Further developments in non-commutative tomography

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Outline

- 1D tomograms: General and related to algebras
- Multidimensional tomograms
- Tomograms as operator symbols
- Rotated time tomography
- Hermite-basis tomography
- Analysis of velocity fluctuations

1D tomograms

- *Tomograms that use the conformal group operators and eigenvectors of their linear combinations:*
- Time-frequency tomogram

$$B_1 = \mu t + i\nu \frac{d}{dt}$$

- Time-scale

$$B_2 = \mu t + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

- Frequency-scale

$$B_3 = i\mu \frac{d}{dt} + i\nu \left(t\frac{d}{dt} + \frac{1}{2} \right)$$

- Time-conformal

$$B_4 = \mu t + i\nu \left(t^2\frac{d}{dt} + t \right)$$

1D tomograms

In general:

$$B_4 = \mu t + i\nu \left(g(t) \frac{d}{dt} + \frac{1}{2} \frac{dg(t)}{dt} \right)$$

the generalized eigenvectors being

$$\psi_g(\mu, \nu, t, X) = |g(t)|^{-1/2} \exp i \left(-\frac{X}{\nu} \int^t \frac{ds}{g(s)} + \frac{\mu}{\nu} \int^t \frac{s ds}{g(s)} \right)$$

Another finite-dimensional Lie algebra

Another finite-dimensional Lie algebra which may be used to construct tomograms, exploring other features of the signals, is generated by $\mathbf{1}$, t and

$$\begin{aligned} \omega &= i\tfrac{d}{dt} \\ D &= i\left(t\tfrac{d}{dt} + \tfrac{1}{2}\right) \\ F &= -\tfrac{1}{2}\left(\tfrac{d^2}{dt^2} - t^2 + 1\right) \\ \sigma &= \tfrac{1}{2}\left(\tfrac{d^2}{dt^2} + t^2 + 1\right) \end{aligned}$$

Of special interest are the tomograms related to the operators

$$B_F = \mu t + \nu F$$

and

$$B_\sigma = \mu t + \nu \sigma$$

As before, the construction of the tomograms relies on finding a complete set of generalized eigenvectors for the operators B_F and B_σ. With $y = t + \frac{\mu}{\nu}$ one defines creation and annihilation operators

Another finite-dimensional Lie algebra

$$a = \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right); \qquad a^{\dagger} = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$$

obtaining

$$B_F = \nu\left(a^{\dagger}a - \frac{\mu^2}{2\nu^2}\right)$$

$$B_\sigma = \nu\left(aa - \frac{\mu^2}{2\nu^2}\right)$$

Therefore for B_F one has an orthonormalized complete set of eigenvectors

$$\psi_n^{(F)}(t) = u_n\left(t + \frac{\mu}{\nu}\right)$$

with a discrete set of eigenvalues $X_n = \nu\left(n + \frac{1}{2}\right) - \frac{\mu^2}{2\nu}$

$$B_F \psi_n^{(F)}(t) = X_n \psi_n^{(F)}(t)$$

Another finite-dimensional Lie algebra

the function u_n is

$$u_n(y) = \left(\pi^{1/2} 2^n n!\right)^{-1/2} \left(y - \frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}$$

and the tomogram $M_f^{(F)}(\mu, \nu, X_n)$

$$M_f^{(F)}(\mu, \nu, X_n) = \left| \int \psi_n^{(F)*}(t) f(t) \, dt \right|^2$$

For B_σ one uses a basis of coherent states

$$\begin{aligned} \phi_\lambda(y) &= e^{\lambda a^\dagger - \lambda^* a} u_0(y) \\ &= e^{\frac{|\lambda|^2}{2}} \sum_{n=0} \frac{\lambda^n}{\sqrt{n!}} u_n(y) \end{aligned}$$

with decomposition of identity

$$\frac{1}{\pi} \int \phi_\lambda(y) \phi_\lambda^*(y) \, d^2\lambda = 1$$

Another finite-dimensional Lie algebra

Then, a set of generalized eigenstates of B_σ is

$$\psi_\lambda^{(\sigma)}(\mu,\nu,t) = \phi_\lambda\left(t+\frac{\mu}{\nu}\right)$$

with eigenvalues

$$B_\sigma\psi_\lambda^{(\sigma)}(\mu,\nu,t) = X_\lambda\psi_\lambda^{(\sigma)}(\mu,\nu,t)$$

$$X_\lambda = \nu\left(\lambda^2 - \frac{\mu^2}{2\nu^2}\right)$$

the tomogram being

$$M_f^{(\sigma)}(\mu,\nu,X_\lambda) = \left|\int \psi_\lambda^{(\sigma)*}(\mu,\nu,t)\, f(t)\, dt\right|^2$$

This tomogram is closely related to the Sudarshan-Glauber P-representation.

