Stochastic solutions of pde's: A tool for localized behavior and parallel computing

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Contents

Stochastic solutions of pde's

- **Stochastic solution** = a stochastic process which, when started from a particular point in the domain, generates after time t a boundary measure which, integrated over the initial condition at $t = 0$, provides a solution of the equation at x and time t.

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- **Example: the heat equation**

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad \text{with} \qquad u(0,x) = f(x)$$

the process is Brownian motion, $dX_t = dB_t$, and the solution

$$u(t,x) = \mathbb{E}_x f(X_t) \tag{1}$$

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- The domain here is $\mathbb{R} \times [0,t)$ and the expectation value in (1) is the inner product $\langle \mu_t, f \rangle$ of the initial condition f with the *measure* μ_t *generated by the Brownian motion at the* $t-$*boundary*.

Stochastic solutions of pde's

- Using the heat kernel the solution is

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\,dy$$

Integration over the domain versus "integration" over paths.

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Integration over the domain versus "integration" over paths.

- Even for linear problems, the stochastic solution approach provides a way to express exact solutions in a way that is not possible with kernels and integral representations:

$$Lf\left(x\right)=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(x)\partial_{ij}f\left(x\right)+\sum_{i=1}^{d}b_{i}(x)\partial_{i}f\left(x\right)$$

$$\left(L+v(x)\right)u(x)=-g(x)\qquad\text{with}\qquad u=0\qquad\text{on}\qquad\partial D$$

$$\boxed{u(x)=\mathbb{E}^{x}\left[\int_{0}^{\tau_D}g(X_s)e^{\int_0^s v(X_r)dr}ds\right]}$$

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- Classical results for linear pde's. Recent work in nonlinear pde's: KPP, Navier-Stokes, Poisson-Vlasov, MHD, etc.

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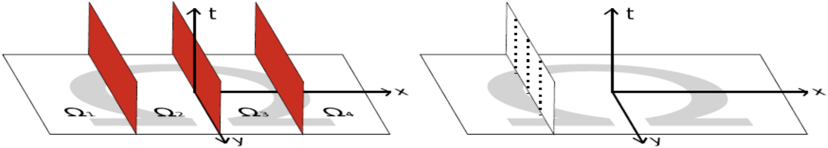
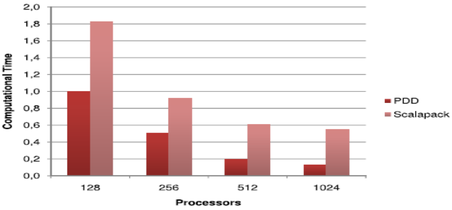
- Stochastic algorithms handle equally well regular and complex boundary conditions.
- **Domain decomposition** using interpolation of localized stochastic solutions and then, in each small domain, a deterministic code. Avoids the communication time problem. Fully parallel.

The probabilistic domain decomposition (PDD) method

(J. Acebrón, A. Rodríguez-Rozas, R. Spigler)

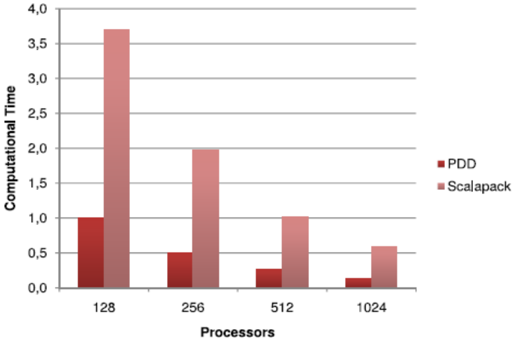


$$u_t = L^2 u_{xx} - u, \quad u(x,0) = \sin\left(\frac{\pi x}{L}\right), \quad u(0,t) = u(L,t) = 0,$$



The probabilistic domain decomposition (PDD) method

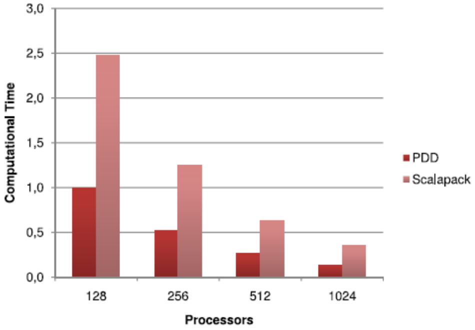
$$u_t = D\,u_{xx} - u + u^2, \quad u(x,0) = 1 - \frac{1}{\left(1 + \exp\frac{x}{\sqrt{6\,D}}\right)^2}$$



The probabilistic domain decomposition (PDD) method

$$u_t = (x^2+1)u_{xx} + [2+\sin(x)]u_x - u + \frac{1}{2}u^2 + \frac{1}{2}u^3,$$

$$u(x,0) = 1 \text{ for } 0 \leq x < 1, \quad \text{and } u(x,0) \equiv 0 \text{ elsewhere on the line}$$



Stochastic solutions: Two construction methods

- **McKean's method:** a probabilistic version of the Picard series.
  First the differential equation is written as an integral equation and rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition
  Then the Picard iteration is interpreted as an evolution and branching proces.
  The stochastic solution is equivalent to importance sampling of a normalized Picard series.

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  Then the Picard iteration is interpreted as an evolution and branching proces.
  The stochastic solution is equivalent to importance sampling of a normalized Picard series.
- **The method of superprocesses:** constructs the boundary measures of a measure-valued stochastic process and obtains the solutions of the differential equation by a scaling procedure.

The KPP equation: McKean's formulation

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + v^2 - v \qquad\qquad v\left(0,x\right) = g\left(x\right)$$

- $G\left(t,x\right) =$ Green's operator for heat equation $\partial_t v(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v(t,x)$

$$G\left(t,x\right) = e^{\frac{1}{2}t\frac{\partial^2}{\partial x^2}}$$

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- KPP in integral form

$$v\left(t,x\right)=e^{-t}G\left(t,x\right)g\left(x\right)+\int_0^t e^{-(t-s)}G\left(t-s,x\right)v^2\left(s,x\right)ds \quad (2)$$

Denoting by $\left(\xi_t,\Pi_x\right)$ a Brownian motion starting from time zero and coordinate x, Eq.(2) may be rewritten as

$$\begin{aligned} v\left(t,x\right) &= \Pi_x\left\{e^{-t}g\left(\xi_t\right)+\int_0^t e^{-(t-s)}v^2\left(s,\xi_{t-s}\right)ds\right\} \\ &= \Pi_x\left\{e^{-t}g\left(\xi_t\right)+\int_0^t e^{-s}v^2\left(t-s,\xi_s\right)ds\right\} \end{aligned}$$

