Non-commutative tomography: A tool for data analysis and signal processing

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Outline

- Integral transforms: linear and bilinear
- Wavelet-type, quasi-distributions and tomograms: Examples and relations
- Tomograms and the conformal group operators
- Aplications:

1. Detection of small signals
2. Filtering and component separation
3. Plasma reflectometry

Tomographic data analysis. General setting

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- **1 - Wavelet-type transform**

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- **3 - Tomographic transform or tomogram**

$$M_f^{(B)}(X) = \langle f \mid \delta(B(\alpha) - X) \mid f \rangle$$

Examples for wavelet-type and quasi-distributions

- **Fourier transform**: is $W_f^{(h)}(\alpha)$ if $U(\alpha)$ is unitary generated by $B_F\left(\overrightarrow{\alpha}\right) = \alpha_1 t + i\alpha_2 \frac{d}{dt}$ and h is a (generalized) eigenvector of the time-translation operator

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- **Bertrand transform:** $Q_f(\alpha)$ for B_W

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- For normalized $\mid f \rangle$,
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- The tomogram is a homogeneous function

  $$M_f^{(B/p)}(X) = |p| M_f^{(B)}(pX)$$

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- $$Q_f^{(B)}(\alpha) = W_f^{(f)}(\alpha),$$
- $$W_f^{(h)}(\alpha) = \frac{1}{4} \int e^{iX} \left[\begin{array}{c} M_{f_1}^{(B)}(X) - iM_{f_2}^{(B)}(X) \\ -M_{f_3}^{(B)}(X) + iM_{f_4}^{(B)}(X) \end{array} \right] dX,$$

with

$$\begin{array}{ll} \mid f_1\rangle = \mid h\rangle + \mid f\rangle; & \mid f_3\rangle = \mid h\rangle - \mid f\rangle; \\ \mid f_2\rangle = \mid h\rangle + i \mid f\rangle; & \mid f_4\rangle = \mid h\rangle - i \mid f\rangle. \end{array}$$

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- Quasidistribution of the Husimi–Kano type

$$H_f^{(b)}(\alpha) = \langle f \mid U(\alpha) \mid f \rangle.$$

The conformal group

- The generators of the conformal group

in $\mathbb{R}^d$

$$\begin{aligned} \omega_k &= i\frac{\partial}{\partial t_k} \\ D &= i\left(t \bullet \nabla + \frac{d}{2}\right) \\ R_{j,k} &= i\left(t_j\frac{\partial}{\partial t_k} - t_k\frac{\partial}{\partial t_j}\right) \\ K_j &= i\left(t_j^2\frac{\partial}{\partial t_j} + t_j\right) \end{aligned}$$

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in $\mathbb{R}$

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\omega &= i\frac{d}{dt} \\
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\end{aligned}$$

Tomograms associated to the conformal group

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- Time-conformal

$$B_4 = \mu t + i\nu \left(t^2 \frac{d}{dt} + t \right)$$

Tomograms associated to the conformal group

- General construction of the tomograms: Let

$$\int dY \left| Y \right\rangle \left\langle Y \right| = 1$$

be a decomposition of the unit, with generalized eigenvectors of the operator B. Then

$$M(\alpha, X) = \int dY \langle f \mid \delta \left(B \left(\alpha \right) - X \right) \left| Y \right\rangle \left\langle Y \right| \mid f \rangle = \left| \langle X \mid f \rangle \right|^2$$

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- Therefore the construction of the tomograms reduces to the calculation of the generalized eigenvectors of each B operator
- $B_1 \psi_1 \left(\mu, \nu, t, X \right) = X \psi_1 \left(\mu, \nu, t, X \right)$

$$\psi_1 \left(\mu, \nu, t, X \right) = \exp i \left(\frac{\mu t^2}{2\nu} - \frac{tX}{\nu} \right)$$

$$\int dt \psi_1^* \left(\mu, \nu, t, X \right) \psi_1 \left(\mu, \nu, t, X' \right) = 2\pi\nu\delta \left(X - X' \right)$$

