Ergodic parameters and dynamical complexity

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Generalized ergodic parameters: A cocycle formulation

- **Motivation:** Lyapunov exponents are global functions of the invariant measure. However, the invariant measure itself contains more information. Ergodic parameters are averages of local fluctuating quantities. The quantities describing the fluctuations are again ergodic parameters, etc. (Ruelle)
- **Cocycles and the Oseledets theorem;** $f : M \to M$ measure preserving transformation of a Lebesgue space $(M, \mathcal{B}, \mu)$). For any measurable function $g : M \to GL(N, \mathbb{R})$ let
$$C(x,n) = g\left(f^{n-1}(x)\right) \cdots g(x)$$
$$C(x,n+k) = C\left(f^k(x), n\right) C(x,k)$$
$C : M \times \mathbb{Z} \to GL(N, \mathbb{R})$ is called a *cocycle* (over f). Any cocycle has this form. g is the *generator* of the cocycle.
- **Theorem (Oseledets):** If $\ln_+ \|g(x)\| \in L^1(M, \mu)$
(i) $\exists$ a decomposition $\mathbb{R}^N = \oplus_{i=1}^{k(x)} E_i(x)$ invariant under $C(x,n)$,
(ii) $\lim_{n\to\infty} \frac{1}{n} \ln \frac{\|C(x,n)v\|}{\|v\|} = \chi_i(x)$ with $\chi_1(x) < \cdots < \chi_{k(x)}(x)$, exists uniformly in $v \in E_i(x) \diagdown \{0\}$.

Lyapunov, conditional exponents and fluctuation moments

- For the **Lyapunov exponent** the cocycle generator is

$$g_1(x) = Df(x) = \exp(\ln(Df(x)))$$

- **Conditional exponents:** (Pecora, Carroll) In the Jacobian Df use partial blocks

  Existence under the same conditions as the Lyapunov exponents (*Phys. Lett. A248 (1998) 167 - 171*)

- Definition: **Lyapunov fluctuation moments** $\chi_i^{(p)}(x)$
  Are the limits $\lim_{n\to\infty} \frac{1}{n} \ln \frac{\|C(x,n)v\|}{\|v\|} = \chi_i(x)$ *when the cocycle generator is*

$$g_p(x) = \exp\left(\ln_+^p(Df(x))\right) \tag{1}$$

Fluctuation moments

Remark : Definition of the logarithm is understood in the framework of the Oseledets-Pesin $\varepsilon-$reduction theorem. For any $\varepsilon > 0$ there is an invertible map $\Gamma_\varepsilon(x) : M \to GL(N, \mathbb{R})$ such that $g_\varepsilon(x) = \Gamma^{-1}(f(x))\, g(x)\, \Gamma(x)$ has block form and in each block $e^{\chi_i(x)-\varepsilon} \leq \left\| g_\varepsilon^i(x)\, v \right\| \leq e^{\chi_i(x)+\varepsilon}$
Then $g_\varepsilon(x)$ generates a cocycle $C_\varepsilon(x, n)$ equivalent to $C(x, n)$.
$\ln_+$ in (1) is therefore computed without ambiguity in each block and

$$\chi_i^{(p)}(x) = \lim_{n\to\infty} \frac{1}{n} \ln \frac{\left\| g_p\left(f^{n-1}(x)\right) \cdots g_p(x)\, v \right\|}{\|v\|}$$

is an ergodic average of the $p-$moment of the local expansion rate.

Proposition

The Lyapunov fluctuation moments $\chi_i^{(p)}(x)$ exist whenever

$$\ln_+ \left\| g_p(x) \right\| \in L^1(M, \mu)$$

Proof : A consequence of the Oseledets multiplicative ergodic theorem

Lyapunov characteristic fluctuation function

- This cocycle construction provides a unified description of the fluctuation ergodic parameters previously considered by several authors (Crisanti, Fujisaka, Froeschle, Vanneste, Oliveira, ...)
- Existence of the fluctuation moments depends on the integrability of

$$\exp\left(\sum k_i \lambda_i^p (x)\right)$$

$\lambda_i(x)$ the local expansion rate at x ; k_i multiplicity of this rate.