Multidimensional tomograms

Several types of multidimensional tomograms may be obtained from generalizations of the one-dimensional ones. Consider a signal $f(t_1, t_2)$. The tomogram will depend on a vector variable $\vec{X} = (X_1, X_2)$ and four real parameters μ_1, μ_2,ν_1, and ν_2. For example, the two-dimensional time-frequency tomogram will be

$$M(\vec{X}, \vec{\mu}, \vec{\nu}) = \frac{1}{4\pi^2 |\nu_1 \nu_2|} \left| \int f(t_1, t_2) e^{\left(\frac{i\mu_1}{2\nu_1} t_1^2 - \frac{iX_1}{\nu_1} t_1 + \frac{i\mu_2}{2\nu_2} t_2^2 - \frac{iX_2}{\nu_2} t_2 \right)} dt_1 \, dt_2 \right|^2$$

From this one may also construct a *center of mass tomogram*

$$M_{\mathrm{cm}}(Y, \vec{\mu}, \vec{\nu}) = \int M(\vec{X}, \vec{\mu}, \vec{\nu}) \, \delta(Y - X_1 - X_2) \, dX_1 \, dX_2$$

$$= \int \delta(Y - X_1 - X_2) \frac{1}{2\pi |\nu_1|} \frac{1}{2\pi |\nu_2|}$$

$$\left| \int f(t_1, t_2) dt_1 \, dt_2 \, \exp \left(\frac{i\mu_1}{2\nu_1} t_1^2 - \frac{iz_1 X_1}{\nu_1} + \frac{i\mu_2}{2\nu_2} t_2^2 - \frac{iz_2 X_2}{\nu_2} \right) \right|^2 dX_1 \, dX_2$$

Multidimensional tomograms

The center of mass tomogram is normalized

$$\int M_{\mathrm{cm}}(X, \vec{\mu}, \vec{\nu})\, dX = 1$$

and a homogeneous function

$$M_{\mathrm{cm}}(\lambda X, \lambda\vec{\mu}, \lambda\vec{\nu}) = \frac{1}{|\lambda|} M_{\mathrm{cm}}(X, \vec{\mu}, \vec{\nu}).$$

The generalization to N channels is straightforward.
As in the one-dimensional case, useful tomograms may be constructed from the operators of Lie algebras. For example, using the generators of the conformal algebra in $\mathbb{R}^d$, $d \geq 2$,

$$\begin{array}{l} \omega_k = i \frac{\partial}{\partial t_k} \\ D = i \left(t \bullet \nabla + \frac{d}{2} \right) \\ R_{j,k} = i \left(t_j \frac{\partial}{\partial t_k} - t_k \frac{\partial}{\partial t_j} \right) \\ K_j = i \left(t_j^2 \frac{\partial}{\partial t_j} + t_j \right) \end{array}$$

Multidimensional tomograms

Let, in two dimensions, $t_1 = t$ and $t_2 = x$. The tomograms corresponding to the operators

$$\begin{aligned} B_\omega &= \mu_1 t + \mu_2 x + \nu_1 \omega_1 + \nu_2 \omega_2 \\ B_D &= \mu_1 t + \mu_2 x + \nu D \\ B_\omega &= \mu_1 t + \mu_2 x + \nu_1 K_1 + \nu_2 K_2 \end{aligned}$$

are straightforward generalizations of the corresponding one-dimensional ones.