Tomograms associated to the conformal group

- $B_2\psi_2(\mu,\nu,t,X) = X\psi_2(\mu,\nu,t,X)$

$$\psi_2(\mu,\nu,t,X) = \frac{1}{\sqrt{|t|}} \exp i\left(\frac{\mu t}{\nu} - \frac{X}{\nu}\log|t|\right)$$

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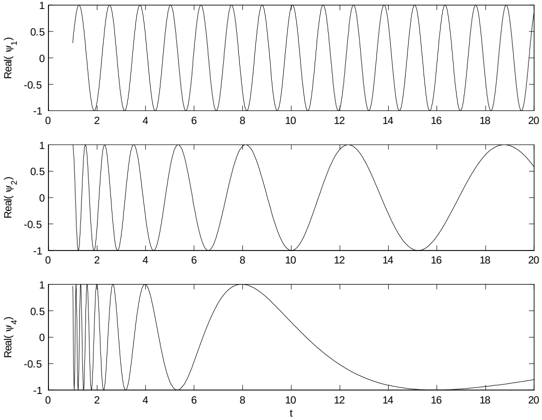
- $B_4\psi_4(\mu,\nu,t,X) = X\psi_4(\mu,\nu,t,X)$

$$\psi_4(\mu,\nu,t,X) = \frac{1}{|t|} \exp i\left(\frac{X}{\nu t} + \frac{\mu}{\nu}\log|t|\right)$$

$$\int dt\psi_4^*(\mu,\nu,t,s)\,\psi_4(\mu,\nu,t,s') = 2\pi\nu\delta(s-s')$$

Tomograms associated to the conformal group

$\mu = 0$



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$$M_1\left(\mu,\nu,X\right)=\frac{1}{2\,\pi|\nu|}\left|\int\exp\left[\frac{i\mu t^2}{2\,\nu}-\frac{itX}{\nu}\right]f\left(t\right)dt\right|^2$$

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$$M_2(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f(t)}{\sqrt{|t|}}e^{\left[i\left(\frac{\mu}{\nu}t-\frac{X}{\nu}\log|t|\right)\right]}\right|^2$$

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$f\left(\omega\right)=$ Fourier transform of $f\left(t\right)$

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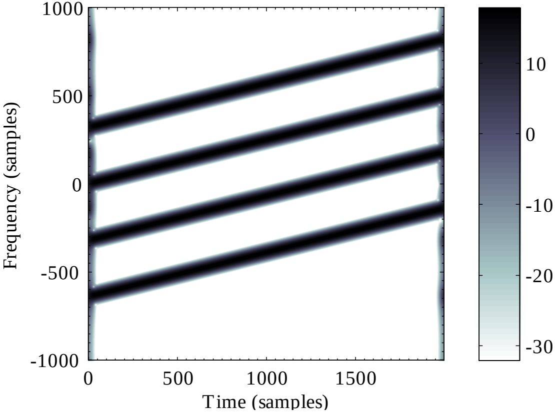
$f(\omega)$ = Fourier transform of $f(t)$

- Time-conformal tomogram

$$M_4(\mu,\nu,X)=\frac{1}{2\pi|\nu|}\left|\int dt\,\frac{f(t)}{|t|}e^{\left[i\left(\frac{X}{\nu t}+\frac{\mu}{\nu}\log|t|\right)\right]}\right|^2$$

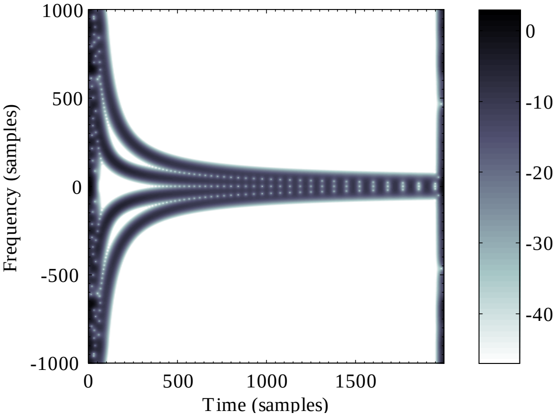
Basis functions of the tomograms in the time-frequency plane

