# Superprocesses and ultradistributions

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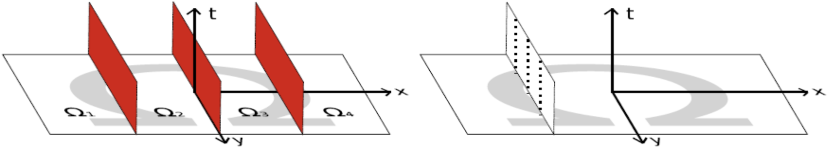
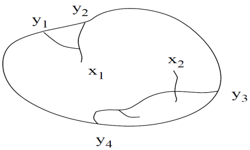
# A parallel computation problem

- Domain decomposition




- Communication time is the main problem afecting computation efficiency
- Probabilistic domain decomposition (PDD) (J. Acébron, R. Spigler)
- Needs a gridless method to find the solution at each point without information about the solution at nearby points.
- **Stochastic solution:** A stochastic process starting from a point $x$ that generates on the boundary a measure which integrated with the boundary condition constructs the solution at $x$.

# Stochastic solutions




- Well-known for linear problems. Example: the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad \text{with} \qquad u(0,x) = f(x)$$

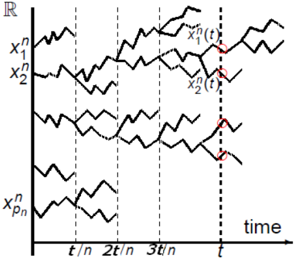
  the process is Brownian motion, $dX_t = dB_t$, and the solution

$$u(t,x) = \mathbb{E}_x f(X_t)$$

- The domain is $\mathbb{R} \times [0,t)$ and the expectation value is the inner product $\langle \mu_t, f \rangle$ of the initial condition $f$ with the *measure* $\mu_t$ *generated by the Brownian motion at the* $t-$*boundary*.
- For nonlinear problems: branching particle processes
- Two methods: MacKean and superprocesses

# Superprocesses

- Superprocesses are scaling limits of infinitely fast branching stochastic processes




  generating a measure-valued process on the boundary

- They are either models for evolving populations or tools to represent the solutions of nonlinear partial differential equations (PDE's)

# Superprocesses

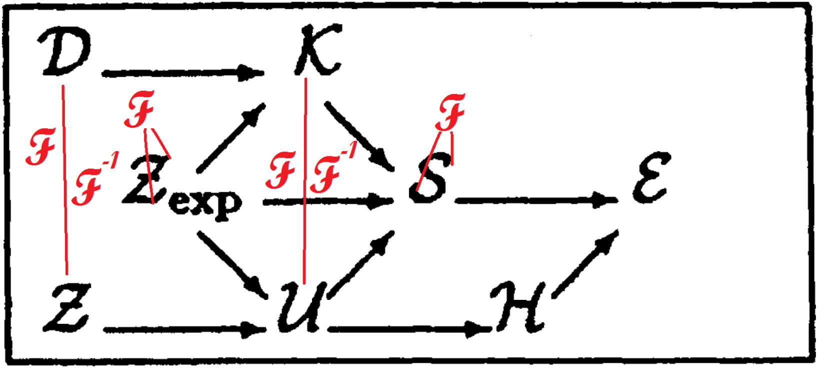
- The figure illustrates the superprocesses that represents the solutio of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^\alpha$$

  for $\alpha = 2$. Likewise superprocesses with different branching schemes may be constructed for $1 < \alpha \leq 2$.
- However, measure-valued superprocesses cannot handle $\alpha > 2$, nor interactions involving derivatives.
- **Suggestion:** Enlarge the configuration space of superprocesses. For example go from a propagating $\delta$ branching to $\delta' s$ to $\delta$ branching to arbitrary derivatives $\delta^{(n)}$ or linear combinations $\sum c_n \delta^{(n)}$. However, as branchings accumulate we might have $\sum_{n=0}^{\infty} c_n \delta^{(n)}$ which is not a distribution.