The KPP equation: McKean's formulation

- **The stochastic solution process:** a Brownian motion plus branching process with exponential holding time T, $P(T > t) = e^{-t}$. At each branching point the particle splits into two, the new particles going along independent Brownian paths. At time $t > 0$ one has n particles located at $x_1(t), x_2(t), \cdots x_n(t)$. The solution is obtained by

$$v(t,x) = \mathbb{E}\left\{g(x_1(t))\, g(x_2(t)) \cdots g(x_n(t))\right\}$$

$|g(x)| \leq 1$

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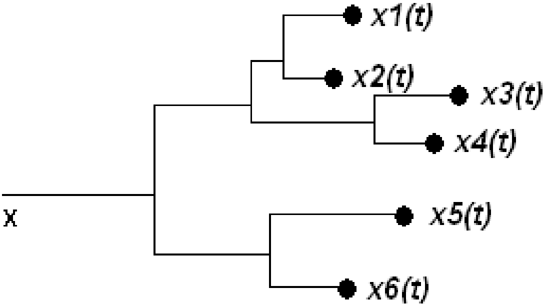
  $|g(x)| \leq 1$

- An equivalent interpretation: a backwards-in-time process from time t at x. When it reaches $t = 0$ samples the initial condition. Generates a measure at the $t = 0$ boundary which is applied to $g(x) = v(0, x)$.

The KPP equation: McKean's formulation



Poisson-Vlasov equation (Cipriano, Floriani, Lima, RVM)

$$\boxed{\frac{\partial f_i}{\partial t} + \overrightarrow{v} \cdot \nabla_x f_i - \frac{e_i}{m_i} \nabla_x \Phi \cdot \nabla_v f_i = 0} \tag{3}$$

$$\Delta_x \Phi = -4\pi \left\{ \sum_i e_i \int f_i \left(\overrightarrow{x}, \overrightarrow{v}, t \right) d^3 v \right\} \tag{4}$$

Fourier transforming Eqs.(3) and (4), with

$$F_i \left(\xi, t \right) = \frac{1}{\left(2\pi \right)^3} \int d^6 \eta f_i \left(\eta, t \right) e^{i \xi \cdot \eta}$$

$\eta = \left(\overrightarrow{x}, \overrightarrow{v} \right)$ and $\xi = \left(\overrightarrow{\xi_1}, \overrightarrow{\xi_2} \right) \doteq \left(\xi_1, \xi_2 \right)$, one obtains

$$\frac{\partial F_i \left(\xi, t \right)}{\partial t} = \overrightarrow{\xi_1} \cdot \nabla_{\xi_2} F_i \left(\xi, t \right) - \frac{4\pi e_i}{m_i} \int d^3 \xi_1^{'} F_i \left(\xi_1 - \xi_1^{'}, \xi_2, t \right) \left(\overrightarrow{\xi_2} \cdot \overrightarrow{\xi_1^{'}} / \left| \xi_1^{'} \right|^2 \right) \sum_j e_j F_j \left(\xi_1^{'}, 0, t \right)$$

Changing variables to

$$\tau = \gamma\left(|\xi_2|\right) t$$

$\gamma\left(|\xi_2|\right)$ is a positive continuous function satisfying

$$\begin{array}{lll} \gamma\left(|\xi_2|\right) = 1 & if & |\xi_2| < 1 \\ \gamma\left(|\xi_2|\right) \geq |\xi_2| & if & |\xi_2| \geq 1 \end{array}$$

$$\begin{aligned} \frac{\partial F_i\left(\xi, \tau\right)}{\partial \tau} &= \frac{\overrightarrow{\xi_1}}{\gamma\left(|\xi_2|\right)} \cdot \nabla_{\xi_2} F_i\left(\xi, \tau\right) - \frac{4\pi e_i}{m_i} \int d^3\xi_1^{'} F_i\left(\xi_1 - \xi_1^{'}, \xi_2, \tau\right) \\ &\times \frac{\overrightarrow{\xi_2} \cdot \hat{\xi_1^{'}}}{\gamma\left(|\xi_2|\right) \left|\xi_1^{'}\right|} \sum_j e_j F_j\left(\xi_1^{'}, 0, \tau\right) \end{aligned}$$

with $\hat{\xi_1} = \frac{\overrightarrow{\xi_1}}{|\xi_1|}$.

Stochastic representation written for the following functions

$$\chi_i\left(\xi_1, \xi_2, \tau\right) = e^{-\lambda\tau} \frac{F_i\left(\xi_1, \xi_2, \tau\right)}{h\left(\xi_1\right)}$$

with λ a constant and $h\left(\xi_1\right)$ a positive function to be specified later. Define

$$\left(\left|\xi_1^{'}\right|^{-1} h * h\right) = \int d^3\xi_1^{'} \left|\xi_1^{'}\right|^{-1} h\left(\xi_1 - \xi_1^{'}\right) h\left(\xi_1^{'}\right)$$

$$p\left(\xi_1, \xi_1^{'}\right) = \frac{\left|\xi_1^{'}\right|^{-1} h\left(\xi_1 - \xi_1^{'}\right) h\left(\xi_1^{'}\right)}{\left(\left|\xi_1^{'}\right|^{-1} h * h\right)}$$

The Poisson-Vlasov equation

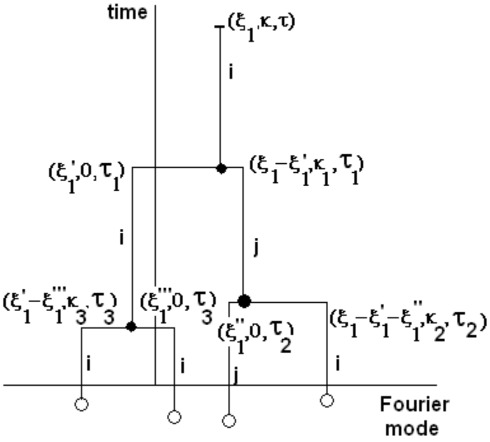
The Fourier transformed equation

$$
\begin{aligned}
&\chi_i(\xi_1, \xi_2, \tau) \\
= \; & \boxed{e^{-\lambda\tau}} \chi_i\left(\xi_1, \xi_2 + \tau \frac{\xi_1}{\gamma(|\xi_2|)}, 0\right) - \frac{8\pi e_i}{m_i \lambda} \frac{\left(|\xi_1|^{-1} h * h\right)(\xi_1)}{h(\xi_1)} \\
& \times \int_0^\tau ds \boxed{\lambda e^{-\lambda s}} \int d^3\xi_1' \boxed{p\left(\xi_1, \xi_1'\right)} \chi_i\left(\xi_1 - \xi_1', \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) \\
& \times \frac{\left(\xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right) \cdot \hat{\xi_1'}}{\gamma\left(\left|\xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right|\right)} \sum_j \boxed{\tfrac{1}{2}} e_j e^{\lambda(\tau - s)} \chi_j\left(\xi_1', 0, \tau - s\right)
\end{aligned} \tag{5}
$$

Notice: Bifurcation occurs at $t' = \frac{\tau - s}{\gamma(|\xi_2|)}$ *and time rescaling always depends on the second argument*

The Poisson-Vlasov equation

The Fourier transformed equation



Eq.(5) has a stochastic interpretation *(an* **exponential process** *plus* **branching** *and* **Bernoulli** *processes)*.