Time-frequency



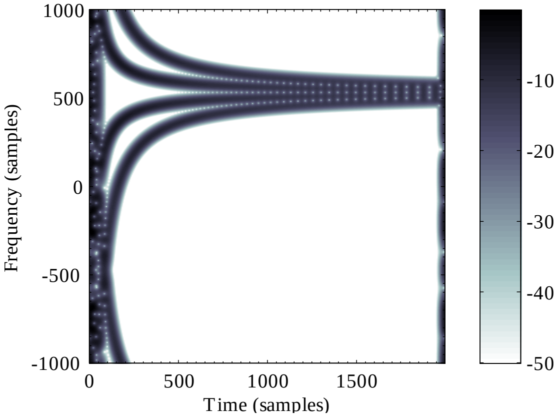
Basis functions of the tomograms in the time-frequency plane

Time-scale



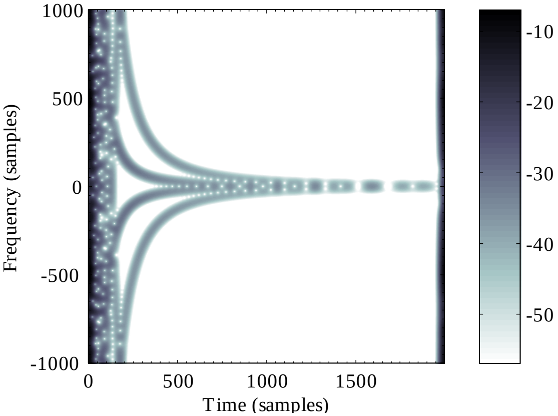
Basis functions of the tomograms in the time-frequency plane

Time-scale



Basis functions of the tomograms in the time-frequency plane

Time-conformal



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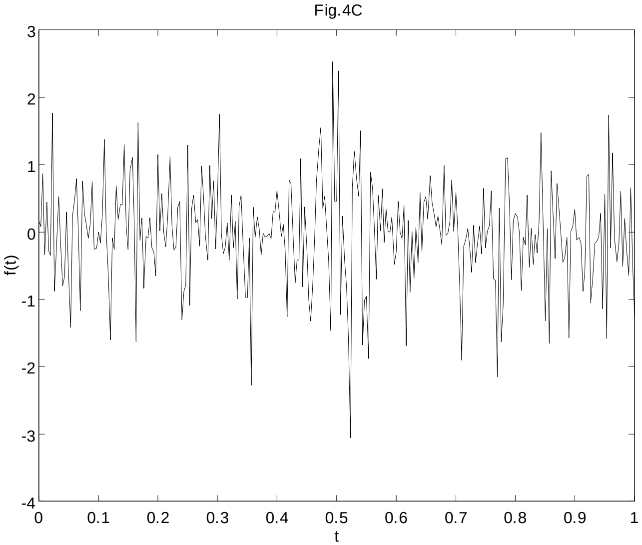
- The. following figures show the signal, its Fourier transform and the tomogram $M_f^{(S)}(s, \mu, \nu)$ ($T = 1$ and $\Omega = 1000$)