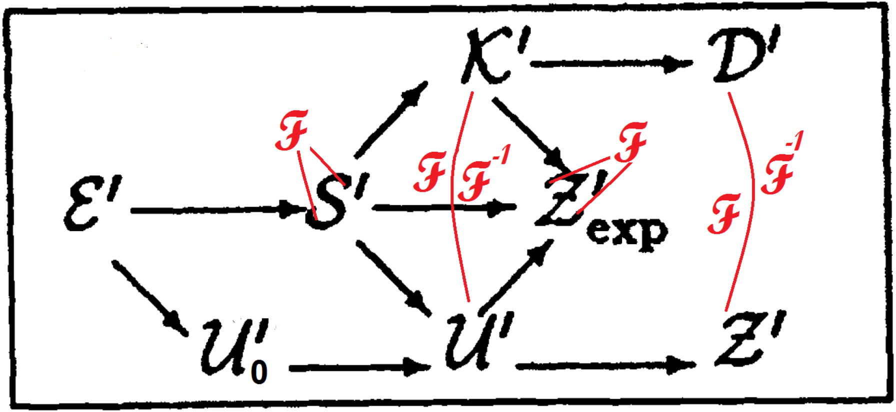
# Distributions and ultradistributions

- The theory of distributions is not just $D'$ and $S'$. There are many other interesting spaces
- Test function spaces, dense embeddings and Fourier maps



- $\mathcal{D} = \cup_K \left\{ \mathcal{D}_K : \varphi \in C^\infty, \text{supp}\,(\varphi) \subset K \right\};$
  $$\|\varphi\|_{(p,K)} = \max_{0 \le r \le p} \left\{ \sup \left| \varphi^{(r)} \right| \right\}$$
- $\mathcal{K} = \cap_{p=0}^{\infty} \mathcal{K}_p; \mathcal{K}_p =$ completion of $C^\infty$ for the norm
  $$\|\varphi\| = \max_{0 \le q \le p} \left\{ \sup \left| e^{p|x|} \varphi^{(q)} \right| \right\}$$
- $\mathcal{S} = \cap \mathcal{S}_{p,r} = \left\{ \varphi \in C^\infty : \|\varphi\|_{p,r} = \sup \left| x^p \varphi^{(r)} \right| \right\}$
- $\mathcal{E} = \varphi \in C^\infty$ with $\omega-$convergence on compacts
- $\mathcal{Z} = \varphi : \mathcal{F}\{\varphi\} \in \mathcal{D}, \varphi(z)$ entire : $\left| z^k \varphi(z) \right| \le C_k e^{a|\mathrm{Im}(z)|}$
- $\mathcal{U} = \cap_{p=0}^{\infty} \mathcal{U}_p; \mathcal{U}_p = \{ \varphi : \mathcal{F}\{\varphi\} \in \mathcal{K}_p \};$
  $$\|\varphi\|_p = \sup_{z \in \Lambda_p} \left\{ \left(1 + |z|^p\right) |\varphi(z)| \right\}$$
- $\mathcal{H} =$ Entire fns. with top. of uniform convergence on compacts of $\mathbb{C}$
- $\mathcal{Z}_{\exp} = \cap_{j=1}^{\infty} \mathcal{Z}_{\exp,j};$
  $$\mathcal{Z}_{\exp,j} = \left\{ \varphi : \|\varphi\|_{\exp,j} = \max_{k \le j} \left\{ e^{j|\mathrm{Re}(z)|} \left| \varphi^{(k)}(z) \right| \right\} \right\}$$

# Distribution spaces



# Distribution spaces

- $\mathcal{D}'$ = Schwartz distributions; locally $\mu(x) = D^k(f(x))$
- $\mathcal{K}'$ = Distributions of exponential type, $\mu(x) = D^k\left(e^{a|x|}f\right)$
- $\mathcal{S}'$ = Tempered distributions
- $\mathcal{E}'$ = Subspace of $\mathcal{D}'$ of distributions of compact support
- $\mathcal{Z}' = \mathcal{D}' \xrightarrow{\mathcal{F}} \mathcal{Z}'; \mathcal{Z}' \xrightarrow{\mathcal{F}^{-1}} \mathcal{D}'$
- $\mathcal{U}'$ = Tempered ultradistributions
- $\mathcal{U}'_0$ = Dual of $\mathcal{H}$, ultradistributions of compact support
- $\mathcal{Z}'_{\text{exp}}$ = Top. dual of $\mathcal{Z}_{\text{exp}}$, contains $\mathcal{U}'$ and $\mathcal{K}'$ as proper subspaces

# Analytic representation. Cauchy-Stieltjes transform

- $f^0(z) = S(f) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt$ if the integral converges
- For a finite order distribution $\mu = D^k f$, $\left[\mu^0\right] \in \mathcal{H}(\mathbb{C}\backslash\mathbb{R}) / \mathcal{H}(\mathbb{C})$

$$\mu_\phi^0 = D_z^k \left(\phi S\left(\frac{f}{\phi}\right)\right)$$

$$\langle \mu | \varphi \rangle = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{\infty} \left(\mu_\phi^0(x + i\varepsilon) - \mu_\phi^0(x - i\varepsilon)\right) \varphi(x)\, dx$$

