Stochastic solutions and singular partial differential equations

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Stochastic solutions of partial differential equations

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- The domain here is $\mathbb{R} \times [0,t)$ and the expectation value is the inner product $\langle \mu_t, f \rangle$ of the initial condition f with the *measure* μ_t *generated by the Brownian motion at the* $t-$*boundary*.

Stochastic solutions of PDE's

- Using the heat kernel the solution would be

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{R}} \frac{1}{\sqrt{t}} \exp\left(-\frac{(x-y)^2}{2t}\right) f(y)\, dy$$

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- Even for linear problems, the stochastic solution approach provides a way to express exact solutions in a way that is not possible with kernels and integral representations: *Example*

$$Lf(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \partial_{ij} f(x) + \sum_{i=1}^{d} b_i(x) \partial_i f(x)$$

$$(L + v(x))\, u(x) = -g(x) \qquad \text{with} \qquad u = 0 \qquad \text{on} \qquad \partial D$$

$$\boxed{u(x) = \mathbb{E}^x \left[\int_0^{\tau_D} g(X_s) e^{\int_0^s v(X_r) dr} ds \right]}$$

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- All these are classical results for linear pde's.

Stoch. solutions for nonlinear PDE's. KPP eq. (McKean)

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + v^2 - v \qquad\qquad v(0,x) = g(x)$$

- $G(t,x)$ = Green's operator for heat equation $\partial_t v(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v(t,x)$

$$G(t,x) = e^{\frac{1}{2} t \frac{\partial^2}{\partial x^2}}$$

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- Write KPP in integral form

$$v(t,x) = e^{-t} G(t,x)\, g(x) + \int_0^t e^{-(t-s)} G(t-s,x)\, v^2(s,x)\, ds$$

Denoting by (ξ_t, Π_x) a Brownian motion starting from time zero and coordinate x, the integral equation may be rewritten as

$$\begin{aligned} v(t,x) &= \Pi_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-(t-s)} v^2(s, \xi_{t-s})\, ds \right\} \\ &= \Pi_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-s} v^2(t-s, \xi_s)\, ds \right\} \end{aligned}$$

The KPP equation

- **The stochastic solution process:** a Brownian motion plus branching process with exponential holding time T, $P\left(T > t\right) = e^{-t}$. At each branching point the particle splits into two, the new particles going along independent Brownian paths. At time $t > 0$ one has n particles located at $x_1\left(t\right), x_2\left(t\right), \cdots x_n\left(t\right)$. The solution is obtained by

$$v\left(t, x\right) = \mathbb{E}\left\{g\left(x_1(t)\right) g\left(x_2(t)\right) \cdots g\left(x_n(t)\right)\right\}$$

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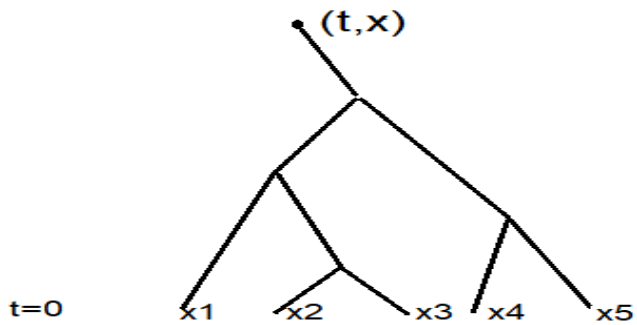
$$v(t,x) = \mathbb{E}\left\{g(x_1(t))\, g(x_2(t)) \cdots g(x_n(t))\right\}$$

- An equivalent interpretation: a backwards-in-time process from time t at x. When it reaches $t = 0$ samples the initial condition. Generates a measure at the $t = 0$ boundary which is applied to $g(x) = v(0,x)$.

$$v(t,x) = \mathbb{E}\left\{v(0,x_1)\, v(0,x_2) \cdots v(0,x_n)\right\}$$



Stochastic solutions: What are they good for?