$e^{-\lambda\tau} =$ survival probability during time τ of the exponential process

$\lambda e^{-\lambda s} ds =$ the decay probability

$p\left(\xi_1, \xi_1^{\prime\prime}\right) d^3\xi_1 =$ branching probability of ξ_1 mode into $\left(\xi_1 - \xi_1^{\prime\prime}, \xi_1^{\prime\prime}\right)$

$\chi\left(\xi_1, \xi_2, \tau\right)$ computed from the expectation value of a multiplicative functional

Convergence of the multiplicative functional:

(A) $\left|\frac{F_i(\xi_1,\xi_2,0)}{h(\xi_1)}\right| \leq 1$

(B) $\left(\left|\xi_1^{\prime}\right|^{-1} h * h\right)\left(\xi_1\right) \leq h\left(\xi_1\right)$, satisfied, for example,

for $h\left(\xi_1\right) = \frac{c}{\left(1+\left|\xi_1\right|^2\right)^2}\left(1 - \theta\left(\left|\xi_1\right| - M\right)\right)$ and $c \leq \frac{1}{3\pi}$

The multiplicative functional of the process $X(\xi_1, \xi_2, \tau)$ is the product of:

- **At each branching point where 2 particles are born**

$$g_{ij}\left(\xi_1, \xi_1^{'}, s\right) = -e^{\lambda(\tau - s)} \frac{8\pi e_i e_j}{m_i \lambda} \frac{\left(\left|\xi_1^{'}\right|^{-1} h * h\right)(\xi_1)}{h(\xi_1)} \frac{\left(\xi_2 + s \frac{\xi_1}{\gamma(|\xi_2|)}\right) \cdot \hat{\xi_1^{'}}}{\gamma\left(\left|\xi_2 + s \frac{\xi_1}{\gamma(|\xi_2|)}\right|\right)}$$

- **When one particle reaches time zero and samples the initial condition**

$$g_{0i}(\xi_1, \xi_2) = \frac{F_i(\xi_1, \xi_2, 0)}{h(\xi_1)}$$

$$\boxed{\chi_i(\xi_1, \xi_2, \tau) = \mathbb{E}\left\{\Pi\left(g_0 g_0^{'} \cdots\right)\left(g_{ii} g_{ii}^{'} \cdots\right)\left(g_{ij} g_{ij}^{'} \cdots\right)\right\}}$$

- Choose $\lambda \geq \left| \frac{8\pi e_i e_j}{\min_i\{m_i\}} \right|$ and $c \leq e^{-\lambda \tau_M} \frac{1}{3\pi} \Longrightarrow$ the absolute value of all coupling constants is bounded by one. τ_M is an upper bound for τ in the successive branchings

$$\tau_M = \frac{\left(Mt + \gamma\left(|\xi_2|\right)\right)^2}{4M}$$

- The branching process, identical to Galton-Watson's, terminates with probability 1 $\Longrightarrow$ no. of inputs to the functional is finite a. s.
- With the bounds on the coupling constants, the multiplicative functional is bounded by one in absolute value almost surely.
- **Th. 1** - *The stochastic process $X\left(\xi_1, \xi_2, \tau\right)$, above described, provides a stochastic solution for the Fourier-transformed Poisson-Vlasov equation $F_i\left(\xi_1, \xi_2, t\right)$ for any arbitrary finite value of the arguments, provided the initial conditions at time zero satisfy the boundedness conditions (A).*

- Instead of renormalizing the time one may write

$$\Theta_i(\xi_1, \xi_2, t) = e^{-t|\xi_2|} \frac{F_i(\xi_1, \xi_2, t)}{h(\xi_1)}$$

$p\left(\xi_1, \xi_1^{'}\right)$ and the conditions on $h(\xi_1)$ are the same as before. The main difference is the survival probability, namely $e^{-t|\xi_2|}$ and $ds\Pi(\xi_1, \xi_2, s)$ the dying probability in time ds

$$\Pi(\xi_1, \xi_2, s) = \frac{|\xi_2 + s\xi_1| \, e^{(t-s)|\xi_2+s\xi_1|-t|\xi_2|}}{N(\xi_1, \xi_2, t)}$$

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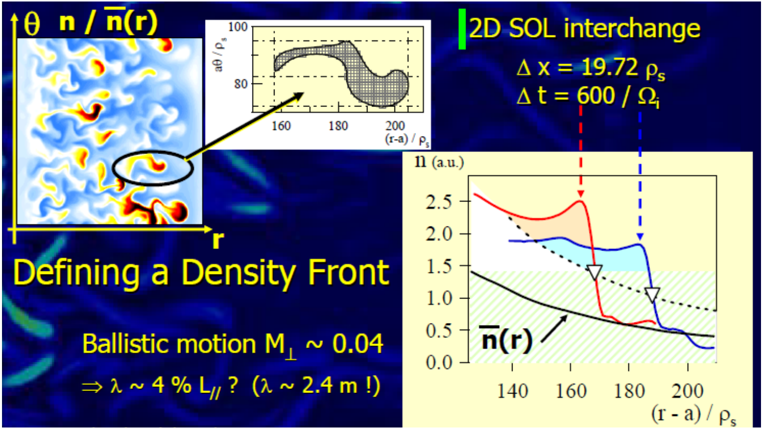
$$\Pi\left(\xi_1, \xi_2, s \right) = \frac{\left| \xi_2 + s\xi_1 \right| e^{(t-s)\left| \xi_2 + s\xi_1 \right| - t\left| \xi_2 \right|}}{N\left(\xi_1, \xi_2, t \right)}$$

- Solutions also for Poisson-Vlasov in an external magnetic field (Fourier and configuration space)

The SOL equations (non-polynomial interactions)

(Ph. Ghendrih, RVM)
Transport and turbulence in the scrape-off layer (SOL) region



The SOL equations

SOLEDGE

$$\begin{aligned}\partial_t N + \frac{1}{q}\partial_\theta \Gamma + \frac{\chi}{\eta} N &= D\partial_r^2 N \\ \partial_t \Gamma + \frac{1}{q}\left(1-\chi\right)\partial_\theta \left(\frac{\Gamma^2}{N} + N\right) + \frac{\chi}{\eta}\left(\Gamma - \Gamma_0\right) &= \nu \partial_r^2 \Gamma\end{aligned}$$

Γ and N are the dimensionless parallel momentum and density, (r, θ) the radial and poloidal coordinates and the mask function χ equals 1 in a region where an obstacle is located and zero elsewhere.