- Tempered ultradistributions: $\mu \in \mathcal{U}'$, $\mu(x) = D^r\left(e^{b|x|} f(x)\right)$

  For $\forall \mu \in \mathcal{U}'$ $\exists \mu^0(z)$ defined and analytic on $\mathbb{C}\backslash\Lambda_{b'}$ for $b' > b$ of polynomial growth on $\overline{\mathbb{C}\backslash\Lambda_{b'}}$

$$\langle \mu | \varphi \rangle = \oint_{\Gamma_{b'}} \mu^0(z)\, \varphi(z)\, dz$$

- For $\mu \in \mathcal{U}_0'$ there is a unique $\mu^0(z)$ vanishing at $\infty$

# Superprocesses

- $\mathcal{U} \subset \mathcal{S}$ , functions in $\mathcal{S}$ that may be extended into the complex plane as entire functions of rapid decrease on strips.
- $\mathcal{U}'$, the dual of $\mathcal{U}$, (Silva's space of tempered ultradistributions), which can also be characterized as the space of all Fourier transforms of distributions of exponential type
- Restrict further to the space $\mathcal{U}_0'$ **of tempered ultradistributions of compact support.**

**Theorem (S. Silva)**

*If $\mu \in \mathcal{U}_0'$, $\mu$ may be expanded in a multipole series*

$$\left[\mu^0\right] = \left[\sum_{i=1}^{\infty} c_n \frac{1}{(z-a)^n}\right] = -\sum_{i=1}^{\infty} (-1)^n \frac{2\pi i}{n!} c_n \delta^{(n)}(z-a)$$

- Therefore at each branching point it is enough to know the action on $\delta^{(n)}(z)$

# Superprocesses

- $(X_t, P_{0,\nu})$ a **branching stochastic process with values** in $\mathcal{U}_0'$ and **transition probability** $P_{0,\nu}$ starting from time 0 and $\nu \in \mathcal{U}_0'$.
- The process satisfies the **branching property** if given $\nu = \nu_1 + \nu_2$

$$P_{0,\nu} = P_{0,\nu_1} * P_{0,\nu_2}$$

  that is, after the branching $\left(X_t^1, P_{0,\nu_1}\right)$ and $\left(X_t^2, P_{0,\nu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $(X_t, P_{0,\nu})$.
- For the **transition operator** $V_t$ operating on functions on $\mathcal{H}$ the branching property is

$$\left\langle V_t f, \nu_1 + \nu_2 \right\rangle = \left\langle V_t f, \nu_1 \right\rangle + \left\langle V_t f, \nu_2 \right\rangle$$

  with $e^{-\langle V_t f, \nu \rangle} \stackrel{\circ}{=} P_{0,\nu} e^{-\langle f, X_t \rangle}$

$$\left\langle V_t f, \nu \right\rangle = -\log P_{0,\nu} e^{-\langle f, X_t \rangle} \qquad f \in \mathcal{H}, \nu \in \mathcal{U}_0'$$

- In the construction of superprocesses on measures, an initial $\delta_x$ branches into other $\delta' s$ with, at most, scaling factors. The restriction to $\mathcal{U}_0'$ generalizes this interpretation with branchings to $\sum c_n \delta^{(n)}$.

- In $M = [0, \infty) \times E$ consider a set $Q \subset M$ and the associated exit process $\xi = (\xi_t, \Pi_{0,x})$ with parameter $k$ defining the lifetime. The process stars from $x \in E$ carrying along an ultradistribution in $\mathcal{U}_0'$.
- At each branching point of the $\xi_t-$process there is a transition ruled by the $P$ probability in $\mathcal{U}_0'$ leading to one or more elements in $\mathcal{U}_0'$. These $\mathcal{U}_0'$ elements are then carried along by the new paths of the $\xi_t-$process. The whole process stops at the boundary $\partial Q$, defining a exit process $(X_Q, P_{0,\nu})$ on $\mathcal{U}_0'$. If the initial $\nu$ is $\delta_x$

$$u(x) = \langle V_Q f, \delta_x \rangle = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$$

$\langle f, X_Q \rangle$ is computed on the (space-time) boundary with the exit ultradistribution generated by the process.

