Stochastic volatility: Models and questions

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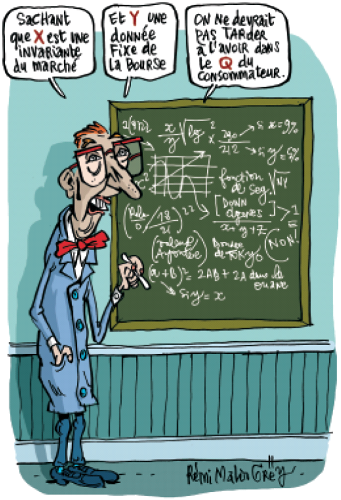
Wanted: A good mathematical model for the market

Un krach des maths



Geometric Brownian motion ?

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB\left(t\right)$$

a basis for most of mathematical finance (Black-Scholes, etc.)

Consequences:

Price changes would be log-normal

$$p\left(\ln \frac{S_T}{S_t}\right) = \frac{1}{\sqrt{2\pi\sigma^2\left(T-t\right)}} \exp\left(-\frac{\left(\ln \frac{S_T}{S_t} - \left(\mu - \frac{\sigma^2}{2}\right)\left(T-t\right)\right)^2}{2\sigma^2\left(T-t\right)}\right)$$

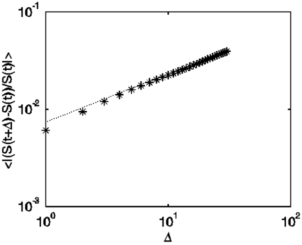
Self-similar, Law(X(at))=Law(a^H X(t)) **with Hurst coefficient** $H = 1/2$

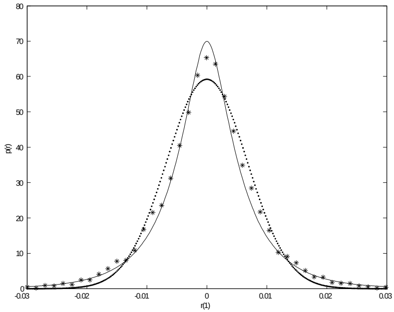
$$E\left|\frac{S\left(t+\Delta\right) - S\left(t\right)}{S\left(t\right)}\right| \approx \Delta^H$$

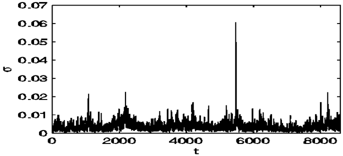
Geometric Brownian motion ?

Empirical tests:

- $p\left(\ln \frac{S_T}{S_t}\right)$ is not lognormal
- Deviations from scaling
- σ is not constant







Stylized market facts

- Returns ($r(t,\Delta) = \frac{S(t+\Delta)-S(t)}{S(t)}$) have nearly no autocorrelation
- The autocorrelations of $|r(t,\Delta)|$ decline slowly with increasing lag Δ. Long memory effect
- Leptokurtosis : asset returns have distributions with fat tails and excess peakedness at the mean
- Autocorrelations of sign $r(t,\Delta)$ are insignificant
- Volatility clustering : tendency of large changes to follow large changes and small changes to follow small changes. Volatility occurs in bursts.
- Volatility is mean-reversing and the distribution is close to lognormal or inverse gamma
- Leverage effect : volatility tends to rise more following a large price fall than following a price rise

Volatility as a function of time

- Uncertainty and risk are the driving factors for investors' behavior.
- When the future is uncertain investors are less likely to invest. Uncertainty (volatility) will be changing in time. The natural approach is to "... *build a forecasting model for variance and make it a well-defined process* ..." (Robert Engle – 1982)
- Structural models of the following type were considered:

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + u$$

$$\beta_1, \beta_2, \beta_3 \ldots \quad \textit{factors}$$

$$u \quad \textit{error term}$$

- Models for the *Conditional variance*
  Homoscedasticity = variance of errors is constant
  Heteroscedasticity = variance of errors is not constant

Volatility as a function of time

Models :

ARCH(q) (Autoregressive conditionally heteroscedastic)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \ldots + \alpha_q u_{t-q}^2$$

GARCH (1,1) (Generalized ARCH)

$$\sigma_t^2 = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta \sigma_{t-1}^2$$

IGARCH (Integrated GARCH)

$$\alpha_1 + \beta = 1$$

Leverage : GJR (Glosten, Jagannathan, Runkle)

$$\sigma_t^2 = \alpha_0 + (\alpha_1 + \gamma I_{t-1}) u_{t-1}^2 + \beta \sigma_{t-1}^2$$

$$I_{t-1} = 1 \quad if \;\; u_{t-1} < 0 \quad ; = 0 \quad otherwise$$

EGARCH (exponential GARCH)

$$\ln(\sigma_t^2) = \omega + \beta \ln(\sigma_{t-1}^2) + \gamma \frac{u_{t-1}}{\sqrt{\sigma_{t-1}^2}} + \alpha \left[\frac{|u_{t-1}|}{\sqrt{\sigma_{t-1}^2}} - \sqrt{\frac{2}{\pi}} \right]$$

Stochastic volatility models

In GARCH models, the conditional volatility is a deterministic function of past quantities. In **Stochastic Volatility Models** it is itself a random process.