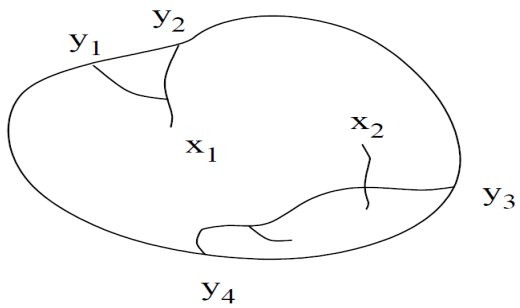
- **Nonlinearity** becomes **branching**.
- New **exact solutions**
- New **numerical algorithms**
  Deterministic algorithms grow exponentially with the dimension d of the space, roughly N^d ($\frac{L}{N}$ the linear size of the grid). The stochastic process only grows with the dimension d.
- Provide **localized solutions**
  Sample paths started from the same point are independent.
  Likewise, paths starting from different points are independent from each other. *The stochastic algorithms are a natural choice for parallel and distributed computation.*
- Stochastic algorithms handle equally well regular and complex boundary conditions.
- **Domain decomposition** using interpolation of localized stochastic solutions and then, in each small domain, a deterministic code.
  Avoids the communication time problem. Fully parallel.

Stochastic solutions: Two construction methods

- **McKean's method:** a probabilistic version of the Picard series.
  First the differential equation is written as an integral equation and rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition
  Then the Picard iteration is interpreted as an evolution and branching proces.
  The stochastic solution is equivalent to importance sampling of a normalized Picard series.
- **The method of superprocesses:** constructs the boundary measures of a measure-valued stochastic process and obtains the solutions of the differential equation by a scaling procedure (see the appendix).

Superprocesses. Branching exit measures (Dynkin)

- Superprocesses are scaling limits of infinitely fast branching stochastic processes generating a measure-valued process on the boundary



- They are either models for evolving populations or tools to represent the solutions of nonlinear partial differential equations (PDE's). The figure illustrates the superprocesses that represents the solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^{\alpha}$$

for $\alpha = 2$. Likewise superprocesses with different branching schemes may be constructed for $1 < \alpha \leq 2$. **However, Dynkin's measure-valued superprocesses cannot handle $\alpha > 2$, nor interactions involving derivatives.**

A list of some results

- **Navier-Stokes**
  E. C. Waymire; *Probability & incompressible Navier-Stokes equations: An overview of some recent developments*, Prob. Surveys 2 (2005) 1-32
- **Vlasov with and without magnetic field and magnetohydrodynamics**
  RVM, F. Cipriano; *Commun. Nonlin. Sci. and Num. Simul. 13 (2008) 221-226, 1736.*
  RVM; *J. of Math. Phys. 51 (2010) 043101*
  E. Floriani, RVM; *J. of Comp. Phys. 242 (2013) 777–789.*
- **Fractional KPP**
  F. Cipriano, H. Ouerdiane, RVM; *Fract. Calculus Appl. Anal. 12 (2009) 47–56.*
- **Extension of superprocesses to ultradistributions**
  RVM; *Stochastics 89 (2017) 896-909.*

Stochastic solutions as generators of regular structures

- Three STEPS:
  1 - Nonlinear PDE $\longleftrightarrow$ Propagation (diffusion) (linear part) $\oplus$ Branching process (nonlinearity=branching)
  2 - The process generates a basis of regular structures (polynomials of the $t = 0$ condition and its derivatives at random points in the domain) to represent the solution at (t, x).
  3 - Computation of functionals in the basis, averages, etc.
  Convergence checks for existence of the functionals, for example $|\partial^n f\,(0, x)| \leq 1$
- If the regularity index of most terms in the equation is sufficiently positive, this method is identical to importance sampling of the Picard iteration.
  However if some terms have negative regularity, for example space-time white noise $(-\frac{d}{2} - 1 - \varepsilon)$ the Picard iteration does not make sense (except with mollification and renormalization, whenever possible).