Detection of signals in noise



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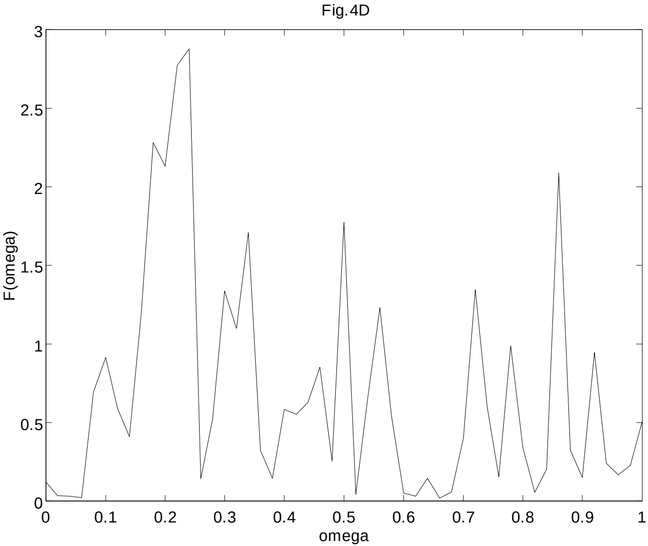

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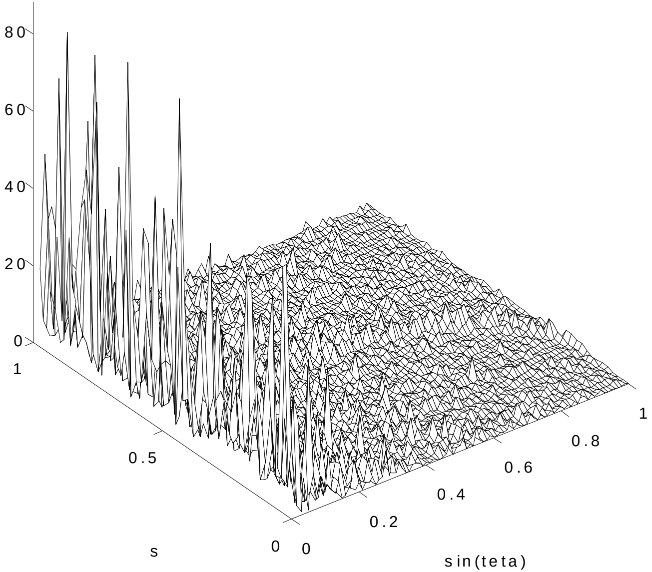
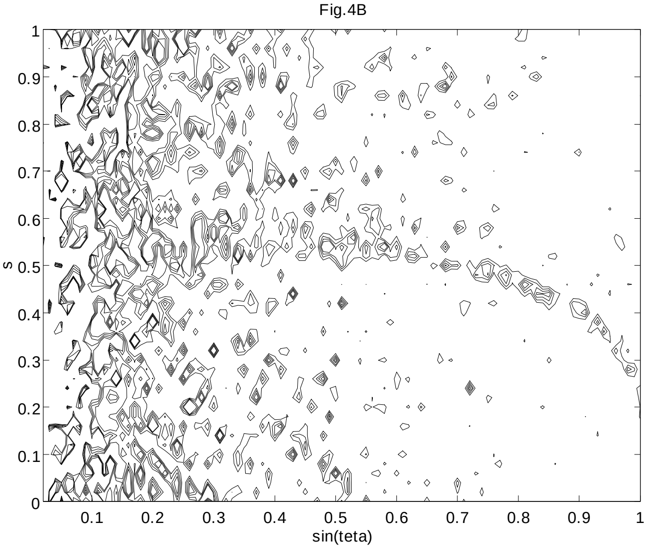


Detection of signals in noise



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- A ridge of small peaks is reliable because the rigorous probability interpretation of $M(\theta, X)$ renders the method immune to spurious effects.

Component decomposition

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  Examples: Bat echolocation, whale sounds, radar, sonar, etc.

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- One possibility: Separation of components using its behavior in the time-frequency plane. Consider the finite-time tomogram

$$M(\theta, X) = \left| \int f(t) \psi_{\theta,X}(t) \, dt \right|^2 = |< f, \psi >|^2$$

with

$$\psi_{\theta,X}(t) = \frac{1}{\sqrt{T}} \exp\left(\frac{-i\cos\theta}{2\sin\theta} t^2 + \frac{iX}{\sin\theta} t \right)$$

$\mu = \cos\theta, \nu = \sin\theta$.