TOKAM2D

$$\begin{aligned}\frac{\partial}{\partial t} n &= S - \{\phi, n\} - \sigma n e^{\Lambda - \phi} + D\Delta_\perp n \\ \frac{\partial}{\partial t}\Delta_\perp \phi &= \sigma\left(1 - e^{\Lambda-\phi}\right) + \nu \Delta_\perp^2 \phi - \{\phi, \Delta_\perp \phi\} - \frac{1}{n} g \partial_y n\end{aligned}$$

$n = \frac{N}{N_0}$ is the normalized density field and $\phi = \frac{eU}{T_e}$ the normalized electric potential. Poisson brackets: $\{f, g\} = \partial_{x_1} f \partial_{x_2} g - \partial_{x_2} f \partial_{x_1} g$, with $x_1 = (r - a)/\rho_s$ the minor radius normalized by the Larmor radius $\rho_s^2 = T_e/m_i$ and $x_2 = a\theta/\rho_s$, a being the plasma radius.

The SOLEDGE equation

Dealing with non-polynomial terms: Taylor expansions and operator labels at the branching points

SOLEDGE ($\chi = 0$)

$$
\begin{aligned}
N(t,r,\theta) &= e^{tD\partial_r^2} N(0,r,\theta) - \frac{1}{q}\int_0^t d\tau e^{\tau D\partial_r^2}\partial_\theta \Gamma(t-\tau,r,\theta) \\
\Gamma(t,r,\theta) &= e^{t\nu\partial_r^2}\Gamma(0,r,\theta) - \frac{1}{q}\int_0^t d\tau e^{\tau\nu\partial_r^2}\partial_\theta\left\{\frac{\Gamma^2}{N}+N\right\}(t-\tau,r,\theta)
\end{aligned}
$$

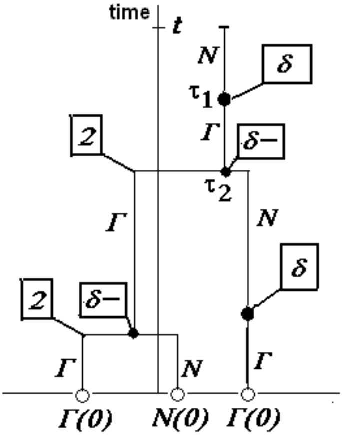
Denote by $\xi_s^{(N)}$ and $\xi_s^{(\Gamma)}$ two Brownian motions in the $r-$coordinate with diffusion coefficients $\sqrt{2D}$ and $\sqrt{2\nu}$. Then the equations may be reinterpreted as defining probabilistic processes for which the expectation values are the functions $N(t,r,\theta)$ and $\Gamma(t,r,\theta)$

The SOLEDGE equation

$$
\begin{aligned}
N(t,r,\theta) &= \mathbb{E}_{(t,r,\theta)}\left[p\frac{1}{p}N\left(0,\xi_t^{(N)},\theta\right)\right. \\
&\left. -\frac{t}{q(1-p)}\int_0^t \frac{1-p}{t}d\tau\partial_\theta\Gamma\left(t-\tau,\xi_\tau^{(N)},\theta\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(t,r,\theta) &= \mathbb{E}_{(t,r,\theta)}\left[p\frac{1}{p}\Gamma\left(0,\xi_t^{(\Gamma)},\theta\right)\right. \\
&\left. -\frac{2t}{q(1-p)}\int_0^t \frac{1-p}{t}d\tau\partial_\theta\left\{\frac{1}{2}\frac{\Gamma^2}{N}+\frac{1}{2}N\right\}\left(t-\tau,\xi_\tau^{(\Gamma)},\theta\right)\right]
\end{aligned}
$$

The SOLEDGE equation



The SOLEDGE equation

The contribution of this sample path to the $N-$expectation value is

$$\partial_\theta^2 \left\{ \left(\partial_\theta \left\{ \frac{\Gamma^2\left(0, r_0^{(1)}, \theta\right)}{N\left(0, r_0^{(2)}, \theta\right)} \right\} \right)^2 \left\{ \partial_\theta \Gamma\left(0, r_0^{(3)}, \theta\right) \right\}^{-1} \right\}$$

times the factor $\left(\frac{1}{p}\right)^3 \frac{4t\tau_1\tau_2^2}{q^4(1-p)^4}$.

If the initial conditions $\left|\frac{\Gamma^2}{N}, N, \Gamma\right|$ and all its derivatives are bound by a constant M, a worst case analysis implies that almost sure convergence of the expectation value is guaranteed for

$$\frac{t}{q} M < 1$$

Fractional processes (F. Cipriano, H. Ouerdiane, RVM)

A fractional version of the KPP equation

$$\boxed{{}_tD_*^{\alpha}u(t,x) = \frac{1}{2}\,{}_xD_\theta^{\beta}u(t,x) + u^2(t,x) - u(t,x)} \qquad (6)$$

${}_tD_*^{\alpha}$ is a Caputo derivative of order α

$${}_tD_*^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-\beta)}\int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} & m-1<\alpha<m \\ \frac{d^m}{dt^m}f(t) & \alpha = m \end{cases}$$

${}_xD_\theta^{\beta}$ is a Riesz-Feller derivative defined through its Fourier symbol

$$\mathcal{F}\left\{{}_xD_\theta^{\beta}f(x)\right\}(k) = -\psi_\beta^\theta(k)\,\mathcal{F}\{f(x)\}(k)$$

with $\psi_\beta^\theta(k) = |k|^\beta e^{i(\text{sign}k)\theta\pi/2}$.
Physically it describes a nonlinear diffusion with growing mass and in our fractional generalization it would represent the same phenomenon taking into account memory effects in time and long range correlations in space.

A fractional nonlinear equation

The first step towards a probabilistic formulation is the rewriting of Eq.(6) as an integral equation.Take the Fourier transform $(\mathcal{F})$ in space and the Laplace transform $(\mathcal{L})$ in time

$$s^{\alpha}\tilde{\hat{u}}(s,k) = s^{\alpha-1}\hat{u}(0^{+},k) - \frac{1}{2}\psi_{\beta}^{\theta}(k)\,\tilde{\hat{u}}(s,k) - \tilde{\hat{u}}(s,k) + \int_{0}^{\infty} dt e^{-st}\mathcal{F}\left(u^{2}\right)$$

where

$$\hat{u}(t,k) = \mathcal{F}(u(t,x)) = \int_{-\infty}^{\infty} e^{ikx}u(t,x)$$

$$\tilde{u}(s,x) = \mathcal{L}(u(t,x)) = \int_{0}^{\infty} e^{-st}u(t,x)$$

This equation holds for $0 < \alpha \leq 1$ or for $0 < \alpha \leq 2$ with $\frac{\partial}{dt}u(0^{+},x) = 0$.