For the operator

$$B_R = \mu_1 t + \mu_2 x + \nu R_{1,2}$$

the eigenstates and the tomogram are:

$$\psi^{(R)}\left(\vec{\mu}, \nu, x, t, X\right) = \exp \frac{i}{\nu}\left(\mu_1 x - \mu_2 t + X \tan^{-1} \frac{t}{x}\right)$$

$$M_f\left(\vec{\mu}, \nu, X\right) = \left| \int \psi^{(R)*}\left(\vec{\mu}, \nu, x, t, X\right) f\left(x, t\right) dxdt \right|^2$$

Tomograms as operator symbols

Tomograms may be described not only as amplitudes of projections on a complete basis of eigenvectors of a family of operators, but also as operator symbols. That is, as a map of operators to a space of functions where the operators non-commutativity is replaced by a modification of the usual product to a star-product.

Let $\hat{A}$ be an operator in Hilbert space $\mathcal{H}$ and $\hat{U}(\vec{x})$, $\hat{D}(\vec{x})$ two families of operators called *dequantizers* and *quantizers*, respectively, such that

$$\mathrm{Tr}\left\{ \hat{U}(\vec{x})\hat{D}(\vec{x}')\right\} = \delta(\vec{x}-\vec{x}') \tag{1}$$

The labels $\vec{x}$ (with components $x_1, x_2, \ldots x_n$) are coordinates in a linear space V where the functions (operator symbols) are defined. Some of the coordinates may take discrete values, then the delta function in (1) should be understood as a Kronecker delta. Provided the property (1) is satisfied, one defines the *symbol of the operator* $\hat{A}$ by the formula

$$f_A(\vec{x}) = \mathrm{Tr}\left\{\hat{U}(\vec{x})\hat{A}\right\}, \tag{2}$$

assuming the trace to exist.

Tomograms as operator symbols

In view of (1), one has the reconstruction formula

$$\hat{A} = \int f_A(x)\hat{D}(\vec{x})\,d\vec{x}$$

The role of quantizers and dequantizers may be exchanged. Then

$$f_A^d(\vec{x}) = \mathrm{Tr}\left\{\hat{D}(\vec{x})\,\hat{A}\right\}$$

is called the dual symbol of $f_A(\vec{x})$ and the reconstruction formula is

$$\hat{A} = \int f_A^d(x)\hat{U}(\vec{x})\,d\vec{x}$$

Symbols of operators can be multiplied using the star-product kernel as follows

$$f_A(\vec{x}) \star f_B(\vec{x}) = \int f_A(\vec{y})f_B(\vec{z})K(\vec{y},\vec{z},\vec{x})\,d\vec{y}\,d\vec{z}$$

the kernel being

$$K(\vec{y},\vec{z},\vec{x}) = \mathrm{Tr}\left\{\hat{D}(\vec{y})\hat{D}(\vec{z})\hat{U}(\vec{x})\right\}$$

Tomograms as operator symbols

The star-product is associative,

$$(f_A(\vec{x}) \star f_B(\vec{x})) \star f_C(\vec{x}) = f_A(\vec{x}) \star (f_B(\vec{x}) \star f_C(\vec{x}))$$

this property corresponding to the associativity of the product of operators in Hilbert space.
With the dual symbols the trace of an operator may be written in integral form

$$\mathrm{Tr}\left\{ \hat{A}\hat{B} \right\} = \int f_A^d(\vec{x}) f_B(\vec{x})\, d\vec{x} = \int f_B^d(\vec{x}) f_A(\vec{x})\, d\vec{x}.$$

For two different symbols $f_A(\vec{x})$ and $f_A(\vec{y})$ corresponding, respectively, to the pairs $(\hat{U}(\vec{x}), \hat{D}(\vec{x}))$ and $(\hat{U}_1(\vec{y}), \hat{D}_1(\vec{y}))$, one has the relation

$$f_A(\vec{x}) = \int f_A(\vec{y}) K(\vec{x}, \vec{y})\, d\vec{y},$$

with intertwining kernel

$$K(\vec{x}, \vec{y}) = \mathrm{Tr}\left\{ \hat{D}_1(\vec{y}) \hat{U}(\vec{x}) \right\}$$

Tomograms as operator symbols

Let now each signal $f\left(t\right)$ be identified with the projection operator Π_f on the function $f\left(t\right)$, denoted by

$$\Pi_f = \left|f\right\rangle \left\langle f\right| \tag{3}$$

Then the tomograms and also other transforms are symbols of the projection operators for several choices of quantizers and dequantizers. Some examples:

