The fractional volatility model: No-arbitrage and risk measures

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Introduction: The fractional volatility model

- Classical Mathematical Finance has, for a long time, been based on the assumption that the price process of market securities may be approximated by geometric Brownian motion (GBM)

$$dS_t = \mu S_t dt + \sigma S_t dB(t) \tag{1}$$

consistent with the fact that in liquid markets the autocorrelation of price changes decays to negligible values in a few minutes.
- Otherwise **GBM has serious shortcomings:**
  - Does not reproduce the empirical leptokurtosis
  - Does not explain why nonlinear functions of the returns exhibit significant positive autocorrelation (volatility clustering)
  - There is an essential memory component and a dynamical model for volatility is needed, σ in Eq.(1) being itself a process.
- This led to many deterministic and stochastic models for the volatility which fit the leptokurtosis but not always the long memory. In contrast with GBM, they mostly lack the kind of nice mathematical properties needed to develop the tools of mathematical finance.

Introduction: The fractional volatility model

- **The fractional volatility model:** a model based on simple mathematical assumptions and reconstructed from the data.
- **Basic hypothesis:**

  (H1) The log-price process $\log S_t$ belongs to a product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$ of which (Ω_1, P_1) is the Wiener space and the second, (Ω_2, P_2), is a probability space to be reconstructed from the data. With $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and $\mathcal{F}_{1,t}$ and $\mathcal{F}_{2,t}$ the σ-algebras in Ω_1 and Ω_2

  $$\log S_t(\omega_1, \omega_2)$$

  (H2) For each fixed ω_2, $\log S_t(\cdot, \omega_2)$ is a square integrable random variable in Ω_1.
- These principles and a careful analysis of the market data led, in an essentially unique way, to the following model:

$$\boxed{\begin{array}{lcl} dS_t & = & \mu S_t\, dt + \sigma_t S_t\, dB(t) \\ \log \sigma_t & = & \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} \end{array}} \tag{2}$$

Introduction: The fractional volatility model

$$\boxed{\begin{array}{lcl} dS_t & = & \mu S_t\, dt + \sigma_t S_t\, dB(t) \\ \log \sigma_t & = & \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} \end{array}}$$

- The data suggests values of H in the range $0.8 - 0.9$.
- The second equation in (2) leads to

$$\sigma(t) = \theta e^{\frac{k}{\delta}\{B_H(t) - B_H(t-\delta)\} - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}} \tag{3}$$

  with $E[\sigma(t)] = \theta > 0$.
- Describes well **the statistics of price returns** for a large $\delta-$range in different markets and also implies **a new option pricing formula**, with "smile" deviations from Black-Scholes.

No-arbitrage and market incompleteness

- Consider a market with an asset obeying the stochastic equations (2) and a risk-free asset A_t

$$dA_t = rA_t \, dt \tag{4}$$

with $r > 0$ constant.

- **Proposition 1:** *The market defined by (2) and (4) is free of arbitrages*
- **Lemma:** *For σ given by (3) one has:*
i) For any integer number n, $\int_0^T \mathbb{E}\left(\sigma_t^n\right) dt < \infty$, where the expectation is with respect to the probability measure P_2;
ii) Assuming that $\mu \in L^\infty\left([0, T], P_1 \times P_2\right)$, for any $t \in [0, T]$ there is a constant $C > 0$ such that $P_1 \times P_2$-a.e.

$$\int_0^t \frac{\left(r - \mu_s\right)^2}{\sigma_s^2} \, ds \leq C$$

No-arbitrage and market incompleteness

- **Proof of the lemma:** The first property follows from

$$\mathbb{E}\left(e^{\lambda(B_H(t)-B_H(t-\delta))}\right) = e^{\frac{\lambda^2}{2}\delta^{2H}}$$

for any complex number λ, while the second one from the Hölder continuity of the fractional Brownian motion B_H of order less than H. More precisely, for each $\alpha \in (0, H)$ there is a constant $C_\alpha > 0$ such that P_2-a.e.

$$|B_H(t) - B_H(s)| \leq C_\alpha \, |t-s|^\alpha$$

and thus $P_1 \times P_2$-a.e.

$$\begin{aligned}
\int_0^t \frac{(r-\mu_s)^2}{\sigma_s^2}\,ds &\leq \frac{(r+\|\mu\|_\infty)^2}{\theta^2}\, e^{\left(\frac{k}{\delta}\right)^2\delta^{2H}} \int_0^t e^{\frac{2k}{\delta}|B_H(s)-B_H(s-\delta)|}\,ds \\
&\leq \frac{T(r+\|\mu\|_\infty)^2}{\theta^2}\, e^{\left(\frac{k}{\delta}\right)^2\delta^{2H}+2kC_\alpha\delta^{\alpha-1}}
\end{aligned}$$