Solving for $\tilde{\hat{u}}(s,k)$ one obtains an integral equation

$$\tilde{\hat{u}}(s,k) = \frac{s^{\alpha-1}}{s^{\alpha}+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\hat{u}(0^{+},k) + \int_{0}^{\infty} dt \frac{e^{-st}}{s^{\alpha}+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\mathcal{F}\left(u^{2}(t,x)\right)$$

A fractional nonlinear equation

Taking the inverse Fourier and Laplace transforms

$$
\begin{aligned}
& u\left(t,x\right) \\
= \; & \boxed{E_{\alpha,1}\left(-t^{\alpha}\right)} \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}\left(k\right)\right)t^{\alpha}\right)}{E_{\alpha,1}\left(-t^{\alpha}\right)}\right)\left(x-y\right)u\left(0,y\right) \\
& + \int_{0}^{t} d\tau \boxed{\left(t-\tau\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\left(t-\tau\right)^{\alpha}\right)} \\
& \int_{-\infty}^{\infty} dy \mathcal{F}^{-1}\left(\frac{E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}\left(k\right)\right)\left(t-\tau\right)^{\alpha}\right)}{E_{\alpha,\alpha}\left(-\left(t-\tau\right)^{\alpha}\right)}\right)\left(x-y\right)u^{2}\left(\tau,y\right)
\end{aligned}
$$

$E_{\alpha,\rho}$ is the generalized Mittag-Leffler function $E_{\alpha,\rho}\left(z\right)=\sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma\left(\alpha j+\rho\right)}$

$$
E_{\alpha,1}\left(-t^{\alpha}\right)+\int_{0}^{t} d\tau \left(t-\tau\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\left(t-\tau\right)^{\alpha}\right)=1
$$

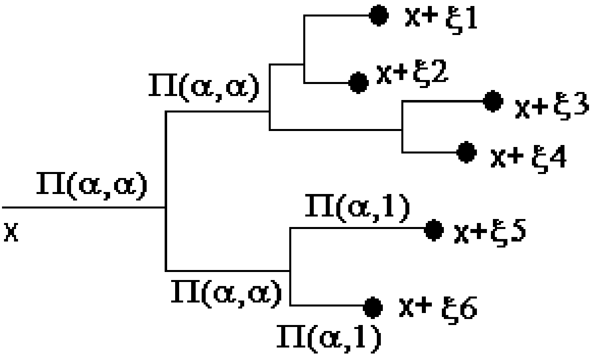
A fractional nonlinear equation

We define the following propagation kernel

$$G_{\alpha,\rho}^{\beta}(t,x) = \mathcal{F}^{-1}\left(\frac{E_{\alpha,\rho}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)t^{\alpha}\right)}{E_{\alpha,\rho}(-t^{\alpha})}\right)(x)$$

$$\begin{aligned} & u(t,x) \\ = \; & \boxed{E_{\alpha,1}(-t^{\alpha})}\int_{-\infty}^{\infty} dy \boxed{G_{\alpha,1}^{\beta}(t,x-y)} u(0^{+},y) \\ & + \int_{0}^{t} d\tau \boxed{(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)} \\ & \int_{-\infty}^{\infty} dy \boxed{G_{\alpha,\alpha}^{\beta}(t-\tau,x-y)} u^{\boxed{2}}(\tau,y) \end{aligned}$$

$E_{\alpha,1}(-t^{\alpha})$ and $(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)$ = survival probability up to time t and the probability density for the branching at time τ (branching process B_{α})


x+ ξ1
x+ξ2
x+ξ3
x+ ξ4
x+ξ5
x+ ξ6
Π(α,α)
Π(α,α)
Π(α,α)
Π(α,1)
Π(α,1)
x

A fractional nonlinear equation

$$u(t,x) = \mathbb{E}_x \left(\varphi_1 \varphi_2 \cdots \varphi_n \right)$$

with

$$\begin{aligned}\varphi_i &= \int dy_1^{(i)} dy_2^{(i)} \cdots dy_{k-1}^{(i)} dy_k^{(i)} G_{\alpha,\alpha}^{\beta}(\tau_1, x-y_1)\, G_{\alpha,\alpha}^{\beta}(\tau_2, y_1-y_2) \cdots \\ &\cdots G_{\alpha,\alpha}^{\beta}(\tau_{k-1}, y_{k-2}-y_{k-1})\, G_{\alpha,1}^{\beta}(\tau_k, y_{k-1}-y_k)\, u\left(0^+, y_k\right)\end{aligned}$$

with $\sum_{i=1}^{k} \tau_j = t$, $k-1$ being the number of branchings leading to particle i

The propagation kernels satisfy the conditions to be the Green's functions of stochastic processes in $\mathbb{R}$:

$$u(t,x) = \mathbb{E}_x \left(u(0^+, x+\xi_1) u(0^+, x+\xi_2) \cdots u(0^+, x+\xi_n) \right)$$

A fractional nonlinear equation

Denote the processes associated to $G^{\beta}_{\alpha,1}(t,x)$ and $G^{\beta}_{\alpha,\alpha}(t,x)$, respectively by $\Pi^{\beta}_{\alpha,1}$ and $\Pi^{\beta}_{\alpha,\alpha}$

Proposition: *The nonlinear fractional partial differential equation (6), with $0 < \alpha \leq 1$, has a stochastic solution, the coordinates $x + \xi_i$ in the arguments of the initial condition obtained from the exit values of a propagation and branching process, the branching being ruled by the process B_α and the propagation by $\Pi^{\beta}_{\alpha,1}$ for the first particle and by $\Pi^{\beta}_{\alpha,\alpha}$ for all the remaining ones.*

A sufficient condition for the existence of the solution is

$$\left| u(0^+, x) \right| \leq 1$$

The Green´s functions and characterization of the processes

The processes $\Pi^{\beta}_{\alpha,1}$ and $\Pi^{\beta}_{\alpha,\alpha}$

$$\mathcal{F}\left\{G^{\beta}_{\alpha,1}\left(t,x\right)\right\}\left(t,k\right)=\frac{E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}\left(k\right)\right)t^{\alpha}\right)}{E_{\alpha,1}\left(-t^{\alpha}\right)}$$

$$\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}\left(t,x\right)\right\}\left(t,k\right)=\frac{E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}\left(k\right)\right)t^{\alpha}\right)}{E_{\alpha,\alpha}\left(-t^{\alpha}\right)}$$