- If the expansion rate variable fails to have moments for large p,

Definition

The *Lyapunov characteristic fluctuation function* $C(\alpha)$ is defined as the $\lim_{n\to\infty} \frac{1}{n} \ln \frac{\|C(x,n)v\|}{\|v\|}$ when the generator of the cocycle is

$$g_\alpha(x) = \exp\left(\exp\left(i\alpha \ln_+ (Df(x))\right)\right)$$

As before, existence of $C(\alpha)$ depends on integrability of $\ln_+ \|g_\alpha(x)\|$ and, because $\exp(i\alpha \ln_+ (Df(x)))$ is bounded, this is always fulfilled.

Correlations and other parameters

Although $C(\alpha)$ contains complete information on the statistical properties of the local fluctuation rate, a full ergodic characterization of the dynamics should also contain information about correlations at different points. The ergodic parameters obtained from the Hessian in a variational formulation

$$A_N = \sum_{\alpha=1}^{d} \sum_{k \geq 1}^{N} \left[x^{\alpha}(t_k) - f^{\alpha}(x(t_{k-1})) \right] \left[x^{\alpha}(t_k) - f^{\alpha}(x(t_{k-1})) \right]$$

$$\frac{1}{2} H_N = \frac{1}{2} \frac{\partial^2 A_N}{\partial x^{\alpha}(t_j) \partial x^{\beta}(t_k)} = \delta_{\alpha,\beta} \delta_{j,k} - (1 - \delta_{k,N}) \delta_{k,j-1} \frac{\partial f^{\alpha}(x(t_k))}{\partial x^{\beta}(t_k)} - (1 - \delta_{j,N}) \delta_{j,k-1} \frac{\partial f^{\beta}(x(t_j))}{\partial x^{\alpha}(t_j)} + \delta_{j,k} (1 - \delta_{j,N}) \frac{\partial f^{\gamma}(x(t_j))}{\partial x^{\beta}(t_j)} \frac{\partial f^{\gamma}(x(t_j))}{\partial x^{\alpha}(t_j)}$$

$$\boxed{\mu_1 = \lim_{N \to \infty} \frac{1}{N} TrH_N ; \; \mu_p = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{p} \binom{p}{k} (-1)^k \, TrH_N^{p-k} \left(\mu_1^{(N)} \right)^k}$$

(*PLA 155 (1991) 388*), already contain some information on the correlations, but a full study of this problem is far from complete.

A tool: correlation cocycles

- Let $f : M \to M$ be a measure preserving transformation of a Lebesgue space $(M, \mathcal{B}, \mu)$). For a measurable function $g : M \to GL(N, \mathbb{R})$ let

  $$C_k(x, n) = g\left(f^{(n-1)k}(x), f^{(n-2)k}(x)\right) \cdots g\left(f^k(x), x\right)$$

  Then

  $$C_k(x, n+p) = C_k\left(f^{pk}(x), n\right) C_k(x, p)$$

  $C_k : M \times \mathbb{Z} \to GL(N, \mathbb{R})$ may be called a *correlation cocycle* (over f)
- If $\ln_+ \left\| g\left(f^k(x), x\right) \right\| \in L^1(M, \mu)$ Oseledets' theorem applies and choosing appropriate functions $g : M \to GL(N, \mathbb{R})$ one obtains *correlation ergodic parameters.*

  Example:

  $$g\left(f^k(x), x\right) = Df\left(f^k x\right) Df(x) - (Df(x))^2$$

Dynamical Rényi entropy and fluctuations of the local expansion rate

- Φ =partition of M; $\left\{\phi_i^{(n)}\right\}$= elements of partition $\Phi_n = \bigvee_{i=0}^{n-1} f^{-i}(\Phi)$
- **Dynamical Rényi entropy of order α**

$$K(\alpha) = \sup_{\Phi} \left\{ \lim_{n\to\infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \sum_i \mu\left(\phi_i^{(n)}\right)^{\alpha} \right\}$$

Related to what some authors (Fujisaka, Benzi-Paladini-Parisi-Vulpiani, Grassberger-Procaccia) call generalized Lyapunov exponents

- An easier to compute (not necessarily equivalent) definition :

$$K_B(\alpha) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \sum_{i_0\cdots i_{n-1}} \left(p\left(i_0\cdots i_{n-1}\right)\right)^{\alpha}$$

$p(i_0 \cdots i_{n-1})$ is the joint probability to be at the box i_0 at time 0, to be at box i_1 at time 1, $\cdots$, and to be at box i_{n-1} at time $n-1$, the sum being over all different blocks of length n.