Component decomposition

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- For all different θ's the $U(\theta)$ are unitarily equivalent operators, hence all the tomograms share the same information. The component separation technique is based on the search for an intermediate value of θ where a good compromise might be found between time localization and frequency information.
- First select a subset X_n in such a way that the corresponding family $\left\{\psi_{\theta,X_n}(t)\right\}$ is orthogonal and normalized,

$$< \psi_{\theta,X_n} \psi_{\theta,X_m} >= \delta_{m,n}$$

  This is possible by taking the sequence

$$X_n = X_0 + \frac{2n\pi}{T} \sin\theta$$

  where X_0 is freely chosen (in general we take $X_0 = 0$)

Component decomposition

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- Multi-component analysis is done by selecting subsets $\mathcal{F}_k$ of the X_n and reconstructing partial signals (k-components) by restricting the sum to

$$f_k(t) = \sum_{n \in \mathcal{F}_k} c_{X_n}^{\theta}(f) \psi_{\theta,X_n}(t)$$

for each k.

Component decomposition. Examples

- $$y(t) = y_1(t) + y_2(t) + y_3(t) + b(t)$$

$$\begin{aligned} y_1(t) &= \exp(i25t), t \in [0, 20] \\ y_2(t) &= \exp(i75t), t \in [0, 5] \\ y_3(t) &= \exp(i75t), t \in [10, 20] \end{aligned}$$

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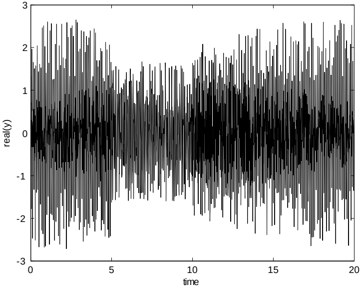
- Real part of the time signal



Component decomposition. Examples

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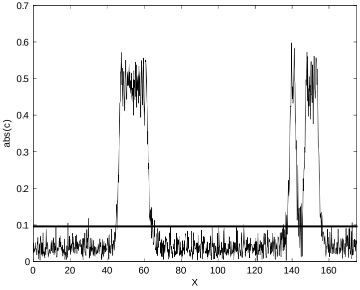
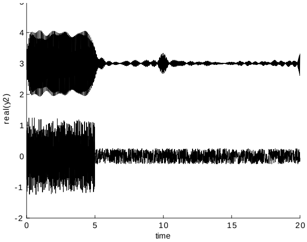
- 

Component decomposition. Examples

- Reconstrution of the $y_2\left(t\right)$



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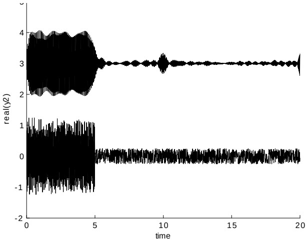
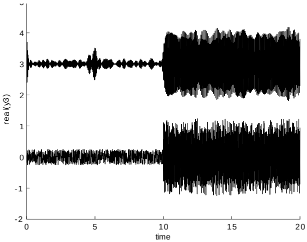


- and $y_3(t)$ components



Component decomposition. Examples

- Sum $y(t) = y_0(t) + y_R(t) + b(t)$ of an "incident" $y_0(t)$ and a "deformed reflected" chirp $y_R(t)$ delayed by $3s$ with white noise added.

$$y_0(t) = e^{i\Phi_0(t)} \qquad y_R(t) = e^{i\Phi_R(t)}$$

$\Phi_0(t) = a_0 t^2 + b_0 t$ and
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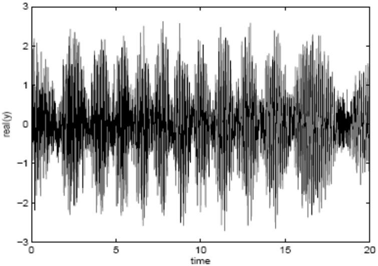

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Component decomposition. Examples

- Comparison of the phase derivatives $\frac{d}{dt}\Phi_0(t)$ and $\frac{d}{dt}\Phi_R(t)$. Except for the three first seconds, the spectrum of the signals $y_0(t)$ and $y_R(t)$ is almost the same