For a propagation kernel $G\left(t,x\right)$ to be the Green's function of a stochastic process, the following conditions should be satisfied:
(i) $G\left(0,x-y\right)=\delta\left(x-y\right)$ or $\mathcal{F}\left\{G\right\}\left(0,k\right)=1\ \forall k$
(ii) $\int dxG\left(t,x\right)=1\ \forall t$ or $\mathcal{F}\left\{G\right\}\left(t,0\right)=1$
(iii) $G\left(t,x\right)$ should be real and ≥ 0

The Green´s functions and characterization of the processes

For the processes $\Pi^{\beta}_{\alpha,1}$ and $\Pi^{\beta}_{\alpha,\alpha}$

(i) $\mathcal{F}\left\{G^{\beta}_{\alpha,1}\right\}(0,k)=\frac{E_{\alpha,1}(0)}{E_{\alpha,1}(0)}=1$ and $\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}\right\}(0,k)=\frac{E_{\alpha,\alpha}(0)}{E_{\alpha,\alpha}(0)}=1$

(ii) $\mathcal{F}\left\{G^{\beta}_{\alpha,1}\right\}(t,0)=\frac{E_{\alpha,1}(-t^{\alpha})}{E_{\alpha,1}(-t^{\alpha})}=1$ and $\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}\right\}(t,0)=\frac{E_{\alpha,\alpha}(-t^{\alpha})}{E_{\alpha,\alpha}(-t^{\alpha})}=1$

(iii) If $\mathcal{F}\left\{G\right\}(t,-k)=\left(\mathcal{F}\left\{G\right\}(t,k)\right)^{*}$ then $G(t,x)$ is real.

Because $\psi^{\theta}_{\beta}(-k)=\left(\psi^{\theta}_{\beta}(k)\right)^{*}$ it follows

$$E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(-k)\right)t^{\alpha}\right)=\left(E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)\right)^{*}$$

$$E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(-k)\right)t^{\alpha}\right)=\left(E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)\right)^{*}$$

implying that both $G^{\beta}_{\alpha,1}(t,x)$ and $G^{\beta}_{\alpha,\alpha}(t,x)$ are real.

The Green´s functions and characterization of the processes

Finally, for the positivity, one notices that for $0 < \alpha \leq 1$ and $\rho \geq \alpha$, $E_{\alpha,\rho}(-x)$ is a completely monotone function. Therefore

$$E_{\alpha,\rho}(-x) = \int_0^\infty e^{-rx} dF(r)$$

with F nondecreasing and bounded. For $G^{\beta}_{\alpha,\rho}(t,x)$ ($\rho = 1$ and $\rho = \alpha$) one has

$$\begin{aligned} G^{\beta}_{\alpha,\rho}(t,x) &= \frac{1}{2\pi E_{\alpha,\rho}(-t^{\alpha})} \int_0^\infty dF(r) \int_{-\infty}^{\infty} dk e^{-ikx} e^{-rt^{\alpha}\left(1+\frac{1}{2}\psi^{\theta}_{\beta}(-k)\right)} \\ &= \frac{1}{2\pi E_{\alpha,\rho}(-t^{\alpha})} \int_0^\infty dF(r)\, e^{-rt^{\alpha}} \int_{-\infty}^{\infty} dk e^{-ikx} e^{-\frac{rt^{\alpha}}{2}\psi^{\theta}_{\beta}(-k)} \end{aligned}$$

We recognize the last integral (in k) as the Green's function of a Levy process. Therefore one has an integral in r of positive quantities implying that $G^{\beta}_{\alpha,1}(t,x)$ and $G^{\beta}_{\alpha,\alpha}(t,x)$ are positive.

The Green´s functions and characterization of the processes

The process B_α

The decaying probability in time $d\tau$ of this process is

$$\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\tau^{\alpha}\right)$$

From

$$\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\tau^{\alpha}\right)d\tau = 1 - E_{\alpha,1}\left(-t^{\alpha}\right)$$

it follows that $E_{\alpha,1}\left(-t^{\alpha}\right)$ is the survival probability up to time t. The process B_α is a fractional generalization of the exponential process.

Superprocesses

- $\mathcal{S}$ = the Schwartz space of functions of rapid decrease on E
- $\mathcal{U} \subset \mathcal{S}$, functions in $\mathcal{S}$ that may be extended into the complex plane as entire functions of rapid decrease on strips.
- $\mathcal{U}'$, the dual of $\mathcal{U}$, (Silva's space of tempered ultradistributions), which can also be characterized as the space of all Fourier transforms of distributions of exponential type
- Restrict further to the space $\mathcal{U}_0'$ **of tempered ultradistributions of compact support.**
- $(X_t, P_{0,\nu})$ a **branching stochastic process with values** in $\mathcal{U}_0'$ and **transition probability** $P_{0,\nu}$ starting from time 0 and $\nu \in \mathcal{U}_0'$.
- The process satisfies the **branching property** if given $\nu = \nu_1 + \nu_2$

$$P_{0,\nu} = P_{0,\nu_1} * P_{0,\nu_2}$$

that is, after the branching $\left(X_t^1, P_{0,\nu_1}\right)$ and $\left(X_t^2, P_{0,\nu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $(X_t, P_{0,\nu})$.

Superprocesses

- For the **transition operator** V_t operating on functions on $\mathcal{U}$ the branching property is

$$\langle V_t f, \nu_1 + \nu_2 \rangle = \langle V_t f, \nu_1 \rangle + \langle V_t f, \nu_2 \rangle$$

with $e^{-\langle V_t f, \nu \rangle} \triangleq P_{0,\nu} e^{-\langle f, X_t \rangle}$

$$\langle V_t f, \nu \rangle = -\log P_{0,\nu} e^{-\langle f, X_t \rangle} \qquad f \in \mathcal{U}, \nu \in \mathcal{U}_0'$$

- In the usual construction of superprocesses on measures, one starts from an initial δ_x which branches into other $\delta' s$ with, at most, some scaling factors. The restriction to $\mathcal{U}_0'$ preserves this pointwise interpretation. Any ultradistribution in $\mathcal{U}_0'$ has a multipole expansion at any point of its support (a series of $\delta' s$ and their derivatives)
- In $M = [0, \infty) \times E$ consider a set $Q \subset M$ and the associated exit process $\xi = (\xi_t, \Pi_{0,x})$ with parameter k defining the lifetime. The process stars from $x \in E$ carrying along an ultradistribution in $\mathcal{U}_0'$ with support on the path.