(Invariant measure absolutely continuous with respect to Lebesgue)

Estimating Rényi entropies from the local expansion rate

Local expansion rate $\Lambda(x) = \prod_{\lambda_i > 0} e^{\lambda_i(x)} \Longrightarrow$ If the system is in box i_0 at time 0, it can go to $\Lambda(i_0)$ boxes in the next step, then to $\Lambda(i_0)\Lambda(i_1)$ boxes, etc. ($\Lambda(i_k)$ the average expansion rate in the box i_k and $\mu(i_0)$ the measure of the box i_0)

$$p(i_0 \cdots i_{n-1}) = \frac{\mu(i_0)}{\Lambda(i_o) \cdots \Lambda(i_{n-2})}$$

$$K_B(\alpha) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \sum_{i_0 \cdots i_{n-1}} \left(\frac{\mu(i_0)}{\Lambda(i_o) \cdots \Lambda(i_{n-2})} \right)^{\alpha}$$

Considering average values and normalizing to obtain $\sum_{i_0 \cdots i_{n-1}} p(i_0 \cdots i_{n-1}) = \mu(i_0)$ in $\alpha \to 1$ limit

$$K_B(\alpha) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{1-\alpha} \frac{1}{n} \ln \left\langle \mu(i_0)^{\alpha} \left(\frac{1}{\Lambda(i_o) \cdots \Lambda(i_{n-2})} \right)^{\alpha - 1} \right\rangle$$

Estimating Rényi entropies from the local expansion rate

In the $\lim_{n\to\infty}$

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{1-\alpha}\frac{1}{n}\ln\left\langle \exp\left((1-\alpha)\sum_{k=0}^{n-2}\ln\Lambda(i_k)\right)\right\rangle$$

$(1-\alpha)K(\alpha)$ is the *pressure function* for the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-2}\ln\Lambda(i_k)$. The Legendre transform

$$I(y) = \sup_{\alpha}\left\{(1-\alpha)y - (1-\alpha)K(\alpha)\right\}$$

is the *deviation function* for the large deviations of the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-2}\ln\Lambda(i_k)$

$$P_n\left\{\frac{1}{n}\sum_{k=0}^{n-2}\ln\Lambda(i_k)\in(y, y+dy)\right\} \asymp \exp(-nI(y))\,dy$$

Estimating Rényi entropies from the local expansion rate

Proposition

(i)The Legendre transform of the (box) dynamical Rényi entropy is the deviation function for the local expansion rate.
(ii) If a weak correlation condition is verified, namely

$$\left\langle \exp\left((1-\alpha)\sum_{k=0}^{n-2}\ln\Lambda(i_k)\right)\right\rangle \prod_{k=0}^{n-2}\left\langle \exp\left((1-\alpha)\ln\Lambda(i_k)\right)\right\rangle^{-1} \leq c_1 e^{c_2 n^{\gamma}}$$

$c_2 > 0$ *and* $\gamma < 1$

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\frac{1}{1-\alpha}\ln\left\langle \exp\left((1-\alpha)\ln\Lambda(i)\right)\right\rangle$$

$$K_B(\alpha) = \lim_{\varepsilon\to 0}\sum_{s=1}^{\infty} k_s(\ln\Lambda)(1-\alpha)^{s-1}$$

where $k_s(\ln\Lambda)$ *are the cumulants of the local expansion rate.*

Ergodic parameters, measures of complexity and self-organization

- **A (well-known) characterization of complexity**
  Excess entropy or *effective measure complexity*
  $p_N(s_1 \cdots s_n)$ the probability to find the block $s_1 \cdots s_n$ of size n

$$H(n) = -\sum_{\{s_i\}} p_n(s_1 \cdots s_n) \log p_n(s_1 \cdots s_n)$$

$h_s = \lim_{n\to\infty} \frac{1}{n} H(n)$ being the *Shannon entropy*.
Excess entropy E (effort needed to construct a model of the system)

$$E = \sum_n \left(\frac{1}{n} H(n) - h_s \right)$$

Is a measure of the diversity of dynamical structures

Ergodic parameters, measures of complexity and self-organization

Finite-time fluctuations in Lyapunov exponents, are a symptom of *diversity of dynamical structures*. $\Rightarrow$ A dynamical version of the excess entropy ?