Stochastic solutions as generators of regular structures

- Nevertheless the diffusion + branching process may still be well defined (Ill defined multiplication becomes branching) - STEP 1.
- It also generates a basis of **regular structures**, for example polynomials of white noise at different space-time points (with probability one) with coefficients given by the function and its derivatives at $t = 0$ - STEP 2.
- It is only STEP 3 that one has to worry about.
- **Singular partial differential equations** are pde's which are singular in the sense that nonlinear functions of the driving terms, or of the solution itself, are ill-defined. *Even the question of what means to be a solution is a non-trivial matter*. An example

$$\partial_t h = \partial_x^2 h + \lambda \left(\partial_x h\right)^2 + \xi,$$

  Kardar-Parisi-Zhang (KPZ), ξ being a space-time white noise.
- If the solution has the same regularity structure as Brownian motion, in a Picard solution, the nonlinear term has powers of the white noise, that is, products of distributions at the same spacetime points.

Stochastic solutions for singular PDE's. The KPZ example

- KPZ, having been studied before by several methods, is a good testing ground for this approach.
  KPZ: **Two approaches:**
  1- The (rescaled) Cole-Hopf transformation

$$Z = e^{-t} e^{\lambda h};\ h = \frac{1}{\lambda}\left(t + \log Z\right), \tag{1}$$

leads to

$$\partial_t Z = \partial_x^2 Z - Z + \lambda Z \xi. \tag{2}$$

The last term still involves two distribution-valued entities.
The first step is to rewrite it as an integral equation,

$$\begin{aligned} Z(t,x) &= e^{-t} e^{t\partial_x^2} Z(0,x) + \int_0^t e^{-(t-s)} e^{(t-s)\partial_x^2} Z(s,x)\, \lambda\xi(s,x) \\ &= e^{-t} e^{t\partial_x^2} Z(0,x) + \int_0^t e^{-s} e^{s\partial_x^2} Z(t-s,x)\, \lambda\xi(t-s,x), \end{aligned}$$

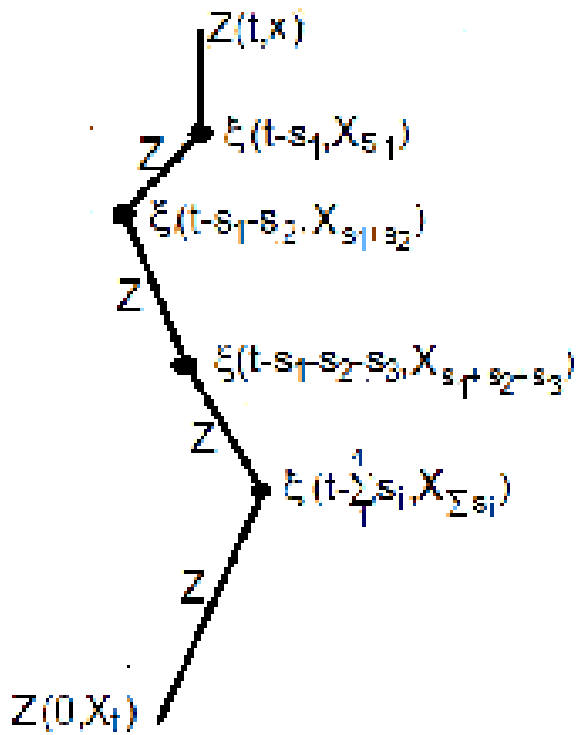
Stochastic solutions for singular PDE's. KPZ equation

- $Z(t,x)$ being distribution-valued, the integrals involve a product of distributions and, when interpreted as usual equalities, would be meaningless. However they become well-defined when interpreted as *a symbolic definition of an iterated stochastic process*.
- The product $Z(s,x)\,\xi(s,x)$ simply means that the $Z-$process samples the driving term at a branching point. This case is the simplest instance of a branching. A term $Z^3(s,x)$, for example, would mean a branching at (s,x) of the process into three similar processes. It is in this sense that, as stated before, **undefined multiplications of distributions are traded off by well defined branchings**.
- Noticing that $e^{t\partial_x^2}$ is the evolution operator for a diffusion process X_t and that $e^{-t}+\int_0^t e^{-s}ds=1$, the second equation may be written as an expectation value over a branching and diffusion process which starts at (t,x) and evolves backwards in time to $t=0$.

$$Z(t,x)=\mathbb{E}_{(t,x)}\left\{e^{-t}Z(0,X_t)+\int_0^t dse^{-s}Z(t-s,X_s)\,\lambda\xi(t-s,X_s)\right\}$$