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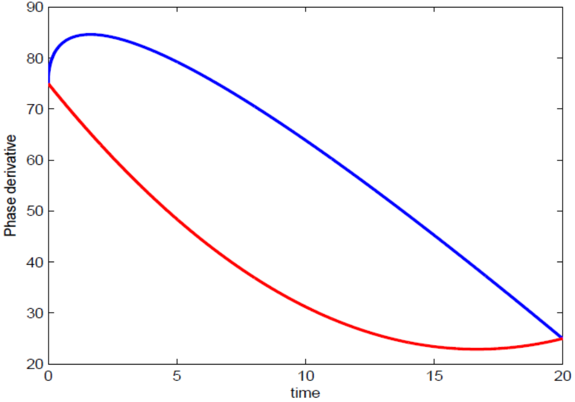

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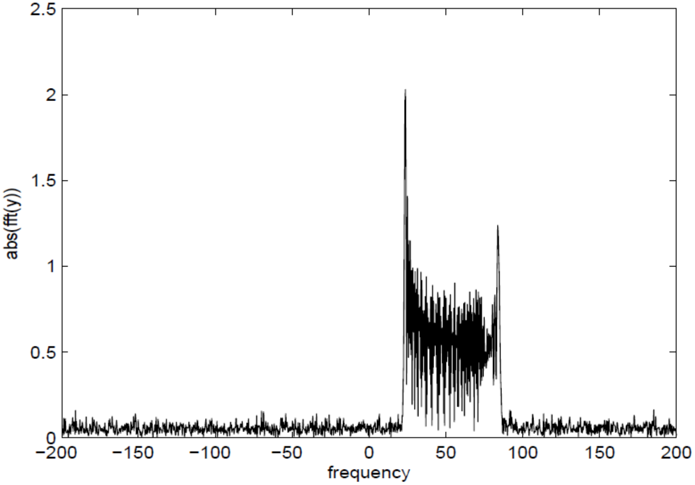
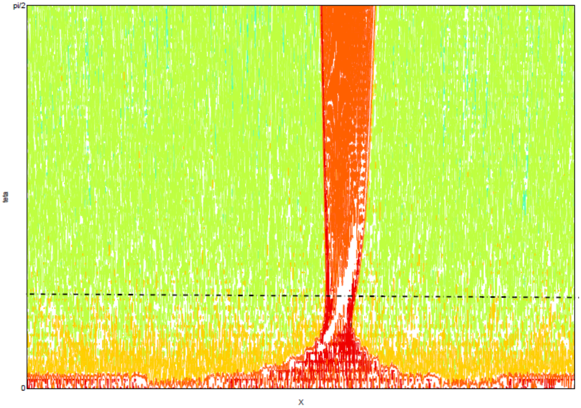
Component decomposition. Examples