Use the large deviation principle

Legendre transform $I(y)$ of $(1-\alpha)K(\alpha)$ is the deviation function of the random variable $Y_n = \frac{1}{n}\sum_{k=0}^{n-2} \ln \Lambda(i_k)$. Average value of Y_n is an estimate of $\frac{1}{n}H(n)$. Therefore, **a dynamical version of the excess entropy** is

$$E_e = \sum_n \left\{ \int_0^\infty y P_n(y)\, dy - y_{I_{\min}} \right\}$$

with $y_{I_{\min}}$ being the value that minimizes $I(y)$ and

$$P_n(y) = \frac{e^{-nI(y)}}{\int_0^\infty e^{-nI(y)} dy}$$

E_e may be computed from the ergodic parameters that define the fluctuations of the local expansion rate. (*arXiv:1008.2664*)

Structure and self-organization

- Structure index

$$S = \frac{1}{N} \sum_{i=1}^{N_+} \left(\frac{\lambda_0}{\lambda_i} - 1 \right)$$

diverges whenever a Lyapunov exponent approaches zero from above (points where long time correlations develop)

- Self-organization (partitions $\Sigma_k = R^k \times R^{m-k}$)

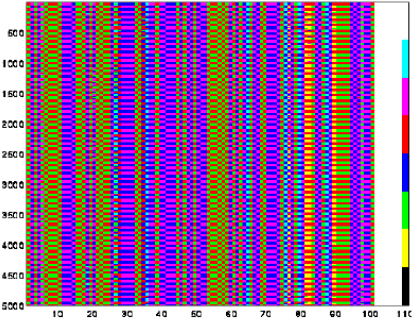
$$I_\Sigma(\mu) = \sum_{k=1}^{N} \{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\}$$

$$h_k(\mu) = \sum_{\xi_i^{(k)}>0} \xi_i^{(k)}; h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)}>0} \xi_i^{(m-k)}; h(\mu) = \sum_{\lambda_i>0} \lambda_i$$

Structure and self-organization: An example

$$x_i(t+1) = (1-c)\, f(x_i(t)) + \frac{c}{N-1} \sum_{k \neq i} f(x_k(t))$$

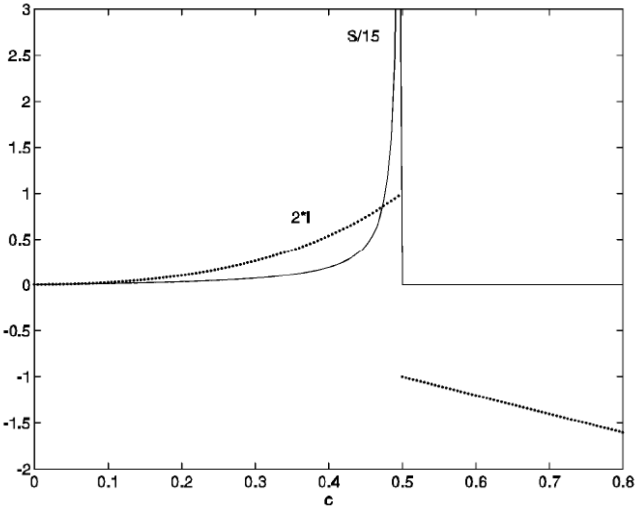
$f(x) = 2x \,(\mathrm{mod}\, 1)$



$c = 0.495$

$c = 0.51$

Structure and self-organization: An example



(*Physica A 276 (2000) 550-571; 295 (2001) 537-561*)

Synchronization and clustering

- **Synchronization**
  (Classical mathematical example: the Kuramoto model)
  A similar, discrete-time oscillators model

$$\boxed{x_i\left(t+1\right) = x_i\left(t\right) + \omega_i + \frac{k}{N-1}\sum_{j=1}^{N} f_\alpha\left(x_j - x_i\right)}$$

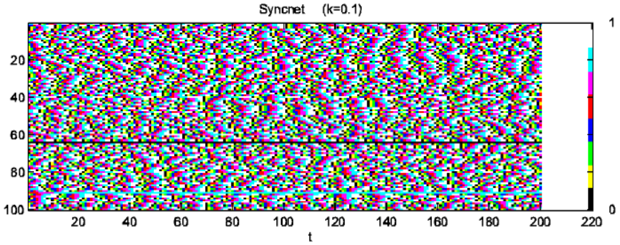
$$p\left(\omega\right) = \frac{\gamma}{\pi\left[\gamma^2 + \left(\omega - \omega_0\right)^2\right]}$$