# Superprocesses

- **The connection to nonlinear pde's** is established by defining the whole process to be a $(\xi, \psi)$-*superprocess* if $u(x)$ satisfies the equation

$$u + G_Q \psi(u) = K_Q f \tag{1}$$

$$G_Q f(r, x) = \Pi_{0,x} \int_0^{\tau} f(s, \xi_s)\, ds \; ; \qquad K_Q f(x) = \Pi_{0,x} 1_{\tau<\infty} f(\xi_\tau)$$

$\psi(u)$ means $\psi(0, x; u(0, x))$ and $\tau$ is the first exit time from $Q$.

- **Construction of the superprocess**: Let $\varphi\left(s,x;z\right)$ be the branching function at time $s$ and point $x$. Then, with $P_{0,x}e^{-\langle f,X_Q\rangle}\doteq e^{-w(0,x)}$

$$e^{-w(0,x)}=\Pi_{0,x}\left[e^{-k\tau}e^{-f(\tau,\xi_\tau)}+\int_0^\tau dske^{-ks}\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right)\right] \quad (2)$$

  $\tau$ is the first exit time from $Q$ and $f\left(\tau,\xi_\tau\right)=\langle f,X_Q\rangle$ computed with the exit boundary ultradistribution. *For measure-valued superprocesses $\varphi\left(s,y;z\right)=c\sum_0^\infty p_n(s,y)z^n$, now it may be a more general function.*

### Lemma

$$\textit{Eq. (2)}=\Pi_{0,x}\left[e^{-f(\tau,\xi_\tau)}+k\int_0^\tau ds\left[\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right)-e^{-w(\tau-s,\xi_s)}\right]\right] \quad (3)$$

Uses $\int_0^\tau ke^{-ks}ds=1-e^{-k\tau}$ and $\Pi_{0,x}1_{s<\tau}\Pi_{s,\xi_s}=\Pi_{0,x}1_{s<\tau}$

# Superprocesses

- Eq.(1) is now obtained by a limiting process. Let in (3) replace $w\left(0,x\right)$ by $\beta w_{\beta}\left(0,x\right)$ and $f$ by $\beta f$. $\beta$ is interpreted as the mass of the particles and when $X_Q \to \beta X_Q$ then $P_{\mu} \to P_{\frac{\mu}{\beta}}$.

  $e^{-\beta w(0,x)} =$

$$\Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta \int_0^\tau ds \left[\varphi_\beta\left(s,\xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right]$$

- *Scaling limit* (first type)

$$u_\beta^{(1)} = \left(1 - e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta^{(1)} = \left(1 - e^{-\beta f}\right)/\beta$$

$$\psi_\beta^{(1)}\left(0,x;u_\beta^{(1)}\right) = \frac{k_\beta}{\beta}\left(\varphi\left(0,x;1-\beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right)$$

$$u_\beta^{(1)}(0,x) + \Pi_{0,x} \int_0^\tau ds \psi_\beta^{(1)} \left( s, \xi_s; u_\beta^{(1)} \right) = \Pi_{0,x} f_\beta^{(1)} (\tau, \xi_\tau)$$

that is

$$u_\beta^{(1)} + G_Q \psi_\beta^{(1)} \left( u_\beta^{(1)} \right) = K_Q f_\beta^{(1)}$$

When $\beta \longrightarrow 0$, $f_\beta^{(1)} \longrightarrow f$ and if $\psi_\beta$ goes to a well defined limit $\psi$ then $u_\beta$ tends to a limit $u$ solution of (1) associated to a superprocess. Also one sees from that in the $\beta \longrightarrow 0$ limit

$$u_\beta^{(1)} \longrightarrow w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$$

The superprocess corresponds to a cloud of particles for which both the mass and the lifetime tend to zero

# Superprocesses on measures

Restrict to measure-valued superprocesses, that is, in terms of paths, to $\delta' s$ propagating along the paths of the $(\xi_t, \Pi_{0,x})$ process and branching to new $\delta$ measures at each branching point.