The *Wigner-Ville function*: is the symbol of $\mid f\rangle\langle f \mid$ corresponding to the dequantizer

$$\hat{U}(\vec{x}) = 2\hat{\mathcal{D}}(2\alpha)\hat{P}, \qquad \alpha = \frac{t+i\omega}{\sqrt{2}}$$

$\hat{P}$ is the inversion operator $\hat{P}f(t) = f(-t)$
and $\hat{\mathcal{D}}(\gamma)$ is a "displacement" operator

$$\hat{\mathcal{D}}(\gamma) = \exp\left[\frac{1}{\sqrt{2}}\gamma\left(t - \frac{\partial}{\partial t}\right) - \frac{1}{\sqrt{2}}\gamma^*\left(t + \frac{\partial}{\partial t}\right)\right]$$

Tomograms as operator symbols

The quantizer operator is

$$\hat{D}(\vec{x}) := \hat{D}(t,\omega) = \frac{1}{2\pi}\hat{U}(t,\omega),$$

t and ω being time and frequency.
The Wigner–Ville function is

$$W(t,\omega) = 2\mathrm{Tr}\left\{\mid f\rangle\langle f \mid \hat{D}(2\alpha)\hat{D}\right\}$$

or, in integral form

$$W(t,\omega) = 2\int f^*(t)\hat{\mathcal{D}}(2\alpha)f(-t)\,dt$$

The *symplectic tomogram* or time-frequency tomogram of $\mid f\rangle\langle f \mid$ corresponds to the dequantizer

$$\hat{U}(\vec{x}) := \hat{U}(X,\mu,\nu) = \delta\left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega}\right),$$

$$\hat{t}f(t) = tf(t), \qquad \hat{\omega}f(t) = -i\frac{\partial}{\partial t}f(t)$$

and $X,\mu,\nu \in R$.

Tomograms as operator symbols

The quantizer of the symplectic tomogram is

$$\hat{D}(\vec{x}) := \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp\left[i\left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega}\right)\right]$$

The *optical tomogram* is the same as above for the case

$$\mu = \cos\theta, \qquad \nu = \sin\theta.$$

Thus the optical tomogram is

$$\begin{aligned} M(X,\theta) &= \mathrm{Tr}\left\{\mid f\rangle\langle f \mid \delta\left(X\hat{1} - \mu\hat{t} - \nu\hat{\omega}\right)\right\} \\ &= \frac{1}{2\pi}\int f^*(t)e^{ikX}\exp\left[ik\left(X - t\cos\theta + i\frac{\partial}{\partial t}\sin\theta\right)\right] f(t)\,dt\,dk \\ &= \frac{1}{2\pi|\sin\theta|}\left|\int f(t)\exp\left[i\left(\frac{\cot\theta}{2}t^2 - \frac{Xt}{\sin\theta}\right)\right]dt\right|^2. \end{aligned}$$

Tomograms as operator symbols

One important feature of the formulation of tomograms as operator symbols is that one may work with deterministic signals $f(t)$ as easily as with probabilistic ones. In this latter case the projector would be replaced by

$$\Pi_p = \int p_\mu \left| f_\mu \right\rangle \left\langle f_\mu \right| d\mu$$

with $\int p_\mu d\mu = 1$, the tomogram being the symbol of this new operator. This also provides a framework for an algebraic formulation of signal processing more general than was done in the past. There, a signal model is a triple $(\mathcal{A}, \mathcal{M}, \Phi)$ $\mathcal{A}$ being an algebra of linear filters, $\mathcal{M}$ a $\mathcal{A}$-module and Φ a map from the vector space of signals to the module. With the operator symbol interpretation both (deterministic or random) signals and (linear or nonlinear) transformations on signals are operators. By the application of the dequantizer they are mapped onto functions, the filter operations becoming star-products.