- At each branching point of the ξ_t-process there is a transition ruled by the P probability in $\mathcal{U}_0'$ leading to one or more elements in $\mathcal{U}_0'$. These $\mathcal{U}_0'$ elements are then carried along by the new paths of the ξ_t-process. The whole process stops at the boundary ∂Q, defining a exit process $(X_Q, P_{0,\nu})$ on $\mathcal{U}_0'$. If the initial ν is δ_x
$$u(x) = \langle V_Q f, \nu \rangle = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$$
$\langle f, X_Q \rangle$ is computed on the (space-time) boundary with the exit ultradistribution generated by the process.
- **Connection to nonlinear pde's** established by defining the whole process to be a (ξ, ψ)-*superprocess* if $u(x)$ satisfies the equation
$$u + G_Q \psi(u) = K_Q f \tag{7}$$
$$G_Q f(r, x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s)\, ds\ ; \qquad K_Q f(x) = \Pi_{0,x} 1_{\tau<\infty} f(\xi_\tau)$$
$\psi(u)$ means $\psi(0, x; u(0, x))$ and τ is the first exit time from Q.

Construction of the superprocess: Let $\varphi(s, x; z)$ be the branching function at time s and point x. Then, with $P_{0,x} e^{-\langle f, X_Q \rangle} \circeq e^{-w(0,x)}$

$$e^{-w(0,x)} = \Pi_{0,x} \left[e^{-k\tau} e^{-f(\tau,\xi_\tau)} + \int_0^\tau ds k e^{-ks} \varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right) \right] \quad (8)$$

τ is the first exit time from Q and $f(\tau, \xi_\tau) = \langle f, X_Q \rangle$ is computed with the exit boundary ultradistribution. *For measure-valued superprocesses*

$$\varphi(s, y; z) = c \sum_0^\infty p_n(s, y) z^n$$

with $\sum_n p_n = 1$*, but now it may be a more general function.*
Using $\int_0^\tau k e^{-ks} ds = 1 - e^{-k\tau}$ and the Markov property $\Pi_{0,x} 1_{s<\tau} \Pi_{s,\xi_s} = \Pi_{0,x} 1_{s<\tau}$ Eq.(8) is converted into

$$e^{-w(0,x)} = \Pi_{0,x} \left[e^{-f(\tau,\xi_\tau)} + k \int_0^\tau ds \left[\varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right) - e^{-w(\tau-s,\xi_s)} \right] \right] \quad (9)$$

- Eq.(7) is now obtained by a limiting process. Let in (9) replace $w(0,x)$ by $\beta w_\beta(0,x)$ and f by βf. β is interpreted as the mass of the particles and when $X_Q \to \beta X_Q$ then $P_\mu \to P_{\frac{\mu}{\beta}}$.

$$e^{-\beta w(0,x)} =$$

$$\Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta \int_0^\tau ds \left[\varphi_\beta\left(s,\xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right]$$

- *Scaling limit* (first type)

$$u_\beta^{(1)} = \left(1 - e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta^{(1)} = \left(1 - e^{-\beta f}\right)/\beta$$

$$\psi_\beta^{(1)}\left(0,x;u_\beta^{(1)}\right) = \frac{k_\beta}{\beta}\left(\varphi\left(0,x;1-\beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right)$$

Superprocesses

$$u_\beta^{(1)}(0,x)+\Pi_{0,x}\int_0^\tau ds\psi_\beta^{(1)}\left(s,\xi_s;u_\beta^{(1)}\right)=\Pi_{0,x}f_\beta^{(1)}(\tau,\xi_\tau)$$

that is

$$u_\beta^{(1)}+G_Q\psi_\beta^{(1)}\left(u_\beta^{(1)}\right)=K_Qf_\beta^{(1)}$$

When $\beta\longrightarrow 0$, $f_\beta^{(1)}\longrightarrow f$ and if ψ_β goes to a well defined limit ψ then u_β tends to a limit u solution of (7) associated to a superprocess. Also one sees from that in the $\beta\longrightarrow 0$ limit

$$u_\beta^{(1)}\longrightarrow w_\beta=-\log P_{0,x}e^{-\langle f,X_Q\rangle}$$

The superprocess corresponds to a cloud of particles for which both the mass and the lifetime tend to zero

Superprocesses on measures

Restrict to measure-valued superprocesses, that is, in terms of paths, to $\delta' s$ propagating along the paths of the $(\xi_t, \Pi_{0,x})$ process and branching to new δ measures at each branching point. Let us construct a superprocess providing a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^\alpha$$

for $1 < \alpha \leq 2$. Comparing with (7) one should have

$$\psi(0, x; u) = u^\alpha$$

Then, with $z = 1 - \beta u_\beta^{(1)}$ one has

$$\begin{aligned} \varphi(0, x; z) &= \textstyle\sum_n p_n z^n = z + \frac{\beta}{k_\beta} u_\beta^{(1)\alpha} = z + \frac{\beta}{k_\beta}\frac{(1-z)^\alpha}{\beta^\alpha} \\ &= z + \frac{1}{k_\beta \beta^{\alpha-1}}\left(1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots\right) \end{aligned}$$

Choosing $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ the terms in z cancel and for $1 < \alpha \leq 2$ the coefficients of all z powers are positive and may be interpreted as branching probabilities p_n into new $\delta' s$

$$p_0 = \frac{1}{\alpha}; \quad p_1 = 0; \quad \cdots \quad p_n = \frac{(-1)^n}{\alpha} \begin{pmatrix} \alpha \\ n \end{pmatrix}; \qquad \sum_n p_n = 1$$

With $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ and $\beta \to 0$ the superprocess provides a solution to

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^\alpha$$

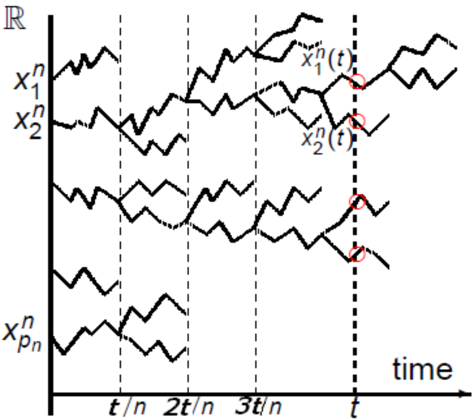
$\alpha = 2$ is an upper bound for this representation, because for $\alpha > 2$ some of the $p_n' s$ would be negative. **For the particular case**

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u^2$$

$$p_1 = 0; \quad p_0 = p_2 = \frac{1}{2}; \quad k_\beta = \frac{2}{\beta}$$

Superprocesses and a nonlinear heat equation

$\alpha = 2$



Superprocesses on measures: other limits

Superprocesses are usually associated with nonlinear pde's in the scaling limit $\beta \longrightarrow 0$. However other limits may also be useful. For example with with $p_n = \delta_{n,2}$, $\beta = 1$ and $k_\beta = 1$ one obtains

$$\begin{aligned}\psi_\beta^{(1)}\left(0, x; u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\sum p_n \left(1 - \beta u_\beta^{(1)}\right)^n - 1 + \beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\beta^2 u_\beta^{(1)2} - \beta u_\beta^{(1)}\right) \longrightarrow u^2 - u\end{aligned}$$

In this case, one is led to the KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2 + u$$

Because $\beta = 1$ instead of $\beta \longrightarrow 0$, the solution is given by $(1 - e^{-w})$ instead of $u_\beta^{(1)} \longrightarrow w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$. Although the solution of KPP may be obtained by another method, interpretation as an exit measure allows for the construction of solutions with arbitrary boundary conditions.