■

No-arbitrage and market incompleteness

Proof of Proposition 1: Let $P := P_1 \times P_2$ be the probability product measure and define the process

$$Z_t = \frac{S_t}{A_t} \tag{5}$$

in the interval $0 \leq t \leq T$, which obeys the equation

$$dZ_t = (\mu_t - r)\, Z_t\, dt + \sigma_t Z_t\, dB_t \tag{6}$$

Now let

$$\eta_t = \exp\left(\int_0^t \frac{r - \mu_s}{\sigma_s}\, dB_s - \frac{1}{2} \int_0^t \frac{|r - \mu_s|^2}{\sigma_s^2}\, ds \right) \tag{7}$$

which by the Lemma fulfills the Novikov condition and thus it is a P-martingale. Moreover, it yields a probability measure P' equivalent to P by

$$\frac{dP'}{dP} = \eta_T$$

No-arbitrage and market incompleteness

- By the Girsanov theorem

$$B_t^* = B_t - \int_0^t \frac{r - \mu_s}{\sigma_s} \, ds$$

is a $P'-$Brownian motion and

$$Z_t = Z_0 + \int_0^t \sigma_s Z_s \, dB_s^*$$

is a P'-martingale. By the fundamental theorem of asset pricing, the existence of an equivalent martingale measure for Z_t implies that there are no arbitrages, that is, $\mathbb{E}_{P'}\left[Z_t | \mathcal{F}_{1,s} \times \mathcal{F}_{2,s}\right] = Z_s$ for $0 \leq s < t \leq T$. ∎

No-arbitrage and market incompleteness

- **Proposition 2:** *The market defined by (2) and (4) is incomplete*
- **Proof:** Here we use an integral representation for the fractional Brownian motion,

$$B_H(t) = C \int_0^t K(t,s)\, dW_s \tag{8}$$

W_t being a Brownian motion independent from B_t, and consider the bi-dimensional Brownian motion (B_t, W_t) on P. Given the P_2-martingale

$$\eta'_t = \exp\left(W_t - \frac{1}{2}t\right)$$

we now use the product $\eta_t \eta'_t$. Due to the Lemma, the Novikov condition is fulfilled insuring that $\eta_t \eta'_t$ is a P-martingale and

$$\frac{dP''}{dP} = \eta_T \eta'_T$$

a probability measure.

No-arbitrage and market incompleteness

- As before, the Girsanov theorem implies that the Z_t process is still a P''-martingale. The equivalent martingale measure not being unique the market is, by definition, incomplete. ■
- Incompleteness of the market is a reflection of the fact that in stochastic volatility models there are two different sources of risk and only one of the risky assets is traded. In this case a choice of measure is how one fixes the volatility risk premium.

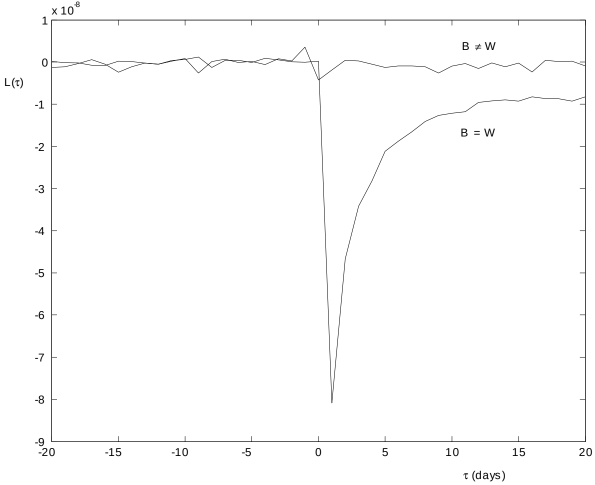
Leverage and the identification of the stochastic generators

- The following nonlinear correlation of the returns

$$L(\tau) = \left\langle \left| r(t+\tau) \right|^2 r(t) \right\rangle - \left\langle \left| r(t+\tau) \right|^2 \right\rangle \langle r(t) \rangle \qquad (9)$$

  is called *leverage* and the *leverage effect* is the fact that, for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus decays to zero whereas for $\tau < 0$ it has almost negligible values.
- In the form of Eqs.(2) the volatility process σ_t affects the log-price, but is not affected by it. Therefore, in its simplest form the fractional volatility model contains no leverage effect.
- Leverage may, however, be implemented in the model in a simple way if one identifies the Brownian processes B_t and W_t in (2) and (8). Identifying the random generator of the log-price process with the stochastic integrator of the volatility, at least a part of the leverage effect is taken into account.