Stochastic solutions for singular PDE's. KPZ equation

$p_t = e^{-t}$ is the surviving probability from time t to time zero and $\rho_s = e^{-s}$ is the probability density for branching between s and $s + ds$. Between branchings the process propagates as a pure diffusion process and at each branching point there is a sampling of the white noise at that space-time point, as well as the creation of a new propagation path for the process. A typical sample path



Stochastic solutions for singular PDE's. KPZ equation

- The contribution of a sample path with n branchings is

$$\begin{aligned} F_n &= \lambda^n \xi\left(t-s_1, X_s\right) \xi\left(t-s_1-s_2, X_{s_1+s_2}\right) \cdots \\ &\quad \cdots \xi\left(t-s_1-s_2 \cdots-s_n, X_{s_1+s_2 \cdots+s_n}\right) \cdots Z\left(0, X_t\right) \end{aligned}$$

- The stochastic construction involves two distinct probability spaces, Ω and Ω', the first being the auxiliary probability space of the Brownian motion X_t used to compute the expectation $\mathbb{E}_{(t,x)}$ and the second the probability space of the driving white noise.
- The expectation value (in Ω) is over the branching and diffusion process, for a particular fixed realization of the white noise $\{\xi(t,x), t \in [0,t]\}$ in Ω'.
- Because the driving term is a white noise, $Z(t,x)$ is also a random variable in Ω'. If one lets the ξ white noise realization change at each branching one would obtain a process $Z'(t,x)$ in $\Omega \otimes \Omega'$. Functionals of the solution are partial averages over Ω.

Stochastic solutions for singular PDE's. KPZ equation

- So far one has avoided the product of distributions arising from the last term by trading off products by branchings. Then, to compute the F_n functional of each sample path, one deals with the product of n white noises which, with probability one, are defined at different space-time points. Therefore one deals with products of n independent Gaussian variables, which are well-defined random variables with distribution given by Meijer G-function. Hence, with probability one, the Z' process and the $F'_n s$ are well-defined.

- A configuration space of non-overlapping Gaussian processes?

Proposition

The F_n functionals over the branching, diffusion and white noise sample paths generate a well-defined process Z' in $\Omega \otimes \Omega'$. A partial expectation over Ω, when it exists, is the $\Omega' -$process solution of Eq.(2). The process solution of the KPZ equation is obtained from the second equation in (1).

Stochastic solutions for singular PDE's. KPZ equation

- A set of regular structures (the $F_n's$) is found to expand the solution. Something else may be needed, when taking averages over the $F_n's$
- For a single realization of the ξ noise, expectation values over the branching are performed explicitly summing all possible sample paths of the process, noticing that in each case the composition of the branching probability $e^{-s}e^{-s_1}\cdots e^{-\left(t-\sum s_i\right)}$ is e^{-t}. The result is

$$\begin{aligned} Z(t,x) &= \mathbb{E}_{(t,x)}\left\{e^{-t}Z(0,X_t)\left(1+\lambda\int_0^t dB(t-s,X_s)\right.\right. \\ &\quad \left.\left.+\lambda^2\int_0^t\int_0^s dB(t-s,X_s)\,dB\left(t-s-s',X_{s+s'}\right)+\cdots\right)\right\} \\ &= \mathbb{E}_{(t,x)}\left\{e^{-t}Z(0,X_t)\,e^{\lambda\int_0^t dB(t-s,X_s)}\right\}, \end{aligned}$$

$B(t,x)$ being the Brownian motion for which the white noise $\xi(t,x)$ is the distributional derivative. Finally

$$\widehat{h}(t,x)=\frac{1}{\lambda}\log\mathbb{E}_{(t,x)}\left\{e^{\lambda h(0,X_t)}e^{\lambda\int_0^t dB(t-s,X_s)}\right\},$$

Stochastic solutions for singular PDE's. KPZ equation

- However this is only a formal expression until one defines what is the meaning of $e^{\lambda \int_0^t dB(t-s,X_s)}$, that is, what is the **signature** of $B(t)$
- If our choice is Itô integration, Itô formula for $\widehat{h}_t = \log W_t$ would imply

$$\partial_t \widehat{h} = \partial_x^2 \widehat{h} + \lambda \left(\partial_x \widehat{h}\right)^2 + \xi - C$$

that is, $\widehat{h}(t,x)$ is not the solution of the KPZ equation but of a *renormalized* KPZ equation with an infinite renormalization constant C. Solution of the original equation would require the choice of a rough path definition for the $B(t)$ signature not containing the 2nd derivative term of the Itô formula.