- **Heston model**

$$dS_t = S_t(\mu dt + \sigma_t dB(t)) \qquad \langle dB dB' \rangle = \rho < 0$$

$$d(\sigma_t^2) = -\Omega(\sigma_t^2 - \sigma_0^2)dt + \gamma\sigma_t dB'(t)$$

- **Two-time scales model** (Perello, Masoliver)

$$dS_t = S_t(\mu dt + e^{\xi_t} dB) \qquad \langle dB dB' \rangle = \rho < 0$$

$$d\xi_t = -\Omega(\xi_t - \xi_{0t})dt + \gamma dB' \qquad \langle dB dB'' \rangle = 0$$

$$d\xi_{0t} = -\Omega_0(\xi_{0t} - \xi_{00})dt + \gamma_0 dB''$$

- **Comte** and **Renault**

$$dS_t = S_t(\mu dt + \sigma_t dB(t))$$
$$d(\ln \sigma_t) = k(\theta - \ln \sigma_t)dt + \gamma dB_H(t)$$

$dB_H(t)$ is fractional Brownian motion

- **The fractional volatility model**

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB(t) \\ \log \sigma_t &= \beta + \frac{k}{\delta}\{B_H(t) - B_H(t-\delta)\} \end{aligned}$$

The Heston model

$$\boxed{dS_t = \mu S_t \, dt + \sigma_t S_t \, dB_t^{(1)}}$$

Changing the variable from price S_t to log-return $r_t = \ln(S_t/S_0)$ and eliminating the drift by introducing $x_t = r_t - \mu t$,

$$dx_t = -\frac{v_t}{2} \, dt + \sqrt{v_t} \, dB_t^{(1)}.$$

$v_t = \sigma_t^2$ is the variance.
A mean-reverting stochastic equation for the variance v_t

$$\boxed{dv_t = -\gamma(v_t - \theta) \, dt + \kappa \sqrt{v_t} \, dB_t^{(2)}}$$

The Wiener processes may be correlated

$$\boxed{dB_t^{(2)} = \rho \, dB_t^{(1)} + \sqrt{1 - \rho^2} \, dZ_t}$$

$\rho \in [-1, 1]$ is the correlation coefficient.

The Heston model

Fokker-Planck equation for the transition probability $P_t(x, v \mid v_i)$ to have log-return x and variance v at time t given the initial log-return $x = 0$ and variance v_i at $t = 0$

$$\begin{aligned} \frac{\partial}{\partial t} P &= \gamma \frac{\partial}{\partial v} \left[(v - \theta) P \right] + \frac{1}{2} \frac{\partial}{\partial x} (vP) \\ &\quad + \rho \kappa \frac{\partial^2}{\partial x \, \partial v} (vP) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (vP) + \frac{\kappa^2}{2} \frac{\partial^2}{\partial v^2} (vP). \end{aligned}$$

An analytical solution for $P_t(x, v \mid v_i)$ was obtained (Dragulescu and Yakovenko). Then, to obtain the pdf of the returns, $P_t(x, v \mid v_i)$ was integrated over the final variance v and averaged over the stationary distribution $\Pi_*(v_i)$ of the initial variance v_i:

$$P_t(x) = \int_0^\infty dv_i \int_0^\infty dv \, P_t(x, v \mid v_i) \, \Pi_*(v_i).$$

The Heston model

Case $\rho = 0$

$$P_t(x) = \frac{e^{-x/2}}{x_0} \int_{-\infty}^{+\infty} \frac{d\tilde{p}}{2\pi}\, e^{i\tilde{p}\tilde{x} + F_{\tilde{t}}(\tilde{p})},$$

$$F_{\tilde{t}}(\tilde{p}) = \frac{\alpha \tilde{t}}{2} - \alpha \ln\left[\cosh\frac{\tilde{\Omega}\tilde{t}}{2} + \frac{\tilde{\Omega}^2 + 1}{2\tilde{\Omega}} \sinh\frac{\tilde{\Omega}\tilde{t}}{2}\right],$$

$$\tilde{\Omega} = \sqrt{1 + \tilde{p}^2}, \quad \tilde{t} = \gamma t, \quad \tilde{x} = x/x_0, \quad x_0 = \kappa/\gamma, \quad \alpha = 2\gamma\theta/\kappa^2.$$

In the long-time limit $\tilde{t} \gg 2$, $P_t(x)$ becomes a function of a single combination z of the two variable x and t (up to the trivial normalization factor N_t and unimportant factor $e^{-x/2}$):

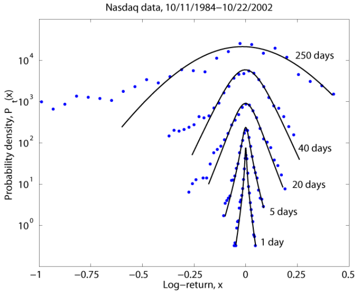
$$P_t(x) = N_t\, e^{-x/2} P_*(z), \quad P_*(z) = K_1(z)/z, \quad z = \sqrt{\tilde{x}^2 + \bar{t}^2},$$

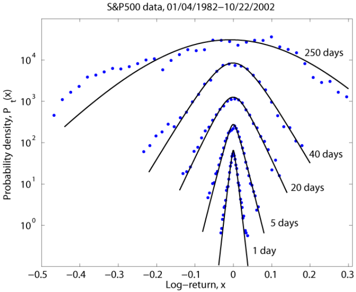
$$\bar{t} = \alpha\tilde{t}/2 = t\theta/x_0^2, \quad N_t = \bar{t} e^{\bar{t}}/\pi x_0,$$

$K_1(z)$ is the first-order modified Bessel function.

The Heston model

$$P_t(x) \propto \begin{cases} \exp\left(-|x|\sqrt{2/\theta t}\right) & \tilde{t} = \gamma t \ll 1 \\ \exp\left(-x^2/2\theta t\right) & \tilde{t} = \gamma t \gg 1 \end{cases}$$





Basic hypothesis:

- **(H1)** *The log-price process* $\log S_t$ *belongs to a probability space* $\Omega \otimes \Omega^{'}$
  The first one, Ω, is the Wiener space
  The second, $\Omega^{'}$, is a probability space to be empirically reconstructed.
  $\omega \in \Omega$, $\omega^{'} \in \Omega^{'}$ and by $\mathcal{F}_t$ and $\mathcal{F}_t^{'}$ the $\sigma-$algebras in Ω and $\Omega^{'}$
  generated by
  the processes up to t. Then,

$$\log S_t \left(\omega, \omega^{'}\right)$$

  (H1) is not limitative.
- **(H2)** The second hypothesis is stronger: *Assume that for each fixed* $\omega^{'}$, $\log S_t \left(\bullet, \omega^{'}\right)$ *is a square integrable random variable in* Ω
  From (H2) it follows that, for each fixed $\omega^{'}$,

$$\frac{dS_t}{S_t}\left(\bullet, \omega^{'}\right) = \mu_t \left(\bullet, \omega^{'}\right) dt + \sigma_t \left(\bullet, \omega^{'}\right) dB\left(t\right)$$