## Theorem (Dynkin)

*For* $1 < \alpha \leq 2$,*there is a superprocess providing a solution to the equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^\alpha$$

**Proof:** Comparing with (1) one should have

$$\psi(0, x; u) = u^\alpha$$

Then, with $z = 1 - \beta u_\beta^{(1)}$ one has

$$\varphi(0, x; z) = \sum_n p_n z^n = z + \frac{\beta}{k_\beta} u_\beta^{(1)\alpha} = z + \frac{\beta}{k_\beta}\frac{(1-z)^\alpha}{\beta^\alpha}$$
$$= z + \frac{1}{k_\beta \beta^{\alpha-1}}\left(1 - \alpha z + \frac{\alpha(\alpha-1)}{2}z^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots\right)$$

Choosing $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ the terms in $z$ cancel and for $1 < \alpha \leq 2$ the coefficients of all $z$ powers are positive and may be interpreted as branching probabilities $p_n$ into new $\delta' s$

$$p_0 = \frac{1}{\alpha}; \quad p_1 = 0; \quad \cdots \quad p_n = \frac{(-1)^n}{\alpha} \left( \begin{array}{c} \alpha \\ n \end{array} \right); \qquad \sum_n p_n = 1$$

With $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ and $\beta \to 0$ the superprocess provides a solution to

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^\alpha$$

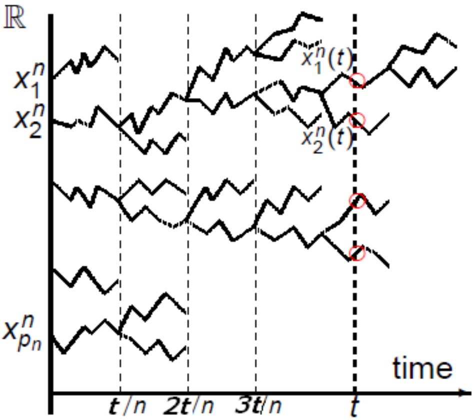
$\alpha = 2$ is an upper bound for this representation, because for $\alpha > 2$ some of the $p_n' s$ would be negative. **For the particular case**

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^2$$

$$p_1 = 0; \quad p_0 = p_2 = \frac{1}{2}; \quad k_\beta = \frac{2}{\beta}$$

# Superprocesses and a nonlinear heat equation

$\alpha = 2$



# Superprocesses on measures: other limits

Superprocesses are usually associated with nonlinear pde's in the scaling limit $\beta \to 0$. However other limits may also be useful.

**Theorem**

*With $p_n = \delta_{n,2}$, $\beta = 1$ and $k_\beta = 1$ the superprocess constructs a solution of the KPP equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2 + u$$

**Proof:**

$$\begin{aligned} \psi_\beta^{(1)}\left(0, x; u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\sum p_n \left(1 - \beta u_\beta^{(1)}\right)^n - 1 + \beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\beta^2 u_\beta^{(1)2} - \beta u_\beta^{(1)}\right) \to u^2 - u \end{aligned}$$

Because $\beta = 1$ instead of $\beta \to 0$, the solution is given by $(1 - e^{-w})$ instead of $u_\beta^{(1)} \to w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$. Interpretation as an exit measure allows for arbitrary boundary conditions.

# Superprocesses on ultradistributions

- Superprocesses on measures allows the construction of solutions for equations which do not possess a natural Poisson clock. It has the severe limitation of requiring a polynomial branching function $\varphi(s, x; z)$. Restricts the nonlinear terms in the pde's to be powers of $u$ $(u^a)$. In addition, these terms must be such that all coefficients in the $z^n$ expansion be positive $(1 < \alpha \leq 2)$.

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- The variable $z$ in $\varphi_{\beta}\left(s,x;z\right)$ is $z=e^{-\beta w\left(\tau-s,\xi_{s}\right)}=P_{0,x}e^{-\langle\beta f,X\rangle}$. When one generalizes to $\mathcal{U}_{0}^{\prime}$, changes of sign and transitions from deltas to their derivatives are allowed. There are basically two new transitions at the branching points:

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- 1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \longrightarrow e^{\langle \beta f, -\delta_x \rangle} = e^{-\beta f(x)}$$

which corresponds to

$$z \longrightarrow \frac{1}{z}$$

- 2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \longrightarrow e^{\left\langle \beta f, \pm\delta'_x \right\rangle} = e^{\mp \beta f'(x)}$$

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$$z \longrightarrow e^{\mp \partial_x \log z}$$

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- Case 1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and case 2) to superprocesses in $\mathcal{U}_0'$.
  How these transformations provide stochastic representations of solutions for other classes of pde's, will be illustrated by two results

## Theorem

*There is a superprocess that, in the $\beta \to 0$ limit, provides a solution to the equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2u^2 - \frac{1}{2}\left(\partial_x u\right)^2$$

**Proof:**

$$\varphi^{(1)}\left(0, x; z\right) = p_1 e^{\partial_x \log z} + p_2 e^{-\partial_x \log z} + p_3 z^2$$