- **In conclusion:** The stochastic solution technique provides a way to generate a set of **regular structures** to expand the solution, but then, when performing resummations, further assumptions are needed in the sense of **rough path theory**.

2 - KPZ - Direct construction

A set of regular expansion structures may also be obtained directly from the KPZ equation without the Cole-Hopf transformation.
Because there is no natural branching clock, add and subtract a linear term

$$\partial_t h(t,x) = \partial_x^2 h(t,x) - \mu h(t,x) + \lambda (\partial_x h)^2 + \mu h(t,x) - \xi(t,x),$$

which allows to write the integral equation as $h(t,x) =$

$$e^{-t\mu} e^{t\partial_x^2} h(0,x) + \int_0^t ds e^{-\mu s} e^{s\partial_x^2} \left\{ \lambda (\partial_x h(t-s,x))^2 + \mu h(t-s,x) - \xi(t-$$

The solution will then be an expectation value of a diffusion and branching process, denoted $DB1$

$$\begin{aligned} h(t,x) &= \mathbb{E}^{DB1} \left\{ e^{-t\mu} h(0,X_t) + \gamma \int_0^t ds \mu e^{-\mu s} \left[\frac{\lambda}{\eta} (\partial_{X_s} h(t-s,X_s))^2 \right.\right. \\ &\left.\left. + \frac{\mu}{\eta} h(t-s,X_s) - \frac{1}{\eta} \xi(t-s,X_s) \right] \right\}, \end{aligned} \tag{3}$$

with $\gamma = \frac{\lambda+\mu+1}{\mu}$; $\eta = \lambda + \mu + 1$

KPZ - Direct construction

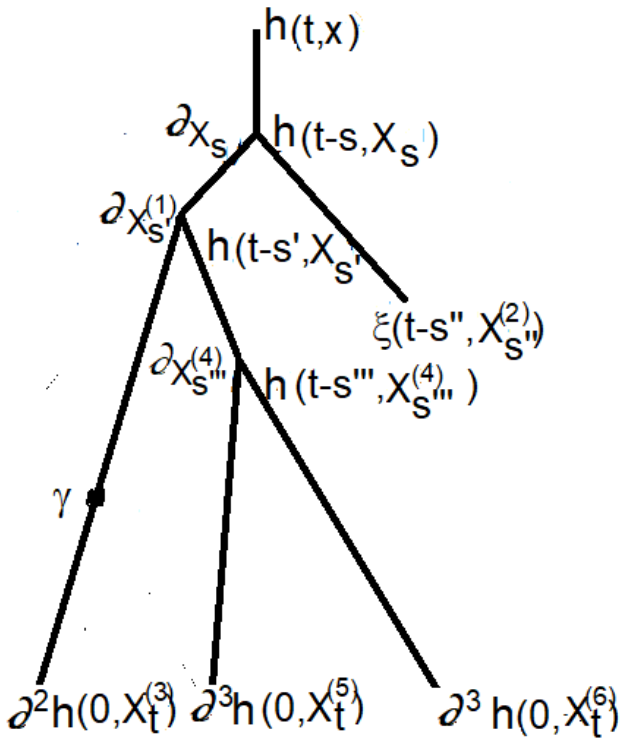
The $DB1-$process starts from (t, x) and diffuses backwards-in-time either to time zero with probability $e^{-t\mu}$ or to a branching point at time s with probability $ds\mu e^{-\mu s}$. At the branching point, with probability $\frac{1}{\eta}$ the process samples the white noise, with probability $\frac{\mu}{\eta}$ the process proceeds undisturbed and with probability $\frac{\lambda}{\eta}$ two new $DB1-$processes are started from $(t-s, X_s)$ and a derivative label ∂_{X_s} is assigned to the branching point. In the subsequent processes both $h(t-s, X_s)$ and the label ∂_{X_s} are transported by the processes

$$e^{s\partial_x^2}\partial_x h(t-s, x) = e^{s\partial_x^2}\partial_x e^{-s\partial_x^2} e^{s\partial_x^2} h(t-s, x).$$