The fractional volatility model

If $\{X_t, \mathcal{F}_t\}$ is a process such that

$$dX_t = \mu_t dt + \sigma_t dB(t)$$

with μ_t and σ_t being $\mathcal{F}_t-$adapted processes, then

$$\mu_t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{ E(X_{t+\varepsilon} - X_t) | \mathcal{F}_t \}$$
$$\sigma_t^2 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ E(X_{t+\varepsilon} - X_t)^2 \middle| \mathcal{F}_t \right\}$$

The process associated to the probability space $\Omega^{'}$ is now inferred from the data. For each fixed $\omega^{'}$ realization in $\Omega^{'}$ one has

$$\sigma_t^2\left(\bullet, \omega^{'}\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ E\left(\log S_{t+\varepsilon} - \log S_t\right)^2 \right\}$$

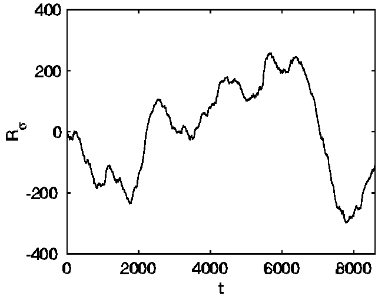
Because each set of market data corresponds to a particular realization $\omega^{'}$, the σ_t^2 process may indeed be reconstructed from the data. Is called the *induced volatility*.

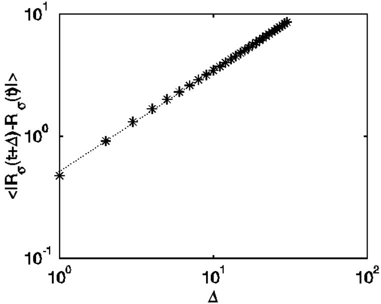
What is the mathematical characterization of the induced volatility process ?

- **Look for scaling properties**. Neither of these hold

$$E\left|\sigma(t+\Delta)-\sigma(t)\right| \sim \Delta^H \qquad E\left|\frac{\sigma(t+\Delta)-\sigma(t)}{\sigma(t)}\right| \sim \Delta^H$$

Instead, the empirical integrated log-volatility is well represented by a relation of the form $\sum_{n=0}^{t/\delta} \log\sigma(n\delta) = \beta t + R_\sigma(t)$ with the $R_\sigma(t)$ process displaying very accurate self-similar properties.





The fractional volatility model

Recall: If a nondegenerate process X_t has finite variance, stationary increments and is self-similar

$$Law\left(X_{at}\right) = Law\left(a^H X_t\right)$$

it has covariance

$$Cov\left(X_s, X_t\right) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right) E\left(X_1^2\right)$$

and the simplest process with these properties is a Gaussian process called *fractional Brownian motion*.

Hence the following *fractional volatility model* is obtained

$$\begin{array}{|lcl|} dS_t & = & \mu S_t dt + \sigma_t S_t dB(t) \\ \log \sigma_t & = & \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} \end{array}$$

δ is the observation time scale and H is in the range $0.8 - 0.9$

The volatility (at resolution δ)

$$\sigma(t) = \theta e^{\frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}}$$

The fractional volatility model. Leverage

Experimentally one finds in actual markets the following nonlinear correlation of the returns

$$L(\tau) = \left\langle \left| r(t+\tau) \right|^2 r(t) \right\rangle - \left\langle \left| r(t+\tau) \right|^2 \right\rangle \langle r(t) \rangle$$

This is called *leverage* or the *leverage effect* and it is found that for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus constantly decays to zero whereas for $\tau < 0$ it has almost negligible values.

In the definition of the fractional volatility model the σ_t acts on the log-price, but is not affected by it.

A modification with leverage: Use

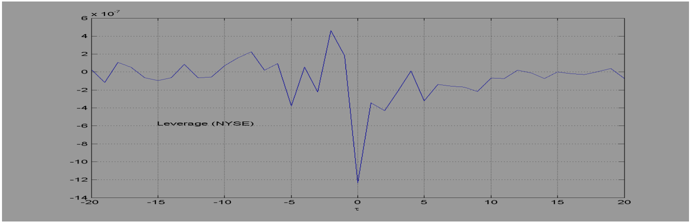
$$B_H(t) = C \left\{ \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB(s) + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \right\}$$

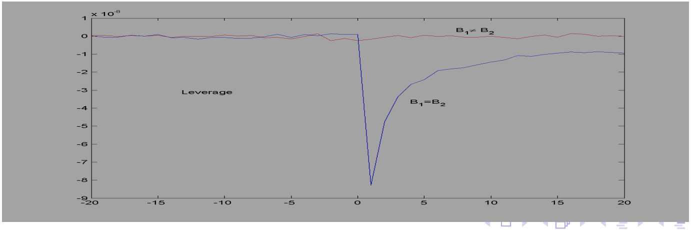
Then, the fractional volatility model may be rewritten as

$$\boxed{\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB^{(1)}(t) \\ \log \sigma_t &= \beta + k' \int_{-\infty}^{t} (t-s)^{H-\frac{3}{2}} dB^{(2)}(s) \end{aligned}}$$

The fractional volatility model. Leverage

$B^{(1)}(s)$ and $B^{(2)}(s)$ are Brownian processes. If $B^{(1)}(s) \neq B^{(2)}(s)$ there is no leverage effect but if $B^{(1)}(s) = B^{(2)}(s)$ one obtains a qualitatively correct leverage.