Rotated-time tomography

- Now consider a version of tomography where a discrete random variable is used as an argument of the probability distribution function. We call this tomography *rotated time tomography*.
- It is a variant of the spin-tomographic approach for the description of discrete spin states in quantum mechanics.
- For a finite duration signal $f(t)$, with $0 \leq t \leq T$,we consider discrete values of time $f(t_m) \equiv f_m$, where with the labeling $m = -j, -j+1, -j+2, \ldots, 0, 1, \ldots, j-1, j$ they are like the components of a spinor $\mid f\rangle$. This means that we split the interval $[0, T]$ onto N parts at time values $t_{-j}, t_{-j+1}, \ldots, t_j$ and replace the signal $f(t)$, a function of continuous time, by a discrete set of values organized as a spinor. By dividing by a factor we normalize the spinor, i.e.,

$$\langle f \mid f\rangle = \sum_{m=-j}^{j} |f_m|^2 = 1$$

Rotated-time tomography

Without loss of generality, consider the "spin" values to be integers, i.e., $j = 0, 1, 2, \ldots$ and use an odd number $N = 2j + 1$ of values.
In this setting, $\mid f\rangle$ being a column vector, we construct the $N \times N$ matrix

$$\rho = \mid f\rangle\langle f \mid$$

with matrix elements

$$\rho_{mm'} = f_m f^*_{m'}.$$

The tomogram is defined as the probability-distribution function

$$\mathcal{M}(m, u) = |\langle m \mid u \mid f\rangle|^2, \qquad m = -j, \ldots, j-1, j$$

where u is the unitary $N \times N$ matrix

$$uu^{\dagger} = 1_N$$

For this matrix we use an unitary irreducible representation of the rotation group (or $SU(2)$)

Rotated-time tomography

with matrix elements

$$
\begin{aligned}
u_{mm'}(\theta) &= \frac{(-1)^{j-m'}}{(m+m')!}\left[\frac{(j+m)!(j+m')!}{(j-m)!(j-m')!}\right]^{1/2}\left(\sin\frac{\theta}{2}\right)^{m-m'}\left(\cos\frac{\theta}{2}\right)^{m+m} \\
&\quad \times\mathcal{F}_{j-m}\left(2m+1, m+m'^2\frac{\theta}{2}\right)
\end{aligned}
$$

$\mathcal{F}_{j-m}$ being a function with Jacobi polynomial structure expressed in terms of hypergeometric function as

$$
\begin{aligned}
\mathcal{F}_n(a,b,t) &= F(-n, a+n, b; t) \\
&= \frac{(b-1)!}{(b+n-1)!}\, t^{1-b}(1-t)^{b-a}\left(\frac{d}{dt}\right)^n\left[t^{b+n-1}(1-t)^{a-b+1}\right]
\end{aligned}
$$

Rotated-time tomography

The dequantizer in the rotated-time tomography is

$$\hat{U}(\vec{x}) \equiv U(m,\vec{n}) = \delta(m1 - u^{\dagger} J_z u) = \delta\left(m1 - \vec{n}\vec{J}\right)$$

where J_z is the matrix with diagonal matrix elements

$$(J_z)_{mm'} = m\delta_{mm'}$$

The vector $\vec{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ determines a direction in 3D space. The matrix was written for $\varphi = 0$ but, if this angle is nonzero, the matrix element has to be multiplied by the phase factor $e^{im\varphi}$.
The quantizer can take several forms. In integral form, it reads

$$\hat{D}(m,\vec{n}) = \frac{2j+1}{\pi}\int_0^{2\pi} \sin^2\frac{\gamma}{2}\exp(-i\vec{J}\vec{n})\gamma\, d\gamma(\cdots)$$

Rotated-time tomography

The tomogram $\mathcal{M}(m,u)$ is a nonnegative normalized probability distribution depending on the direction $\vec{n}$, i.e., $\mathcal{M}(m,u) \geq 0$ and

$$\sum_{m=-j}^{j} \mathcal{M}(m,u) = 1$$

To compute the tomogram for a given direction with angles $\varphi = 0$ and θ, one has to estimate

$$\mathcal{M}(m,\theta) = \sum_{m'',m'=-j}^{j} u^{*}_{mm'}(\theta) f_m f^{*}_{m''} u_{m''m}(\theta)$$

where the matrix $u_{m''m}(\theta)$ given above. The following form for the matrix $u_{m'm}(\theta)$ is more convenient for numerical calculations:

$$u_{m'm}(\theta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2} \left(\cos\frac{\theta}{2}\right)^{m'+m} \left(\sin\frac{\theta}{2}\right)^{m'-m} P^{m'-m,m'+m}_{j-m'}$$

where $P^{a,b}_{n}$ are Jacobi polynomials.