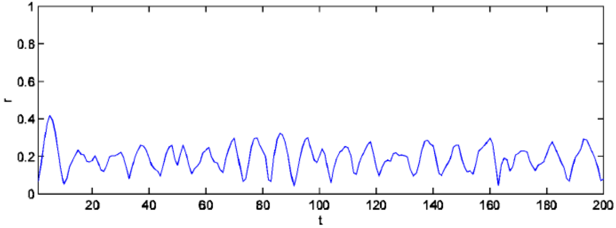
$$f_\alpha\left(x_j - x_i\right) = \alpha\left(x_j - x_i\right)\left(\mathrm{mod}\,.1\right)$$

- **Order parameter**

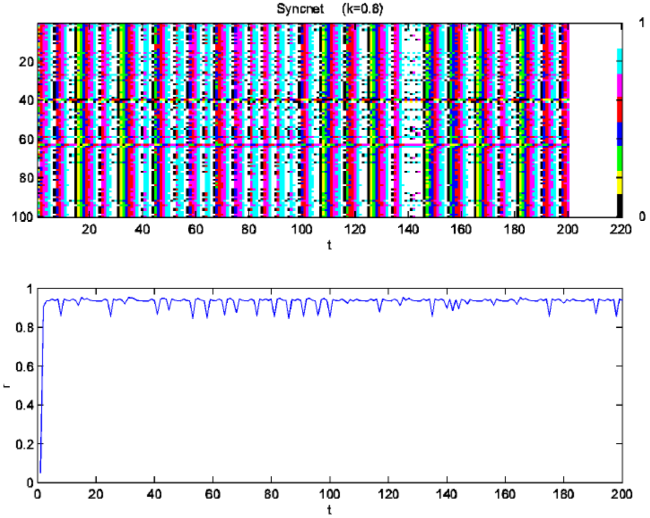
$$r\left(t\right) = \left|\frac{1}{N}\sum_{j=1}^{N} e^{i2\pi x_j(t)}\right|$$

Synchronization and clustering





Synchronization and clustering



Synchronization and clustering

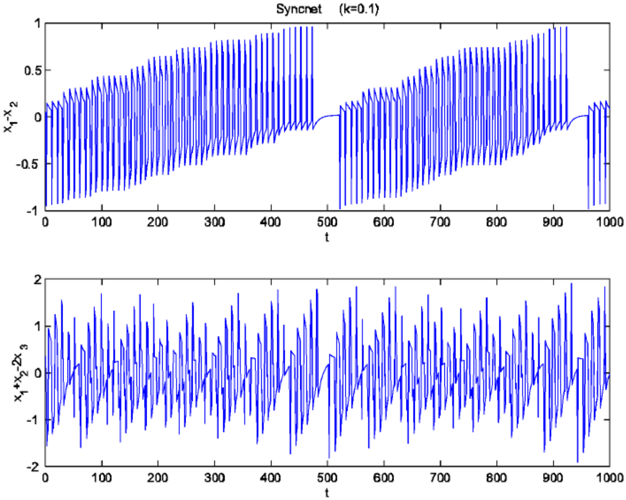
- The Lyapunov spectrum controls the dynamical self-organization of the system
- In this case

$$\begin{aligned}\lambda_1 &= 0 \\ \lambda_i &= \log\left(1-\alpha k\left(\frac{N}{N-1}\right)\right) \qquad (N-1) \text{ times}\end{aligned}$$

$N-1$ contracting directions for any $k \neq 0$
"One-dimensional" system!

- $\Rightarrow$ Strong dynamical correlations even before synchronization

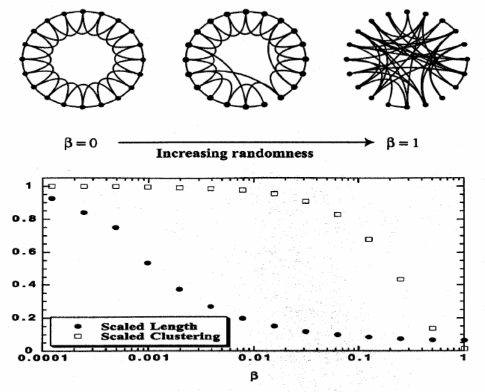
Synchronization and clustering



(*Int. J. Bifurcation and Chaos 15 (2005) 1185-1213*)

Dynamical characterization of network topology

- The small-world phase



- Is there a small-world – random transition ?