The fractional volatility model. Statistics of returns

At each fixed time $\log \sigma_t$ is a Gaussian random variable with mean β and variance $k^2 \delta^{2H-2}$. Then,

$$p_\delta(\sigma) = \frac{1}{\sigma} p_\delta(\log \sigma) = \frac{1}{\sqrt{2\pi}\sigma k \delta^{H-1}} \exp\left\{ -\frac{(\log \sigma - \beta)^2}{2k^2 \delta^{2H-2}} \right\}$$

therefore

$$P_\delta\left(\log \frac{S_{t+\Delta}}{S_t}\right) = \int_0^\infty d\sigma p_\delta(\sigma) p_\sigma\left(\log \frac{S_{t+\Delta}}{S_t}\right)$$

with

$$p_\sigma\left(\log \frac{S_{t+\Delta}}{S_t}\right) = \frac{1}{\sqrt{2\pi\sigma^2\Delta}} \exp\left\{ -\frac{\left(\log\left(\frac{S_{t+\Delta}}{S_t}\right) - \left(\mu - \frac{\sigma^2}{2}\right)\Delta\right)^2}{2\sigma^2\Delta} \right\}$$

The probability distribution of the returns might depend both on the time lag Δ and on the observation time scale δ used to construct the volatility process. This latter dependence might actually be very weak

The fractional volatility model. Statistics of returns

Closed form

$$P_\delta\left(r\left(\Delta\right)\right)=\frac{1}{4\pi\theta k\delta^{H-1}\sqrt{\Delta}}\int_0^\infty dx x^{-\frac{1}{2}}e^{-\frac{1}{C}\left(\log x\right)^2}e^{-\lambda x}$$

$$r\left(\Delta\right)=\log S_{t+\Delta}-\log S_t\ ,\ \theta=e^\beta,\ \lambda=\frac{\left(r\left(\Delta\right)-r_0\right)^2}{2\Delta\theta^2}$$

$$r_0=\left(\mu-\frac{\sigma^2}{2}\right)\Delta\ ,\ C=8k^2\delta^{2H-2}$$

Then

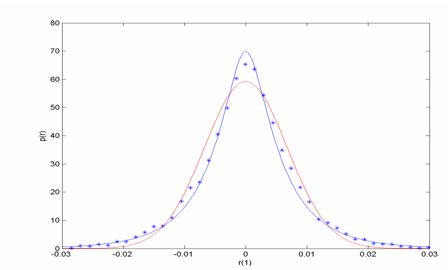
$$\boxed{P_\delta\left(r\left(\Delta\right)\right)=\frac{1}{4\pi\theta k\delta^{H-1}\sqrt{\Delta}}\frac{1}{\sqrt{\lambda}}\left(e^{-\frac{1}{C}\left(\log\lambda-\frac{d}{dz}\right)^2}\Gamma\left(z\right)\right)\Big|_{z=\frac{1}{2}}}$$

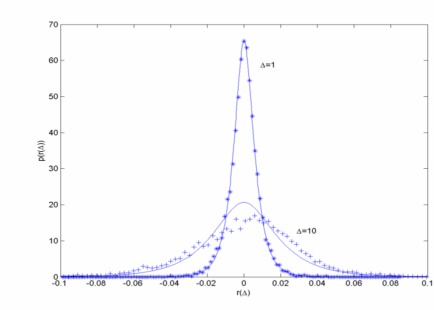
with asymptotic behavior, for large returns

$$\boxed{P_\delta\left(r\left(\Delta\right)\right)\sim\frac{1}{\sqrt{\Delta\lambda}}e^{-\frac{1}{C}\log^2\lambda}}$$

The fractional volatility model. Statistics of returns

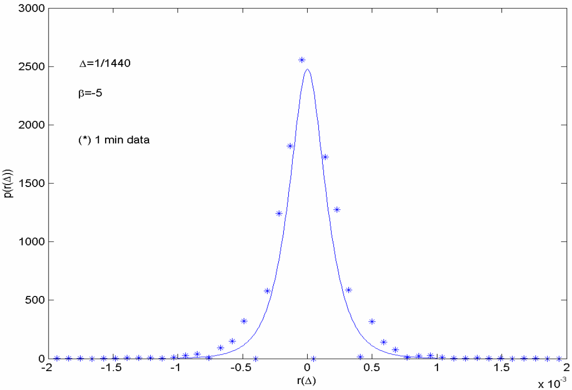
Qualitatively it resembles the double exponential distribution recognized by Silva, Prange and Yakovenko as a new stylized fact in market data. Shown, by Dragulescu and Yakovenko, to follow from the Heston model. Differs from Heston's in that volatility is driven by a process with memory. The analytic form of the distribution and the asymptotic behavior are different. $H = 0.83$, $k = 0.59$, $\beta = -5$, $\delta = 1$, $\Delta = 1$ and $\Delta = 10$ data.





The fractional volatility model. Statistics of returns

Same parameters but $\Delta = \frac{1}{440}$ (one minute). The prediction of the model is compared with one-minute data of USDollar-Euro market for a couple of months in 2001. The result is surprising, because one would not expect the volatility parametrization to carry over to such a different time scale and also because one is dealing with different markets.



The fractional volatility model. Option pricing

Assuming risk neutrality, the value $V(S_t, \sigma_t, t)$ of an option is the present value of the expected terminal value discounted at the risk-free rate

$$V(S_t, \sigma_t, t) = e^{-r(T-t)} \int V(S_T, \sigma_T, T)\, p(S_T | S_t, \sigma_t)\, dS_T$$

$V(S_T, \sigma_T, T) = \max[0, S - K]$ and the conditional probability for the terminal price depends on S_t and σ_t. K is the strike price, T the maturity time and S_t and σ_t the price and volatility of the underlying security.

$$p(S_T | S_t, \sigma_t) = \int p\left(S_T | S_t, \overline{\log \sigma}\right) p\left(\overline{\log \sigma} \middle| \log \sigma_t\right) d\left(\overline{\log \sigma}\right)$$

$\overline{\log \sigma}$ being the random variable

$$\overline{\log \sigma} = \frac{1}{T-t} \int_t^T \log \sigma_s ds$$

$\overline{\log \sigma}$ is the mean volatility from time t to the maturity time T conditioned to an average value $\log \sigma_t$ at time t.