Because the propagation process here is a simple diffusion, the transport of the labels is very straightforward and at the final step, when the processes reach time zero, the derivative is applied at the same point as the point reached by the processes. To use this method of labelled branching trees allows for a simple construction of stochastic solutions for equations involving derivatives or even nonpolynomial interactions.

KPZ - Direct construction

A sample path of the $DB1-$process.



The expansion struture generated by this sample path is

$$-\gamma^4 \partial^2 h\left(0, X_t^{(3)}\right) \partial^3 h\left(0, X_t^{(5)}\right) \partial^3 h\left(0, X_t^{(6)}\right) \xi\left(t-s'', X_{s''}^{(2)}\right).$$

KPZ - Direct construction

At each branching point there is a coupling constant γ. When more than one white noise contribution appears in the multiplicative functional, with probability one they sample the white noises at different space-time points. One has a well-defined product of independent Gaussian random variables. As before one deals with two probability spaces Ω and Ω'. The expectation value in Eq.(3) is an expectation value over the $DB1-$process for each fixed realization of the white noise process, hence an expectation in Ω. When a different realization of the white noise is sampled whenever it appear in the branching tree one obtains a $DB1'-$ process in $\Omega \otimes \Omega'$.

Proposition

The partial expectation (in Ω) of the multiplicative functional of the $DB1'-$process in $\Omega \otimes \Omega'$(diffusion, branching and independent white noises at each branching) generates a solution of the KPZ equation

Similar results for

$$\partial_t \Phi = \triangle \Phi - \Phi^3 + \xi$$

Stochastic solutions and rough paths

Stochastic solution = An integral equation $h(t,x) = e^{-t\mu} e^{t\partial_x^2} h(0,x) + \int_0^t ds e^{-\mu s} e^{s\partial_x^2} \left\{ \lambda \left(\partial_x h(t-s,x)\right)^2 + \mu h(t-s,x) - \xi(t-s,x) \right\}$ interpreted as defining a stochastic process

$$h(t,x) = \mathbb{E}\left\{ e^{-t\mu} h(0,X_t) + \gamma \int_0^t ds \mu e^{-\mu s} \left[\frac{\lambda}{\eta} \left(\partial_{X_s} h(t-s,X_s)\right)^2 + \frac{\mu}{\eta} h(t-s,X_s) - \frac{1}{\eta} \xi(t-s,X_s) \right] \right\},$$

generating a set of regular structures to expand the solution versus a (formal) **solution in terms of iterated integrals**

$$h(t,x) = h(0,x) + \int_0^t ds_1 \left\{ \partial_x^2 h(s_1,x) + \lambda \left(\partial_x h\right)^2 (s_1,x) + \xi(s_1,x) \right\}$$

$\int_0^t ds_1 \left(\int_0^{s_1} ds_2 \left(\partial_x h\right)^2 (s_2,x) \right)^2$; $\int_0^t ds_1 \int_0^{s_1} dB(s_2,x) \left(\partial_x h\right)^2 (s_2,x)^2$; etc.

With h of the same regularity as B these integrals are not defined as Young integrals $\Rightarrow$ *Itô Calculus* (Semimartingale noises) *or Rough path theory*

Stochastic solutions and rough paths

- Rough path methods generate **enhanced paths $(X, \mathbb{X})$** to define solutions for each noise realization. In the stochastic solution method the solution is expanded into structures associated to **a (labelled) tree process**. Even if a particular realization is chosen for the noise, there still is a probabilistic element in the choice of the bifurcation paths. **The solution expansion structure is well defined if, with probability one, each one of its paths is well defined**.
- - *Stochastic solutions and singular partial differential equations*; Commun. Nonlinear Sci. Num. Simul. 125 (2023) 107406 + references therein
- For comparison with the rough path approach:
  - P. Friz, M. Hairer; *A course on rough paths, with an introduction to regularity structures*, Springer 2014.