- Frequency representation



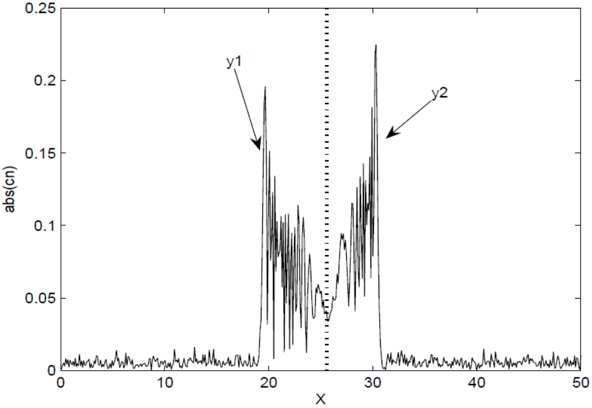
Component decomposition. Examples

- Tomogram of the chirps signal



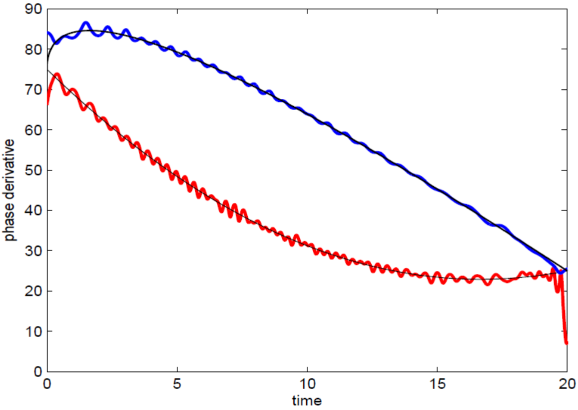
Component decomposition. Examples

Separable spectrum at $\theta = \frac{\pi}{5}$



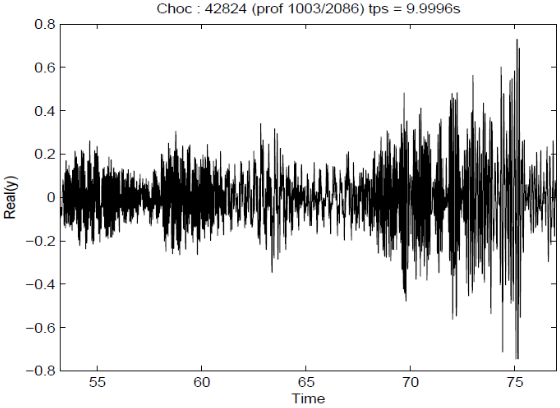
Component decomposition. Examples

The phase derivative



Component decomposition. Examples

- Reflectometry signal



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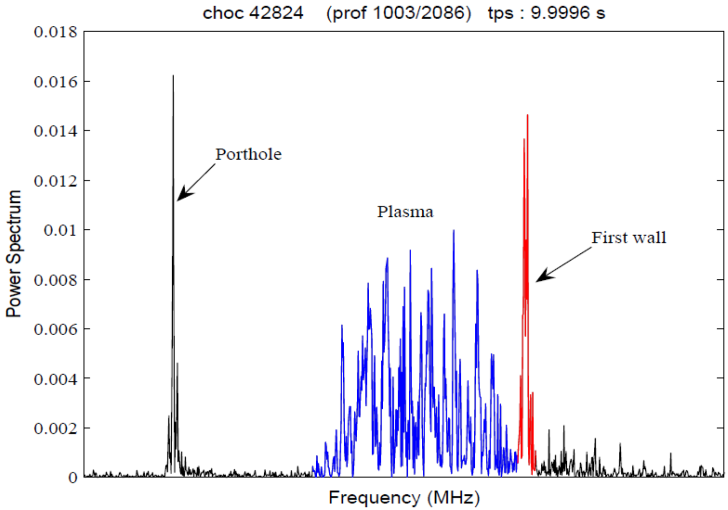
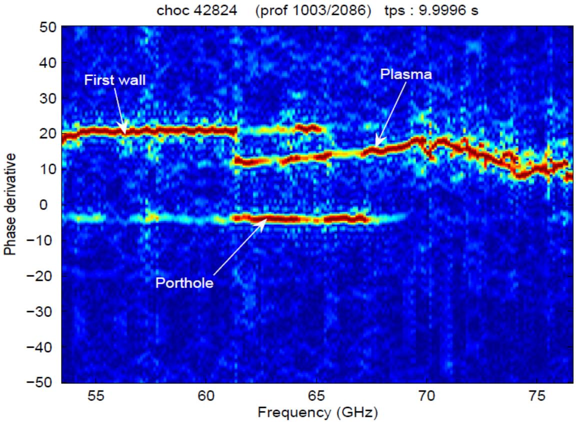
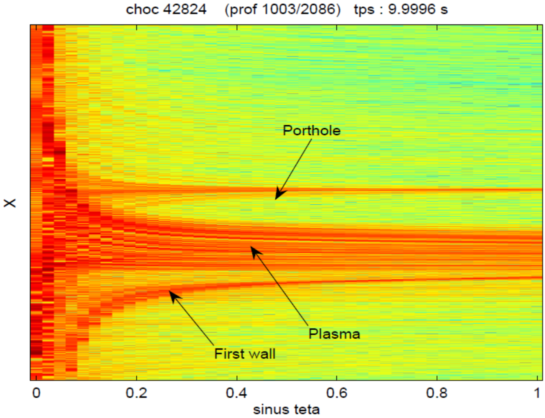