Leverage and the identification of the stochastic generators



- Let us now consider the market (2) and (4) with B_t appearing in (2) replaced by the standard Brownian motion W_t which appears in the integral representation (8).
- **Proposition 3:** *This new market is free of arbitrages*
- **Proof:** In this case $P_1 = P_2$. Since the item ii) in the Lemma still holds for the product measure $P_1 \times P_2$ replaced by the probability measure P_2, with this change of probability measure the proof of this result follows as in the proof of Proposition 1. $\blacksquare$

Risk measures

- Let $\delta S = S_{t+\Delta} - S_t$ and
$$r(\Delta) = \log S_{t+\Delta} - \log S_t$$
be the return corresponding to a time lag Δ.
- The value at risk (VaR) Λ^* and the expected shortfall E^* are
$$\int_{-S}^{-\Lambda^*} P_\Delta(\delta S)\, d(\delta S) = P^*$$
$$E^* = \frac{1}{P^*}\int_{-S}^{-\Lambda^*} (-\delta S)\, P_\Delta(\delta S)\, d(\delta S)$$
S being the capital at time zero, P^* the probability of a loss Λ^* and $P_\Delta(\delta S)$ the probability of a price variation δS in the time interval Δ.

In terms of the returns these quantities are

$$\int_{-\infty}^{\log\left(1-\frac{\Lambda^*}{S}\right)} P(r(\Delta))\, d(r(\Delta)) = P^*$$

Risk measures

$$E^* = \frac{S}{P^*} \int_{-\infty}^{\log\left(1-\frac{\Lambda^*}{S}\right)} \left(1 - e^{r(\Delta)}\right) P\left(r\left(\Delta\right)\right) d\left(r\left(\Delta\right)\right)$$

- For the fractional volatility model the probability distribution of the returns in a time interval Δ, is obtained from

$$P\left(r\left(\Delta\right)\right) = \int_0^\infty p_\delta\left(\sigma\right) p_\sigma\left(r\left(\Delta\right)\right) d\sigma \tag{10}$$

$$p_\delta\left(\sigma\right) = \frac{1}{\sqrt{2\pi}\sigma k \delta^{H-1}} \exp\left\{-\frac{\left(\log\sigma - \beta\right)^2}{2k^2\delta^{2H-2}}\right\} \tag{11}$$

$$p_\sigma\left(r\left(\Delta\right)\right) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}} \exp\left\{-\frac{\left(r\left(\Delta\right) - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)^2}{2\sigma^2\Delta}\right\} \tag{12}$$

Using (10) - (12) Λ^* and E^* are computed. The figures show results for $P^* = 0.01$ (99%VaR) and time lags from 1 to 30 days, using $H = 0.83; k = 0.59, \beta = -5, \delta = 1$ from typical market daily data.

Risk measures

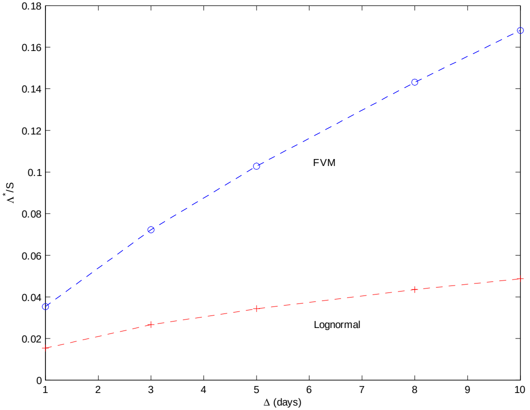


Figure: VaR in the fractional volatility model compared with the lognormal with the same average volatility

Risk measures

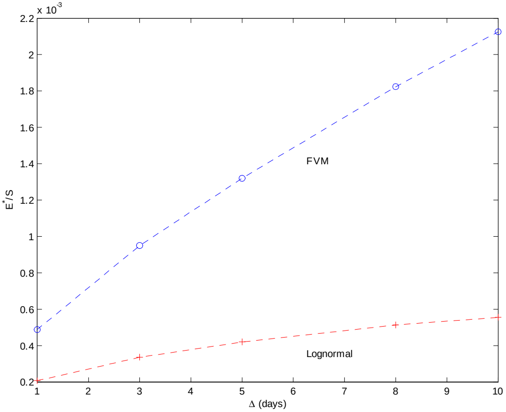


Figure: Expected shortfall in the fractional volatility model compared with the lognormal with the same average volatility

Remarks and conclusions

- FVM describes well returns distribution and modifications to Black-Scholes.
- Universality through different markets. Related to limit-order book dynamics.
- Mathematical consistency. No-arbitrage.
- Leverage with identification of the generators of the stochastic processes. Completeness?