Appendix

Superprocesses as branching exit measures

Superprocesses: Branching exit measures (Dynkin, etc.)

- $(E, \mathcal{B})$ a measurable space, $M_+(E)$ the space of finite measures in E and $\left(X_t, P_{0,\mu}\right)$ a branching stochastic process with values in $M_+(E)$ and transition probability $P_{0,\mu}$ starting from time 0 and measure μ.
- The process satisfies a *branching property* if given $\mu = \mu_1 + \mu_2$

$$P_{0,\mu} = P_{0,\mu_1} * P_{0,\mu_2}$$

that is, after the branching, $\left(X_t^1, P_{0,\mu_1}\right)$ and $\left(X_t^2, P_{0,\mu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $\left(X_t, P_{r0,\mu}\right)$.

- For the *transition operator* V_t operating on functions on E this is

$$V_t f(\mu_1 + \mu_2) = V_t f(\mu_1) + V_t f(\mu_2)$$

where $e^{-\langle V_t f, \mu\rangle} \doteq P_{0,\mu} e^{-\langle f, X_t\rangle}$ or

$$V_t f(\mu) = -\log P_{0,\mu} e^{-\langle f, X_t\rangle}$$

V_t is the *log-Laplace semigroup* associated to X_t. If $\mu_0 = \delta_x$

$$V_t f(x) = -\log P_{0,x} e^{-\langle f, X_t\rangle}$$

Superprocesses: Branching exit measures

- In $S = [0, \infty) \times E$ consider a set $Q \subset S$ and the associated branching exit process $\left(X_Q, P_\mu\right)$ composed of a propagating Markov process in E, $\xi = (\xi_t, \Pi_{0,x})$, a set of probabilities $p_n(t, x)$ describing the branching and a parameter k defining the lifetime.

$$u(x) = V_Q f(x) = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \tag{4}$$

$\langle f, X_Q \rangle$ is the integral of the function f on the (space-time) boundary with the boundary exit measure generated by the process.

- This branching exit process is called a $(\xi, \psi)-$*superprocess* if

$$u + G_Q \psi(u) = K_Q f \tag{5}$$

G_Q and K_Q are the Green and the Poisson operators,

$$G_Q f(r, x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s)\, ds$$

$$K_Q f(x) = \Pi_{0,x} 1_{\tau < \infty} f(\xi_\tau)$$

$\psi(u)$ means $\psi(0, x; u(0, x))$ and τ is the exit time from Q.

Superprocesses: The construction

Let $\varphi(s,x;z)$ be the offspring generating function at time s and point x

$$\varphi(s,x;z) = c\sum_0^\infty p_n(s,x)z^n$$

where $\sum_n p_n = 1$ and c denotes the branching intensity.

$$P_{0,x}e^{-\langle f,X_Q\rangle} \triangleq \tag{6}$$

$$e^{-w(0,x)} = \Pi_{0,x}\left[e^{-k\tau}e^{-f(\tau,\xi_\tau)} + \int_0^\tau dske^{-ks}\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right)\right]$$

The measure-valued process starts from δ_x at time 0, τ is the first exit time from Q and $f(\tau,\xi_\tau)$ the value of a function in the boundary ∂Q. Using $\int_0^\tau ke^{-ks}ds = 1-e^{-k\tau}$ and the Markov property $\Pi_{0,x}1_{s<\tau}\Pi_{s,\xi_s} = \Pi_{0,x}1_{s<\tau}$, Eq.(6) for $e^{-w(0,x)}$ is converted into

$$e^{-w(0,x)} = \Pi_{0,x}\left[e^{-f(\tau,\xi_\tau)} + k\int_0^\tau ds\left[\varphi\left(s,\xi_s;e^{-w(\tau-s,\xi_s)}\right) - e^{-w(\tau-s,\xi_s)}\right]\right]$$

Superprocesses: The limiting procedure

Replace $w(0,x)$ by $\beta w_\beta(0,x)$ and f by βf. β may be interpreted as the mass of the particles and when the measure-valued process $X_Q \to \beta X_Q$ then $P_\mu \to P_{\frac{\mu}{\beta}}$.