Dynamical characterization of network topology

- Define a dynamical system on the network

$$x_i(t+1) = \Sigma_{k=1}^{N} W_{ik} f(x_k(t))$$

$$f(x) = \alpha x \pmod 1$$

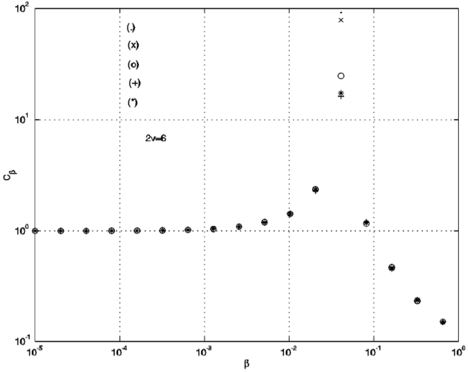
$$W_{ik} = \begin{cases} 1 - \frac{n_v(i)}{2v} c & \text{if } i = k \\ \frac{c}{2v} & \text{if } i \neq k \text{ and } k \in n_v(i) \\ 0 & 0 \text{ otherwise} \end{cases}$$

- and an order parameter using Lyapunov and conditional exponents

$$C_\beta = \left| \frac{h_0^* - h_0}{h_\beta^* - h_\beta} \right| ; \quad h_\beta^* = \sum_{i=1}^{N} \left(\frac{1}{d_i} \sum_{\lambda_\beta^* > 0} \lambda_\beta^*(j) \right) ; \quad h_\beta = \sum_{\lambda_\beta > 0} \lambda_\beta(j)$$

$$\beta_{c2} = 0.04 \qquad C_\beta \sim |\beta - \beta_{c2}|^{-\delta} \qquad \delta_1 = 1.14 \quad \delta_2 = 0.93$$

Dynamical characterization of network topology



The mismatch between local and global dynamics defines the transition (*Phys. Lett. A319 (2003) 285-289*)

Self-organized criticality

- Most SOC models display:
  - Unstable behavior of the local dynamics
  - Extremal dynamics
- **Theorem**
  If the single-agent dynamics has positive Lyapunov exponents and the global dynamics is extremal with finite range, then, in the $N \rightarrow \infty$ limit, the Lyapunov spectrum converges to 0^{+}
- Proof: In the $T \rightarrow \infty$ limit, used to compute the Lyapunov spectrum, the tangent maps have only a nontrivial finite size block during an average time of order $(2r+1)\frac{T}{N}$
- With the Lyapunov spectrum converging to 0^{+} there are no dynamical scales. Thus in the $N \rightarrow \infty$ limit the system is "tuned" to SOC
- A sufficient condition for SOC

Self-organized criticality

- A "detuned" model

$$x_i(t+1) = \Gamma_i\left(\overrightarrow{x}\right) x_i(t) + \left(1 - \Gamma_i\left(\overrightarrow{x}\right)\right) f(x_i(t))$$

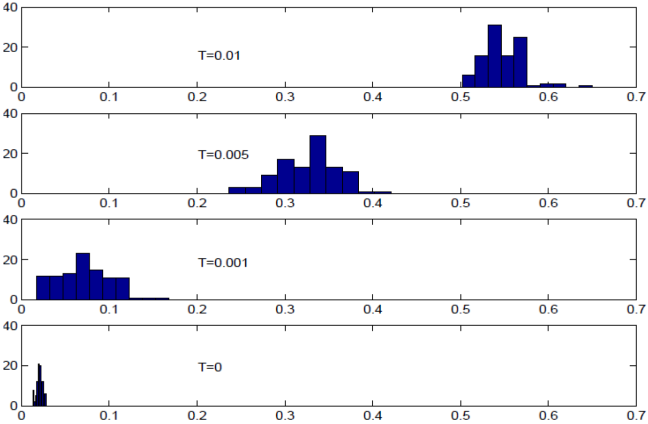
$f(x) = kx \pmod{.1}$

- $\Gamma_i\left(\overrightarrow{x}\right)$ is nearly zero if i corresponds to the minimum x value or to one of its $2n_v$ neighbors and is nearly one otherwise
- For example

$$\Gamma_i\left(\overrightarrow{x}\right) = \prod_{j=i-n_v}^{j=i+n_v} \left(1 - \frac{e^{-\frac{x_j}{T}}}{\sum_{k=1}^{N} e^{-\frac{x_k}{T}}}\right)$$

- Is a "finite temperature" Bak-Sneppen model. Has scaling laws only in the $T \to 0$ limit
- Computation of the Lyapunov spectrum illustrates the theorem

Self-organized criticality



(*Physica D 214 (2006) 182*)