The fractional volatility model. Option pricing

Then,

$$V\left(S_t,\sigma_t,t\right)=\int C\left(S_t,e^{\overline{\log\sigma}},t\right)p\left(\overline{\log\sigma}\right|\log\sigma_t\left.\right)d\left(\overline{\log\sigma}\right)$$

$$C\left(S_t,e^{\overline{\log\sigma}},t\right)=\int e^{-r(T-t)}V\left(S_T,\sigma_T,T\right)p\left(S_T|S_t,\overline{\log\sigma}\right)dS_T$$

$C\left(S_t,e^{\overline{\log\sigma}},t\right)$ is the Black-Scholes price for an option with average volatility $e^{\overline{\log\sigma}}$, known to be

$$C\left(S_t,\sigma,t\right)=S_t\left(a+b\right)N\left(a,b\right)-Ke^{-r(T-t)}\left(a-b\right)N\left(a,-b\right)$$

$$a=\tfrac{1}{\sigma}\left(\frac{\log\frac{S_t}{K}}{\sqrt{T-t}}+r\sqrt{T-t}\right)\qquad b=\tfrac{\sigma}{2}\sqrt{T-t}$$

$$N\left(a,b\right)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{\infty}dye^{-\frac{y^2}{2}(a+b)^2}$$

Instead of $V\left(S_t,\sigma_t,t\right)$, it would be more correct to write $V\left(S_t,\sigma_{\leq t},t\right)$. However, Markov properties of the processes are not assumed, only their Gaussian nature.

$$\overline{\log \sigma} = \log \sigma_t + \frac{1}{T-t} \int_t^T \frac{k}{\delta} ds \int_t^s \left(dB_H(\tau) - dB_H(\tau - \delta) \right)$$

Notice that, because we want to compute the conditional probability of $\overline{\log \sigma}$ given $\log \sigma_t$ at time t, σ_t is not a process but simply the value of the argument in the $V(S_t, \sigma_t, t)$ function.
As a $t-$dependent process the double integral is a centered Gaussian process. Therefore, given $\log \sigma_t$ at time t, $\overline{\log \sigma}$ is a Gaussian variable with conditional mean $E\left\{ \overline{\log \sigma} \middle| \log \sigma_t \right\} = \log \sigma_t$ and variance

$$\begin{aligned} \alpha^2 &= E\left\{ \left(\overline{\log \sigma} - \log \sigma_t \right)^2 \middle| \log \sigma_t \right\} \\ &= \frac{k^2}{\delta^2 (T-t)} \left\{ \frac{1}{2(T-t)} I_1 + I_2 \right\} + k^2 \delta^{2H-2} \end{aligned}$$

$$I_1 = \frac{2}{(2H+1)(2H+2)} \left\{ \begin{array}{c} (T-t+\delta)^{2H+2} + (T-t-\delta)^{2H+2} \\ -2(T-t)^{2H+2} - 2\delta^{2H+2} \end{array} \right\}$$

$$I_2 = \frac{1}{2H+1}\left\{2\left(T-t\right)^{2H+1} - \left(T-t+\delta\right)^{2H+1} - \left(T-t-\delta\right)^{2H+1}\right\}$$

In general, for option pricing purposes, $\delta \ll \left(T-t\right)$ and one may approximate $\alpha^2 \simeq \frac{k^2}{\delta^{2-2H}}\left(1-\left(2H-1\right)\left(\frac{\delta}{T-t}\right)^{2-2H}\right)$
Then

$$p\left(\overline{\log\sigma}\middle|\log\sigma_t\right) = \frac{1}{\sqrt{2\pi}\alpha}\exp\left\{\frac{-\left(\overline{\log\sigma}-\log\sigma_t\right)^2}{2\alpha^2}\right\}$$

Finally

$$V\left(S_t,\sigma_t,t\right) = \int_{-\infty}^{\infty} d\xi C\left(S_t, e^{\xi}, t\right) p\left(\xi\middle|\log\sigma_t\right)$$

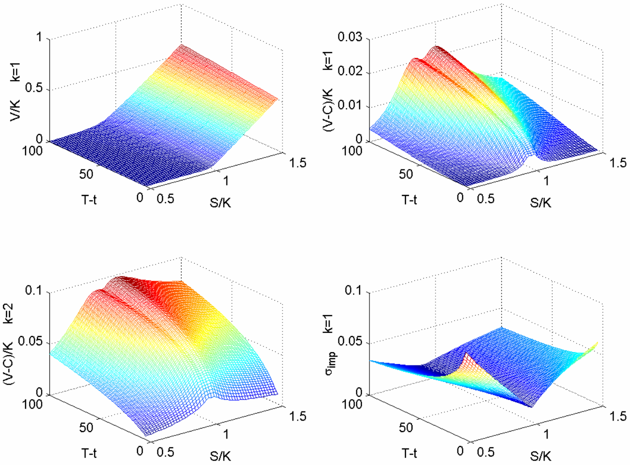
The fractional volatility model. Option pricing

One obtains

$$\begin{aligned} V\left(S_t,\sigma_t,t\right) &= \{S_t\left[aM\left(\alpha,a,b\right)+bM\left(\alpha,b,a\right)\right] \\ &\quad -Ke^{-r(T-t)}\left[aM\left(\alpha,a,-b\right)-bM\left(\alpha,-b,a\right)\right]\} \end{aligned}$$

$$\begin{aligned} M\left(\alpha,a,b\right) &= \frac{1}{2\pi\alpha}\int_{-1}^{\infty}dy\int_0^{\infty}dxe^{-\frac{\log^2 x}{2\alpha^2}}e^{-\frac{y^2}{2}\left(ax+\frac{b}{x}\right)^2} \\ &= \frac{1}{4\alpha}\sqrt{\frac{2}{\pi}}\int_0^{\infty}dx\frac{e^{-\frac{\log^2 x}{2\alpha^2}}}{ax+\frac{b}{x}}erfc\left(-\frac{ax}{\sqrt{2}}-\frac{b}{\sqrt{2}x}\right) \end{aligned}$$