Rotated-time tomography

In principle, one could use not only this unitary matrix but arbitrary unitary matrices. They contain a larger number of parameters (equal to $N^2 - 1$) and can provide additional information on the signal structure. How the time-rotated tomogram explores the time-frequency plane is, as before, illustrated by spectrograms of the eigenstates (see figures). For $m = 0$, $u_{m'm}(\theta)$ reduces to the set of normalized associated Legendre functions $L_j^{m'}$:

$$u_{m',0}(\theta) = \sqrt{\frac{2}{2j+1}} L_j^{m'}(\cos(\theta)).$$

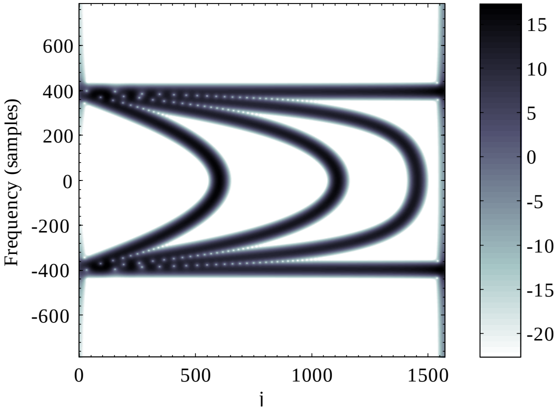
The normalized associated Legendre functions are related to the unmormalized ones $P_j^{m'}$ through:

$$L_j^{m'}(\cos(\theta)) = \sqrt{\frac{2j+1}{2} \frac{(j-m')}{(j+m')}} P_j^{m'}(\cos\theta).$$

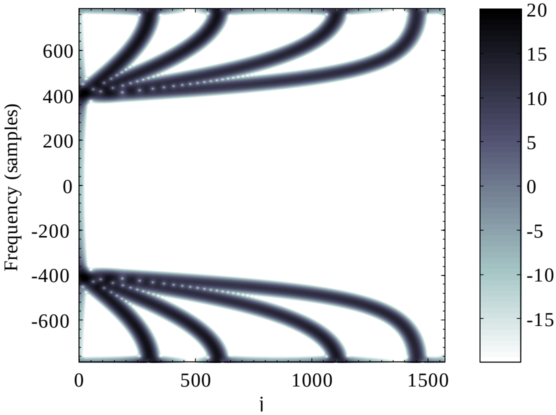
Rotated-time tomography

In the tomogram, θ is the parameter labelling the vectors of the basis associated to $m = 0, m'$. The index j is the variable. In order to illustrate the effect of this tomogram, we computed numerically some vectors in the time-frequency plane. In the discrete setting, If we choose $m' = N$, where N is the number of points, the $\{L_j^N\}_j$ form an orthonormal basis of the discrete time-frequency plane. Hence the projection on the eigenvectors of the rotated tomogram with $m = 0, m' = N$ can be seen as the projection on the bended lines in the time-frequency plane. This tomogram should be adapted for the study of functions which possess certain symetry in the time-frequency plane.

Rotated-time tomography



Rotated-time tomography



Hermite-basis tomography

Here we consider a dequantizer

$$\hat{U}(n,\alpha) = \hat{\mathcal{D}}(\alpha) \mid n\rangle\langle n \mid \hat{\mathcal{D}}^{\dagger}(\alpha), \qquad \alpha = |\alpha| e^{i\theta_\alpha}$$

and a quantizer

$$\hat{D}(n,\alpha) = \frac{4}{\pi(1-\lambda^2)} \left(\frac{\lambda+1}{\lambda-1}\right)^n \hat{\mathcal{D}}(\alpha) \left(\frac{\lambda-1}{\lambda+1}\right)^n \hat{\mathcal{D}}(-\alpha)$$

where $-1 < \lambda < 1$ is an arbitrary parameter and n is related to the order of an Hermite polynomial. This is analogous to the use of a photon number basis in quantum optics.
For any signal $f(t)$, one has the probability distribution (tomogram)

$$\mathcal{M}_f(n,\alpha) = \mathrm{Tr} \mid f\rangle\langle f \mid \hat{U}(n,\alpha)$$

and, from the tomogram, the signal is reconstructed by

$$\mid f\rangle\langle f \mid = \sum_{n=0}^{\infty} \int d^2\alpha \mathcal{M}(n,\alpha) \hat{D}(n,\lambda)$$