This branching function means that at the branching point, with probability $p_1$ a derivative is added to the propagating ultradistribution, with probability $p_2$ a derivative is added plus a change of sign and with probability $p_3$ the ultradistribution branches into two identical ones. Using the transformation and scaling limit one has, for small $\beta$

$$\begin{aligned} z \quad &\rightarrow \quad e^{\mp\partial_x \log z} = e^{\mp\partial_x \log\left(1-\beta u_\beta^{(1)}\right)} \\ &= \quad 1 \pm \beta\partial_x u_\beta^{(1)} + \frac{\beta^2}{2}\left\{\left(\partial_x u_\beta^{(1)}\right)^2 \pm \partial_x u_\beta^{(1)2}\right\} + O\left(\beta^3\right) \end{aligned}$$

# Superprocesses on ultradistributions

$$z \longrightarrow z^2 = \left(1 - \beta u_\beta^{(1)}\right)^2 = 1 - 2\beta u_\beta^{(1)} + \beta^2 u_\beta^{(1)2}$$

Computing $\psi_\beta\left(0, x; u_\beta^{(1)}\right)$ with $p_1 = p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$ one obtains

$$\begin{aligned}\psi_\beta^{(1)}\left(0, x; u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\varphi^{(1)}\left(0, x; 1 - \beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\frac{1}{8}\beta^2\left(\partial_x u_\beta^{(1)}\right)^2 + \frac{1}{2}\beta^2 u_\beta^{(1)2} + O\left(\beta^3\right)\right)\end{aligned}$$

meaning that, with $k_\beta = \frac{4}{\beta}$, the superprocess provides, in the $\beta \longrightarrow 0$ limit, a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2u^2 - \frac{1}{2}\left(\partial_x u\right)^2$$

Now a different scaling limit will be used, namely

$$u_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta w_\beta} - e^{-\beta w_\beta}\right) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta f} - e^{-\beta f}\right)$$

Notice that, as before, $u_\beta^{(2)} \to w_\beta$ and $f_\beta^{(2)} \to f$ when $\beta \to 0$. In this case with $z = e^{\beta w_\beta}$ one has

$$\begin{aligned} z &= -2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 - 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{z} &= 2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 + 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned}$$

For the integral equation one has

$$u_\beta^{(2)}(0,x)+\Pi_{0,x}\int_0^\tau ds\psi_\beta^{(2)}\left(s,\xi_s;u_\beta^{(2)}\right)=\Pi_{0,x}f_\beta^{(2)}(\tau,\xi_\tau)$$

with

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right)=k_\beta\left(\frac{1}{2\beta}\left(\varphi(0,x;z)-\varphi\left(0,x;\frac{1}{z}\right)\right)-u_\beta^{(2)}\right)$$

**Theorem**

*There is a superprocess providing a solution to the equation*

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u^3$$

**Proof:** Let

$$\varphi^{(2)}(0,x;z) = p_1 z^2 + p_2 \frac{1}{z}$$

This branching function means that with probability $p_1$ the ultradistribution branches into two identical ones and with probability $p_2$ it changes its sign. Therefore, in this case, one is simply extending the superprocess construction to signed measures.

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right) = k_\beta \left\{ -p_1 8 u_\beta^{(2)} \left(1 + \frac{1}{2}\beta^2 u_\beta^{(2)2}\right) + p_2 u_\beta^{(2)} - u_\beta^{(2)} + O\left(\beta^4\right) \right\}$$

and with $p_1 = \frac{1}{10}; p_2 = \frac{9}{10}$ and $k_\beta = \frac{5}{2\beta^2}$ one obtains in the in the $\beta \longrightarrow 0$ limit

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right) \longrightarrow -u_\beta^{(2)3}$$

meaning that this superprocess provides a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^3$$

# References


- E. B. Dynkin; *Diffusions, Superdiffusions and Partial Differential Equations,* AMS Colloquium Pubs., Providence 2002.
- R. F. Hoskins and J. Sousa Pinto; *Theories of generalized functions*, 2nd edition, Horwood Pubs., Philadelphia 2005.
- RVM; *Stochastic solutions of nonlinear pde's: McKean versus superprocesses* in "Chaos Complexity and Transport", X. Leoncini and M. Leonetti (eds.), World Scientific Pub. 2012.
- RVM; *Stochastic solutions of nonlinear pde's and an extension of superprocesses* in "Stochastic and Infinite Dimensional Analysis", C. Bernido et al. (eds.), Birkhauser 2016.