Plots: $V\left(S_t,\sigma_t,t\right)$ in the range $T-t\in[5,100]$ and $S/K\in[0.5,1.5]$ as well as $\left(V\left(S_t,\sigma_t,t\right)-C\left(S_t,\sigma_t,t\right)\right)/K$ for $k=1$ and $k=2$. Other parameters fixed at $\sigma=0.01, r=0.001, \delta=1, H=0.8$. Compared with the implied volatility required in the BS model to reproduce the same results (implied volatility surface corresponding to $V\left(S_t,\sigma_t,t\right)$ for $k=1$). Predicts a smile effect with the smile increasing as maturity approaches.


V/K k=1
1
0.5
0
100
50
T-t
0
0.5
1
1.5
S/K
(V-C)/K k=1
0.03
0.02
0.01
0
(V-C)/K k=2
0.1
0.05
0
σ_{imp} k=1
0.1
0.05
0

Arbitrage with the price process driven by fractional Brownian motion

Two assets: A risk-free investment

$$dS_0(t) = rS_0(t)\,dt$$

and a risky one

$$dS_1(t) = \mu S_1(t)\,dt + \sigma S_1(t)\,dB_H(t)$$

What is the meaning of $dB_H(t)$? For $H > \frac{1}{2}$ there two possible notions of integration

\# **Forward** (or **pathwise**) integration $(d^- B_H(t))$

$$\int_0^T \phi(t,\omega)\,d^- B_H(t) = \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k)\left(B_H(t_{k+1}) - B_H(t_k)\right)$$

This integral exists for $H > \frac{1}{2}$ because $B_H(t)$ has finite $q-$variation for $q \geq 1/H$

The **Skorohod integral** (uses the chaos expansion)

1- For $B(t)$

$$u(t,\omega) = \sum_{n=0}^{\infty} I_n(f_n(t,\bullet))$$

$$I_n(f) = n! \int_R \left(\int_{-\infty}^{\tau_n} \cdots \left(\int_{-\infty}^{\tau_2} f(\tau_1, \cdots, \tau_n)\, dB(\tau_1) \right) \cdots dB(\tau_n) \right)$$

with Hermite functions

$$u(t,\omega) = \sum_{\alpha \in \mathcal{I}} c_\alpha^{(t)} \mathcal{H}_\alpha(\omega)$$

$$\mathcal{H}_\alpha(\omega) == h_{\alpha_1}(\langle \omega, \xi_1 \rangle) \cdots h_{\alpha_n}(\langle \omega, \xi_n \rangle)$$

$$\xi_n = \pi^{-1/4} ((n-1)!)^{-1/2} h_n\left(\sqrt{2}x\right) e^{-x^2/2}$$

Then

$$\int_R u(t,\omega)\, \delta B(t) = \sum_{n=0}^{\infty} I_{n+1}\left(\widetilde{f_n}\right) = \int_R u(t,\omega) \diamondsuit W(t)\, dt$$

Arbitrage

where

$$\tilde{f}_n(t_1, \cdots, t_{n+1}) = \{f_n(t_1, \cdots, t_{n+1}) + \cdots + f_n(\cdots, t_{i-1}, t_{i+1}, \cdots, t_i) \\ \cdots + f_n(t_1, \cdots, t_{n+1}, t_1)\} \frac{1}{n+1}$$

$$(F \Diamond G)(\omega) = \sum_{\alpha, \beta \in \mathcal{I}} c_\alpha^{(F)} c_\beta^{(G)} \mathcal{H}_{\alpha+\beta}(\omega)$$

$$W(t) = \sum_{k=1}^{\infty} \xi_k \mathcal{H}_{\varepsilon^{(k)}}(\omega)$$

2 - For $B_H(t)$

$$\int_R u(t, \omega) \delta B_H(t) = \int_R u(t, \omega) \Diamond W_H(t)\, dt$$

$$W_H(t) = \sum_{k=1}^{\infty} M \xi_k \mathcal{H}_{\varepsilon^{(k)}}(\omega)$$

$\varepsilon^{(k)} = (0, 0, \cdots, 1) \in R^k$ $\qquad \mathcal{F}\{Mf(y)\} = c_H |y|^{\frac{1}{2}-H} \mathcal{F}\{f(y)\}$

Arbitrage

Arbitrage with pathwise integration

Two assets: A risk-free investment

$$dS_0(t) = rS_0(t)\,dt$$

and a risky one

$$dS_1(t) = \mu S_1(t)\,dt + \sigma S_1(t)\,dB_H(t)$$

The **wealth** process

$$V^\theta(t) = \theta_0(t)\,S_0(t) + \theta_1(t)\,S_1(t)$$

Self-financing portfolio

$$dV^\theta(t) = \theta(t) \bullet dS(t)$$

A portfolio is called an **arbitrage** if the wealth process satisfies:

$$\begin{array}{rcll} V^\theta(0) & = & 0 & \\ V^\theta(T) & \geq & 0 & a.s. \\ P\left(V^\theta(T) > 0\right) & > & 0 & \end{array}$$

Arbitrage

1) $dB_H(t) = d^- B_H(t)$ (Rogers, Shiryaev)

$$S_1(t) = S_1(0)\exp(\sigma B_H(t) + \mu t)$$

Let $\mu = r, \sigma = 1 = S_1(0)$ and construct the portfolio

$$\theta_0(t) = 1 - e^{2B_H(t)} \qquad \theta_1(t) = 2\left(e^{B_H(t)} - 1\right)$$