Hermite-basis tomography

One has $\mathcal{M}(n,\alpha) \geq 0$ and

$$\sum_{n=0}^{\infty} \mathcal{M}_f(n,\alpha) = 1 \tag{4}$$

for any complex α. For an arbitrary operator $\hat{A}$, one has

$$\hat{I}\hat{A} = \sum_{n=0}^{\infty} \int d^2\alpha \hat{D}(n,\alpha) \mathrm{Tr}\left(\hat{U}(n,\alpha)\hat{A}\right), \tag{5}$$

where $\hat{I}$ is the identity operator.
The explicit form of the tomogram for a signal function $f(t)$ is

$$\mathcal{M}_f(n,\lambda) = \left|\langle f \mid \hat{\mathcal{D}}(\alpha) \mid n\rangle\right|^2 = \left|\int f^*(t) f_{n,\alpha}(t)\, dt\right|^2 \tag{6}$$

where

$$f_{n,\alpha}(t) = \hat{\mathcal{D}}(\alpha)\left[\pi^{-1/4}(2^n n!)^{-1/2} e^{-t^2/2} H_n(t)\right] \tag{7}$$

$H_n(t)$ being an Hermite polynomial.

Thus, one has

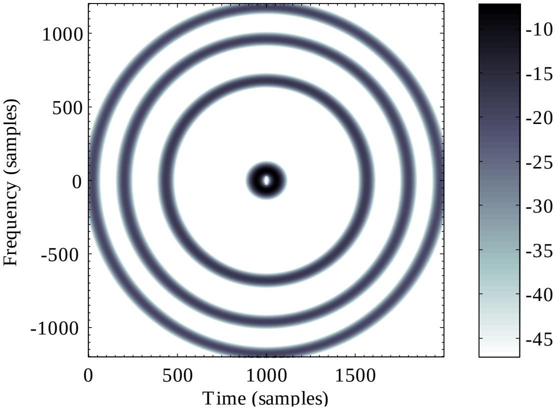
$$f_{n,\alpha}(t) = \pi^{-1/4}(2^n n!)^{-1/2} e^{-(\alpha^2-\alpha^{*2})/4} e^{[(\alpha-\alpha^*)t]/\sqrt{2}} e^{-\tilde{t}^2/2} H_n(\tilde{t})$$

and

$$\tilde{t} = t - \frac{\alpha+\alpha^*}{\sqrt{2}}\,.$$

For fixed $|\alpha|$ the tomogram is a function of the discrete set $n = 0, 1, \ldots$ and the phase factor θ_α.
How the Hermite basis tomogram explores the time-frequency plane is, as before, illustrated by spectrograms of the eigenstates. In the particular case where $\alpha = 0$, the functions $f_{n,0}$ are the Hermite functions. Their time-frequency representation has been calculated on the figure. It shows that the tomogram at $\alpha = 0$ is suited for rotation invariant functions in the time-frequency plane. One can see that: for real α this pattern is shifted in time and for purely imaginary α the pattern is shifted in frequency. The pattern can be shifted in both time and frequency by choosing the appropriate complex value for α.

Hermite-basis tomography



A new application: Turbulent velocity fluctuations

Here we report briefly on an analysis by the tomographic technique of a velocity fluctuation signal of a turbulent flow in a wind tunnel. It illustrates the fact that the choice of the pair of non-commuting operators in tomogram, should be adapted to the signal under study. As before we use finite-time tomograms in the interval $(t_0, t_0 + T)$ and a set of X_n's leading to an orthonormalized set of eigenstates.
The sets of orthonormalized eigenstates for the finite-time time-scale tomogram $M_2(\mu, \nu, X)$ and for the finite-time time-conformal tomogram $M_4(\mu, \nu, X)$ are

$$M_2(\theta, X) = \left| \int_{t_0}^{t_0+T} f^*(t) \psi_{\theta,X}^{(2)}(t)\, dt \right|^2 = \left| < f, \psi^{(2)} > \right|^2$$

$$\psi_{\theta,X}^{(2)}(t) = \frac{1}{\sqrt{\log|t_0+T| - \log|t_0|}} \frac{1}{\sqrt{|t|}} \exp i \left(\frac{\cos\theta}{\sin\theta} t - \frac{X}{\sin\theta} \log|t| \right)$$

$$X_n = X_0 + \frac{2n\pi}{\log|t_0+T| - \log|t_0|} \sin\theta \qquad n \in \mathbb{Z}$$

