) + \int_{0}^{\infty} dt \frac{e^{-st}}{s^{\alpha} + \frac{1}{2}\psi_{\beta}^{\theta}(k)}\mathcal{F}(u^{2}(t,x))$$

# A fractional nonlinear equation

Taking the inverse Fourier and Laplace transforms

$$
\begin{aligned}
& u(t,x) \\
= \; & \boxed{E_{\alpha,1}\left(-t^{\alpha}\right)} \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)t^{\alpha}\right)}{E_{\alpha,1}\left(-t^{\alpha}\right)}\right)(x-y)\, u(0,y) \\
& + \int_{0}^{t} d\tau \boxed{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)} \\
& \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)(t-\tau)^{\alpha}\right)}{E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)}\right)(x-y)\, u^{2}(\tau,y)
\end{aligned}
$$

$E_{\alpha,\rho}$ is the generalized Mittag-Leffler function $E_{\alpha,\rho}(z)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+\rho)}$

$$
E_{\alpha,1}\left(-t^{\alpha}\right)+\int_{0}^{t} d\tau\,(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)=1
$$

# A fractional nonlinear equation

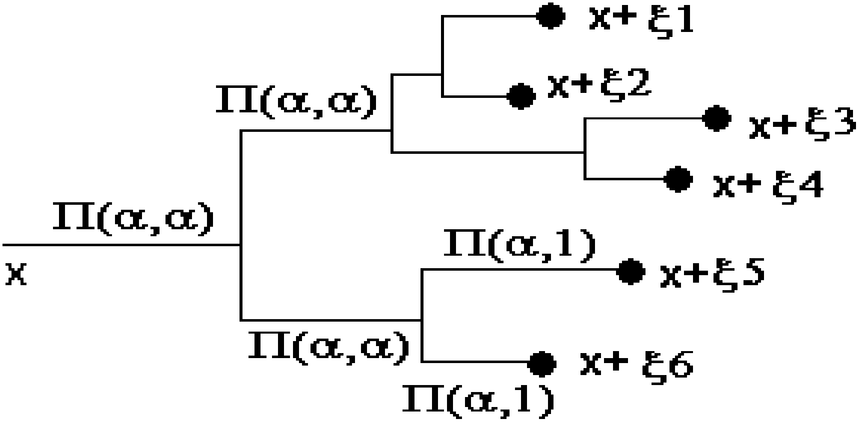
We define the following propagation kernel

$$G^{\beta}_{\alpha,\rho}(t,x) = \mathcal{F}^{-1}\left(\frac{E_{\alpha,\rho}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)}{E_{\alpha,\rho}(-t^{\alpha})}\right)(x)$$

$$\begin{aligned} & u(t,x) \\ = \; & \boxed{E_{\alpha,1}(-t^{\alpha})}\int_{-\infty}^{\infty} dy \boxed{G^{\beta}_{\alpha,1}(t,x-y)} u(0^{+},y) \\ & + \int_0^t d\tau \boxed{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)} \\ & \int_{-\infty}^{\infty} dy \boxed{G^{\beta}_{\alpha,\alpha}(t-\tau,x-y)} u^{\boxed{2}}(\tau,y) \end{aligned}$$

$E_{\alpha,1}(-t^{\alpha})$ and $(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)$ = survival probability up to time $t$ and the probability density for the branching at time $\tau$ (branching process $B_{\alpha}$)

# A fractional nonlinear equation



# A fractional nonlinear equation

$$u(t,x)=\mathbb{E}_x\left(\varphi_1\varphi_2\cdots\varphi_n\right) \tag{35}$$

with

$$\begin{aligned}\varphi_i &= \int dy_1^{(i)}dy_2^{(i)}\cdots dy_{k-1}^{(i)}dy_k^{(i)}\,G_{\alpha,\alpha}^{\beta}\left(\tau_1,x-y_1\right)G_{\alpha,\alpha}^{\beta}\left(\tau_2,y_1-y_2\right)\cdots \\ &\cdots G_{\alpha,\alpha}^{\beta}\left(\tau_{k-1},y_{k-2}-y_{k-1}\right)G_{\alpha,1}^{\beta}\left(\tau_k,y_{k-1}-y_k\right)u\left(0^+,y_k\right)\end{aligned} \tag{36}$$

with $\sum_{i=1}^{k}\tau_j=t,\ k-1$ being the number of branchings leading to particle $i$

The propagation kernels satisfy the conditions to be the Green's functions of stochastic processes in $\mathbb{R}$:

$$u(t,x)=\mathbb{E}_x\left(u(0^+,x+\xi_1)u(0^+,x+\xi_2)\cdots u(0^+,x+\xi_n)\right) \tag{37}$$

# A fractional nonlinear equation

Denote the processes associated to $G_{\alpha,1}^{\beta}(t,x)$ and $G_{\alpha,\alpha}^{\beta}(t,x)$, respectively by $\Pi_{\alpha,1}^{\beta}$ and $\Pi_{\alpha,\alpha}^{\beta}$

**Theorem 3:** *The nonlinear fractional partial differential equation (4), with* $0<\alpha\leq 1$, *has a stochastic solution, the coordinates* $x+\xi_i$ *in the arguments of the initial condition obtained from the exit values of a propagation and branching process, the branching being ruled by the process* $B_\alpha$ *and the propagation by* $\Pi_{\alpha,1}^{\beta}$ *for the first particle and by* $\Pi_{\alpha,\alpha}^{\beta}$ *for all the remaining ones.*

*A sufficient condition for the existence of the solution is*

$$\left|u(0^+,x)\right|\leq 1 \tag{38}$$