It is self-financing

$$d\theta_0(t)\, S_0(t) + d\theta_1(t)\, S_1(t) = 0$$

and

$$\begin{aligned} V^\theta(t) &= \theta_0(t)\, S_0(t) + \theta_1(t)\, S_1(t) \\ &= \left(1 - e^{2B_H(t)}\right) e^{rt} + 2\left(e^{B_H(t)} - 1\right) e^{(B_H(t)+rt)} \\ &= e^{rt}\left(e^{B_H(t)} - 1\right)^2 > 0 \end{aligned}$$

for a.a. (t, ω)

Arbitrage

2) $dB_H(t) = \delta B_H(t)$ (Elliott and van der Hoek, Hu and Oksendal)

$$S_1(t) = S_1(0)\exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right)$$

$$\begin{aligned} V(T) &= V(0) + \int_0^T \theta(t) \bullet \delta S(t) = V(0) + \int_0^T r\theta_0(t) S_0(t)\, dt \\ &+ \int_0^T \mu\theta_1(t) \diamondsuit S_1(t)\, dt + \int_0^T \sigma\theta_1(t) \diamondsuit S_1(t)\, \delta B_H \end{aligned}$$

Change of measure

$$\breve{B}_H(t) = \frac{\mu - r}{\sigma} t + B_H(t)$$

$$V(t) = e^{rt} V(0) + e^{rt} \int_0^t e^{-rs} \sigma\theta_1(s) \diamondsuit S_1(s)\, \delta\breve{B}_H$$

leads to

$$E\{V^\theta(T)\} = e^{rT} V^\theta(0)$$

No arbitrage

Arbitrage

However: (Björk and Hult, Sottinen and Valkeila, Nualart and Taqqu)
The Skorohod integral approach requires either that the portfolio value be

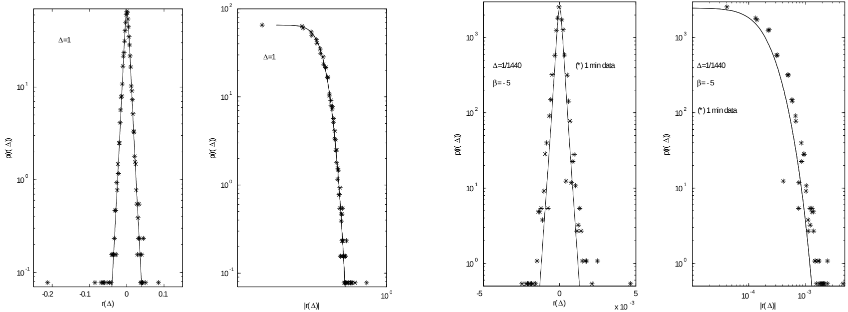
$$V^{\theta}(t) = \theta_0(t)\, S_0(t) + \theta_1(t) \Diamond S_1(t)$$

or that the self-financing condition be

$$dV^{\theta}(t) = \theta(t) \Diamond dS(t)$$

or both. This might not be reasonable from an economic point of view (for example positive portfolio with negative Wick value, etc.)
Question: In the fractional volatility model the fractional noise is on the volatility, not on the price. Is there an arbitrage?

How many market regimes ?



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Two agent-based models are considered.

- In the first the traders strategies play a determinant role.

- In the second the determinant effect is the limit-order book dynamics, the agents having a random nature.

Comparison with agent-based models

A market model with self-adapted or fixed strategies

- The dominance of two types of strategies was to a large extent determined by the initial conditions.
- Different types of return statistics corresponded to the relative importance of either "value investors" or "technical traders".
- The occurrence of market bubbles correspondes to situations where technical trader strategies were well represented.
- Consider a set of investors playing *against* the market (in addition to the impact of this group of investors, the other factors are represented by a stochastic process)

$$z_{t+1} = f\left(z_t, \omega_t\right) + \eta_t$$

  $(z_t = \log p_t)$, ω_t is the total investment made by the group of traders and η_t the stochastic process that represents all the other factors.

$s =$ amount of *stock*

$m =$ *cash*

$p_t =$ price of the traded asset at time t,

Comparison with agent-based models

The purpose of the investors is to increase the total wealth $m_t + p_t \times s_t$
Each investor payoff at time t is
$\Delta_t^{(i)} = \left(m_t^{(i)} + p_t \times s_t^{(i)}\right) - \left(m_0^{(i)} + p_0 \times s_0^{(i)}\right)$.

- **Market impact**

$$z_{t+1} - z_t = \frac{\omega_t}{\lambda_0 + \lambda_1 |\omega_t|^\alpha} + \eta_t$$

$\alpha = \frac{1}{2}$

- **Agent strategies**

The difference (misprice) between price and perceived value v_t

$$\xi_t - z_t = \log(v_t) - \log(p_t)$$

The price trend

$$z_t - z_{t-1} = \log(p_t) - \log(p_{t-1})$$

A non-decreasing function $f(x)$ such that $f(-\infty) = 0$ and $f(\infty) = 1$.
Example $f_1(x) = \theta(x)$ or $f_2(x) = \frac{1}{1+\exp(-\beta x)}, \beta > 0)$.