A new application: Turbulent velocity fluctuations

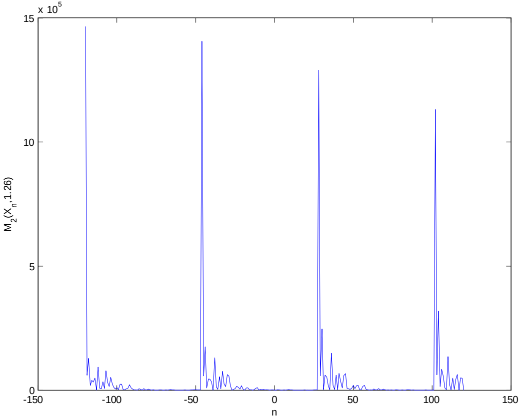
and

$$M_4(\theta, X) = \left| \int_{t_0}^{t_0+T} f^*(t)\psi^{(4)}_{\theta,X}(t)\, dt \right|^2 = \left| < f, \psi^{(4)} > \right|^2$$

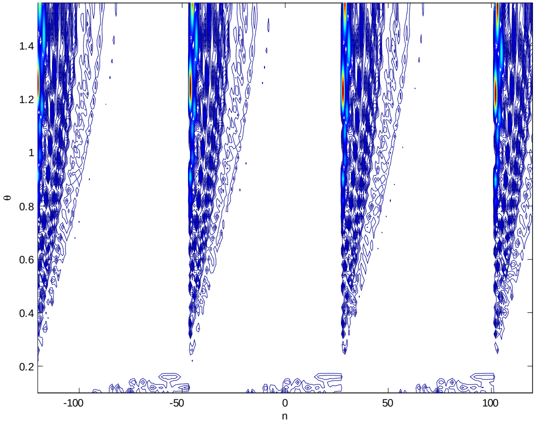
$$\psi^{(4)}_{\theta,X}(t) = \sqrt{\frac{t_0 (t_0 + T)}{T}} \frac{1}{|t|} \exp i \left(\frac{\cos\theta}{\sin\theta} \log|t| + \frac{X}{t\sin\theta} \right)$$

$$X_n = X_0 + \frac{t_0 (t_0 + T)}{T} 2\pi n \sin\theta \qquad\qquad n \in \mathbb{Z}$$

A new application: Turbulent velocity fluctuations



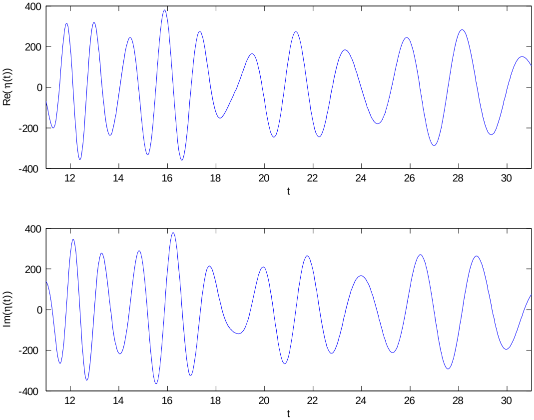
A new application: Turbulent velocity fluctuations



A new application: Turbulent velocity fluctuations

Analyzing the turbulent velocity fluctuations signal with these tomograms, one notices that except for some features on the frequency axis corresponding to some dominating frequencies, no interesting structures are put into evidence when one use the time-frequency tomogram. The situation is more interesting for the time-scale tomogram $M_2(\theta, X)$. In the figure one shows a contour plot for $M_2(\theta, X)$ corresponding to a section of 1000 data points. For intermediate regions of θ one notices, a strong concentration of energy in a few regions. This is put into evidence by a cut at $\theta = 1.26$. Projecting out the signal corresponding to these regions with the corresponding $\psi^{(2)}_{\theta,X}(t)$'s at this θ, one sees that although the signal has many complex features most of the energy is concentrated in fairly regular structures. The next figure shows the structure $\eta(t)$ corresponding to the second peak.

A new application: Turbulent velocity fluctuations



References

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