Component decomposition. Examples

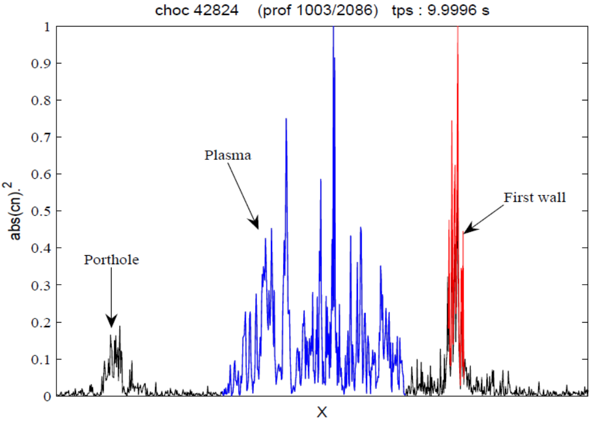


Component decomposition. Examples

- Tomogram of the reflectometry signal



- "Spectrum" at $\theta = \pi - \frac{\pi}{5}$



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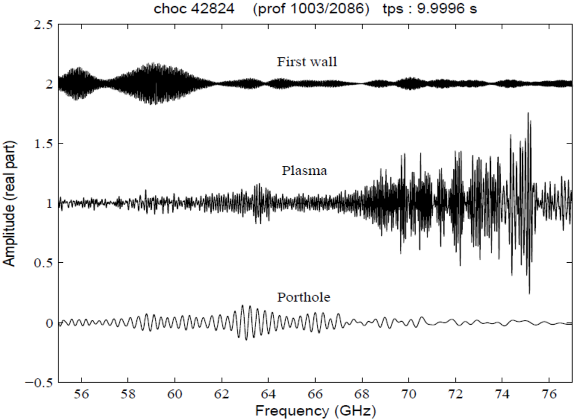

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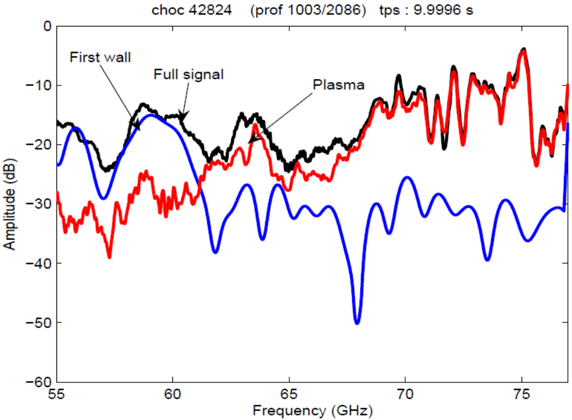

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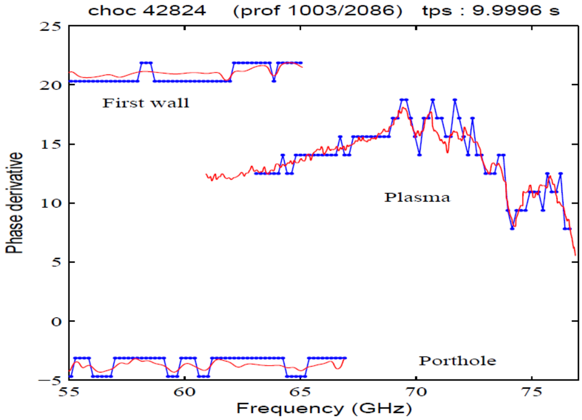

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