Comparison with agent-based models

- Information about misprice and price trend coded on a four-component vector

$$\gamma_t = \begin{pmatrix} f(\xi_t - z_t) f(z_t - z_{t-1}) \\ f(\xi_t - z_t)\left(1 - f(z_t - z_{t-1})\right) \\ \left(1 - f(\xi_t - z_t)\right) f(z_t - z_{t-1}) \\ \left(1 - f(\xi_t - z_t)\right)\left(1 - f(z_t - z_{t-1})\right) \end{pmatrix}$$

- The strategy of each investor is a four-component vector $\alpha^{(i)}$ with entries -1, 0, or 1.
  -1 means to sell, 1 means to buy and 0 means to do nothing.
  At each time, the investment of agent i is $\alpha^{(i)} \cdot \gamma$.
  Fundamental (value-investing strategy) $\alpha^{(i)} = (1, 1, -1, -1)$
  Pure *trend-following* strategy $\alpha^{(i)} = (1, -1, 1, -1)$.
  Total number of possible strategies $3^4 = 81$.
  Strategies labelled by numbers $n^{(i)} = \sum_{k=0}^{3} 3^k \left(\alpha_k^{(i)} + 1\right)$
  (Fundamental = 72, Trend-following = 60)

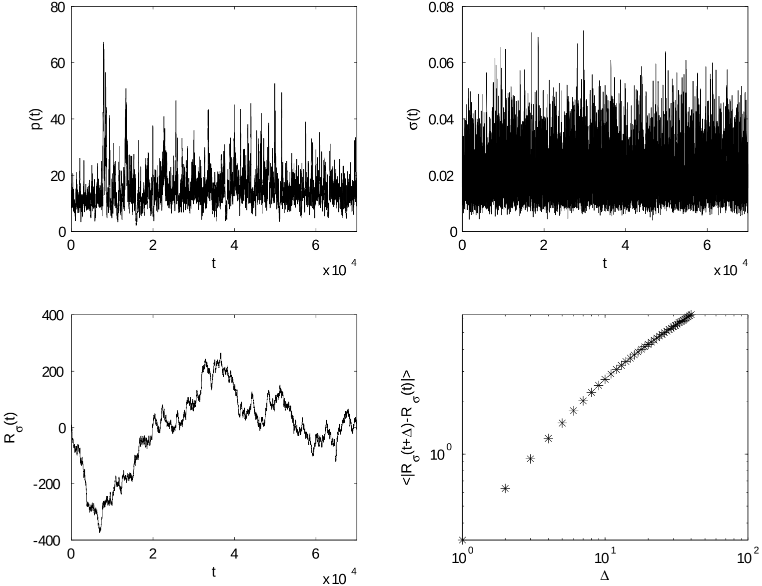
Comparison with agent-based models

Evolution dynamics:

- After r time steps, s agents copy the strategy of the s best performers and, at the same time, have some probability to mutate that strategy.
- The model was run with different initial conditions and with or without evolution of the strategies.
- When the model is run with evolution the asymptotic steady-state behavior depends on the initial conditions.
- Simulation without evolution, with a fixed 50% of fundamental strategies (no. 72) and 50% of trend-following (no. 60), one sees a large number of bubbles and crashes in the price evolution and the price increments distribution has fat tails.

To compare with the behavior of the fractional volatility model: $\sigma_t^2 = \frac{1}{|T_0 - T_1|} var\left(\log p_t\right)$, the parameters in $\sum_{n=0}^{t/\delta} \log \sigma\left(n\delta\right) = \beta t + R_\sigma\left(t\right)$ and $\left|R_\sigma\left(t+\Delta\right) - R_\sigma\left(t\right)\right|$ were estimated from model simulation.

Comparison with agent-based models



Comparison with agent-based models

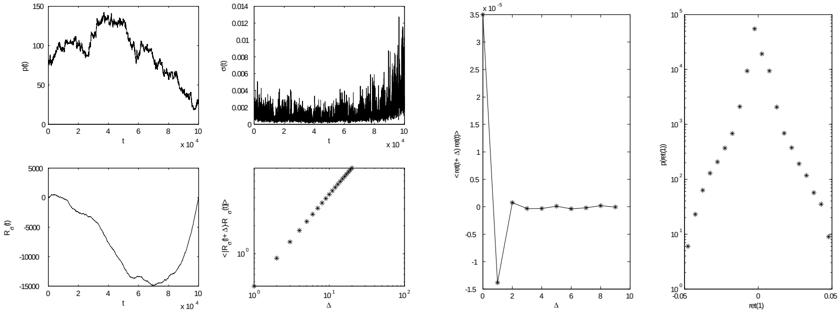
- Notice the lack of scaling behavior of $R_\sigma(t)$ with an asymptotic exponent 0.55, denoting the lack of memory of the volatility process. This might already be evident from the time behavior of $R_\sigma(t)$ in the lower left plot.
- Also, although the returns have fat tails in this case, they are of different shape from those observed in the market data. Similar conclusions are obtained with other combinations of agent strategies.
- In conclusion: **It seems that the features of the fractional volatility model (which are also those of the bulk market data) are not easily captured by a choice of strategies in an agent-based model.**
- Agents' reactions and strategies are very probably determinant during market crisis and market bubbles.

Comparison with agent-based models

A limit-order book

- *Asks* and *bids* arrive at random on a window $[p - w, p + w]$ around the current price p.
- Every time a *buy* order arrives it is fulfilled by the closest non-empty ask slot, the new current price being determined by the value of the ask that fulfills it.
- If no ask exists when a buy order arrives it goes to a cumulative register to wait to be fulfilled. The symmetric process occurs when a *sell* order arrives, the new price being the bid that buys it.
- Because the window around the current price moves up and down, asks and bids that are too far away from the current price are automatically eliminated.
- Sell and buy orders, asks and bids all arrive at random.
- The only parameters of the model are the width w of the limit-order book and the size n of the asks and bids, the sell and buy orders being normalized to one.

Comparison with agent-based models



Comparison with agent-based models

- Model run for different widths w and liquidities n. Although the exact values of the statistical parameters depend on w and n, the statistical nature of the results is essentially the same. In the figure $n = 2$, the limit-order book divided into $2w + 1 = 21$ discrete price slots with $\Delta p = 0.1$.
- The scaling properties of $R_\sigma(t)$ are quite evident from the lower right plot in the figure, the Hurst coefficient being 0.96.
- Conclusion: **the main statistical properties of the market data (fast decay of the linear correlation of the returns, non-Gaussianity and volatility memory) are already generated by the dynamics of the limit-order book with random behavior of the agents.**
- A large part of the market statistical properties (in normal business-as-usual days) depends more on the nature of the price fixing financial institutions than on particular investor strategies.

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