Superprocesses on ultradistributions

- Superprocesses on measures allows the construction of solutions for equations which do not possess a natural Poisson clock. It has the severe limitation of requiring a polynomial branching function $\varphi(s, x; z)$. Restricts the nonlinear terms in the pde's to be powers of u (u^a). In addition, these terms must be such that all coefficients in the z^n expansion be positive $(1 < \alpha \leq 2)$.

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- The variable z in $\varphi_\beta(s, x; z)$ is $z = e^{-\beta w(\tau - s, \xi_s)} = P_{0,x} e^{-\langle \beta f, X \rangle}$. When one generalizes to $\mathcal{U}_0'$, changes of sign and transitions from deltas to their derivatives are allowed. There are basically two new transitions at the branching points:

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- The variable z in $\varphi_\beta(s, x; z)$ is $z = e^{-\beta w(\tau - s, \xi_s)} = P_{0,x} e^{-\langle \beta f, X \rangle}$. When one generalizes to $\mathcal{U}'_0$, changes of sign and transitions from deltas to their derivatives are allowed. There are basically two new transitions at the branching points:
- 1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \longrightarrow e^{\langle \beta f, -\delta_x \rangle} = e^{-\beta f(x)}$$

which corresponds to

$$z \longrightarrow \frac{1}{z}$$

- 2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \longrightarrow e^{\left\langle \beta f, \pm \delta'_x \right\rangle} = e^{\mp \beta f'(x)}$$

which corresponds to

$$z \longrightarrow e^{\mp \partial_x \log z}$$

Superprocesses on ultradistributions

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  which corresponds to

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- Case 1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and the second to superprocesses in $\mathcal{U}'_0$.
  How these transformations provide stochastic representations of solutions for other classes of pde's, will be illustrated by two examples

Superprocesses on ultradistributions: Examples

$$\varphi^{(1)}(0,x;z) = p_1 e^{\partial_x \log z} + p_2 e^{-\partial_x \log z} + p_3 z^2$$

This branching function means that at the branching point, with probability p_1 a derivative is added to the propagating ultradistribution, with probability p_2 a derivative is added plus a change of sign and with probability p_3 the ultradistribution branches into two identical ones. Using the transformation and scaling limit one has, for small β

$$\begin{aligned} z &\rightarrow e^{\mp\partial_x \log z} = e^{\mp\partial_x \log\left(1-\beta u_\beta^{(1)}\right)} \\ &= 1 \pm \beta\partial_x u_\beta^{(1)} + \frac{\beta^2}{2}\left\{\left(\partial_x u_\beta^{(1)}\right)^2 \pm \partial_x u_\beta^{(1)2}\right\} + O\left(\beta^3\right) \end{aligned}$$

$$z \rightarrow z^2 = \left(1-\beta u_\beta^{(1)}\right)^2 = 1 - 2\beta u_\beta^{(1)} + \beta^2 u_\beta^{(1)2}$$

Computing $\psi_\beta\left(0,x;u_\beta^{(1)}\right)$ with $p_1=p_2=\frac{1}{4}$ and $p_3=\frac{1}{2}$ one obtains

$$\begin{aligned}\psi_\beta^{(1)}\left(0,x;u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\varphi^{(1)}\left(0,x;1-\beta u_\beta^{(1)}\right)-1+\beta u_\beta^{(1)}\right)\\ &= \frac{k_\beta}{\beta}\left(\frac{1}{8}\beta^2\left(\partial_x u_\beta^{(1)}\right)^2+\frac{1}{2}\beta^2 u_\beta^{(1)2}+O\left(\beta^3\right)\right)\end{aligned}$$

meaning that, with $k_\beta=\frac{4}{\beta}$, the superprocess provides, in the $\beta\longrightarrow 0$ limit, a solution to the equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-2u^2-\frac{1}{2}\left(\partial_x u\right)^2$$

Superprocesses on ultradistributions: Examples

For the second example a different scaling limit will be used, namely

$$u_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta w_\beta} - e^{-\beta w_\beta}\right) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta f} - e^{-\beta f}\right)$$

Notice that, as before, $u_\beta^{(2)} \to w_\beta$ and $f_\beta^{(2)} \to f$ when $\beta \to 0$. In this case with $z = e^{\beta w_\beta}$ one has

$$\begin{aligned} z &= -2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 - 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{z} &= 2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 + 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned}$$

For the integral equation one has

$$u_{\beta}^{(2)}(0,x)+\Pi_{0,x}\int_{0}^{\tau}ds\psi_{\beta}^{(2)}\left(s,\xi_{s};u_{\beta}^{(2)}\right)=\Pi_{0,x}f_{\beta}^{(2)}(\tau,\xi_{\tau})$$

with

$$\psi_{\beta}^{(2)}\left(0,x;u_{\beta}^{(2)}\right)=k_{\beta}\left(\frac{1}{2\beta}\left(\varphi(0,x;z)-\varphi\left(0,x;\frac{1}{z}\right)\right)-u_{\beta}^{(2)}\right)$$

Superprocesses on ultradistributions: Examples

Let now

$$\varphi^{(2)}\left(0,x;z\right)=p_1 z^2+p_2\frac{1}{z}$$

This branching function means that with probability p_1 the ultradistribution branches into two identical ones and with probability p_2 it changes its sign. Therefore, in this case, one is simply extending the superprocess construction to signed measures.

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right)=k_\beta\left\{-p_1 8u_\beta^{(2)}\left(1+\frac{1}{2}\beta^2 u_\beta^{(2)2}\right)+p_2 u_\beta^{(2)}-u_\beta^{(2)}+O\left(\beta^4\right)\right.$$

and with $p_1=\frac{1}{10}; p_2=\frac{9}{10}$ and $k_\beta=\frac{5}{2\beta^2}$ one obtains in the in the $\beta\rightarrow 0$ limit

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right)\rightarrow -u_\beta^{(2)3}$$

meaning that this superprocess provides a solution to the equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u^3$$

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