$$e^{-\beta w(0,x)} = \Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta \int_0^\tau ds\left[\varphi_\beta\left(s,\xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right]$$

Defining

$$u_\beta = \left(1-e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta = \left(1-e^{-\beta f}\right)/\beta$$

$$\psi_\beta(r,x;u_\beta) = \frac{k_\beta}{\beta}\left(\varphi\left(r,x;1-\beta u_\beta\right) - 1 + \beta u_\beta\right)$$

Superprocesses: The limiting procedure

One obtains

$$u_\beta(0,x) + \Pi_{0,x} \int_0^\tau ds \psi_\beta(s, \xi_s; u_\beta) = \Pi_{0,x} f_\beta(\tau, \xi_\tau)$$

that is

$$u_\beta + G_Q \psi_\beta(u_\beta) = K_Q f_\beta \tag{7}$$

When $\beta \to 0$, $f_\beta \to f$ and if ψ_β goes to a well defined limit ψ then u_β tends to a limit u, solution of (5) associated to a superprocess. Also in the $\beta \to 0$ limit

$$u_\beta \to w_\beta = -\log P_{0,x} e^{-\langle f, X_Q \rangle}$$

If to obtain with (7) the equation we want, the limiting choice is $\beta \to 0$ and $k_\beta \to \infty$, the superprocess corresponds to a cloud of particles for which both the mass and the lifetime tend to zero.

Superprocesses and nonlinear heat equation

The KPP equation may also be interpreted as a superprocess with $\beta \to 1$ and $k_\beta \to 1$. However, the main interest of superprocesses is that with the $\beta \to 0$, $k_\beta \to \infty$ limit stochastic solutions are constructed for other equations, in particular equations without the Poisson clock which is present in the KPP equation. For example for

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2 \quad \text{one has} \quad \psi(0,x;u) = u^2$$

$$\begin{aligned} \psi_\beta(0,x;u_\beta) &= \frac{k_\beta}{\beta}\left(\beta u_\beta - 1 + \sum_{n=0}^{2} p_0 + p_1(1-\beta u_\beta) + p_2(1-\beta u_\beta)^2\right) \\ &= u_\beta^2 \end{aligned}$$

leads to $p_1 = 0; p_0 = p_2 = \frac{1}{2}; k_\beta = \frac{2}{\beta}$. In this case, $\beta \to 0$, the solution is given by (4) and the superprocess a scaling limit ($n \to \infty$ in the figure). Mass and lifetime of the particles tend to zero and at each branching they either die without offspring or have two children.

Superprocesses for more general interactions

The construction may be generalized for interactions u^α with $1 < \alpha \leq 2$. With $z = 1 - \beta u_\beta$ one has

$$\varphi(0,x;z) = \sum_n p_n z^n = z + \frac{\beta}{k_\beta} u_\beta^\alpha = z + \frac{\beta}{k_\beta}\frac{(1-z)^\alpha}{\beta^\alpha} = z + \frac{1}{k_\beta \beta^{\alpha-1}} \times \left(1 - \alpha z + \frac{\alpha(\alpha-1)}{2} z^2 - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots\right)$$

Choosing $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ the terms in z cancel and for $1 < \alpha \leq 2$ the coefficients of all the remaining z powers are positive and may be interpreted as branching probabilities. **It would not be so for $\alpha > 2$.**

$$p_0 = \frac{1}{\alpha}; \quad p_1 = 0; \quad \cdots \quad p_n = \frac{(-1)^n}{\alpha}\begin{pmatrix} \alpha \\ n \end{pmatrix} \quad n \geq 2$$

with $\sum_n p_n = 1$. With this choice of branching probabilities, $k_\beta = \frac{\alpha}{\beta^{\alpha-1}}$ and $\beta \to 0$ one obtains a superprocess which provides a solution to the equation $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^\alpha$ for $1 < \alpha \leq 2$.

Superprocesses for more general interactions

For a generalization to superprocesses on signed measures and ultradistributions, which allows for the construction of stochastic solutions of much more general equations refer to:
Superprocesses on ultradistributions; Stochastics 89 (2017) 896-909.