The mathematics of randomness and fluctuations

R. Vilela Mendes
CMFcIO, Universidade de Lisboa
Academia das Ciências de Lisboa

Uncertainty, fluctuations, volatility

- In the modeling of the natural world, randomness, either intrinsic or as a result of lack of knowlegde of all the variables, plays a central role. However **there are many different kinds of randomness, that is what mathematical theories say**.
- Closely associated to randomness is the role of fluctruations. For example, fluctuations are the hallmark of biological systems in action. Compared to man-made machineries, biological systems fluctuate at various levels, from an individual molecule to the whole cell as a system. The critical role of fluctuations is evident at the transcription initiation by the RNA polymerase and the assembly of the ribosome. Also in population dynamics, ecology, creation of order through disorder, etc. Some titles:
  Order Through Disorder: The Characteristic Variability of Systems;
  https://doi.org/10.3389/fcell.2020.00186
  Randomness and Perceived-Randomness in Evolutionary Biology:
  https://www.jstor.org/stable/20115499

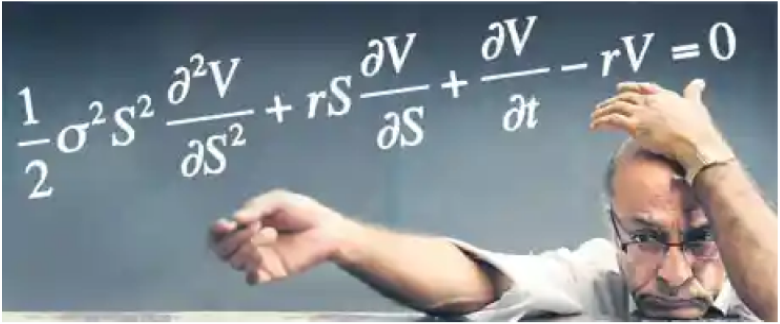
Uncertainty, fluctuations, volatility

The Impact of Environmental Fluctuations on Evolutionary Fitness Functions: DOI: 10.1038/srep15211

- Fluctuations, which usually come under the name of **volatility** also play an important role in social sciences.
  Shaken and stirred : explaining growth volatility: The World Bank, ISBN 978-0-8213-4981-6. - 2001, p. 191-211

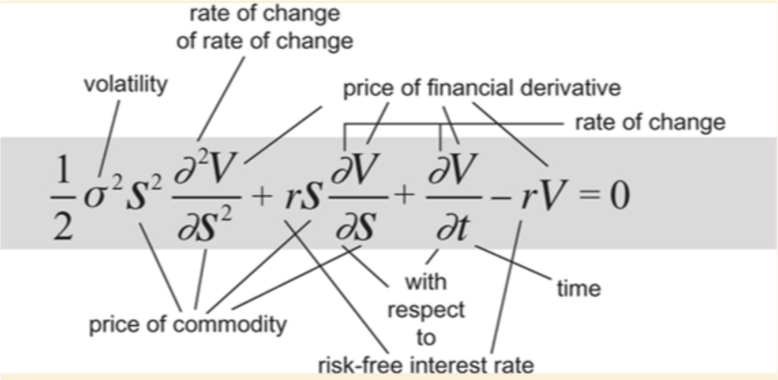
Volatility in economics and its impact

The mathematical equation that caused the banks to crash



The Guardian, February 12, 2012

The Black-Scholes-Merton equation



Recipe for Disaster: The Formula That Killed Wall Street

Wired, February 23, 2009

- **David Li's Gaussian copula formula**

$$C_{n,R}(u_1, ..., u_n) = \Phi_{n,R}(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n))$$

$C_{n,R}$ - joint distribution function
$\Phi_{n,R}$ - joint cumulative distribution function of a multivariate Gaussian with correlation matrix R
Φ^{-1} - inverse cumulative distribution function of a standard univariate normal distribution
Used by the derivatives departments of investment banks to price CDO's and credit rating agencies (Moody's, Standard & Poor's and Fitch)
Based essentially in the same assumptions as BS.

The central limit theorem (CLT)

Let $X_1, X_2, \cdots, X_n, \cdots$ be independent random variables with means $\{\mu_k\}$ and variances $\{\sigma_k^2 < \infty\}$. Let $\mu = \frac{1}{n}\sum_{k=1}^{n} \mu_k$ and $B^2 = \sum_{k=1}^{n} \sigma_k^2$, then the distribution of $S = \frac{X_1+X_2+\cdots+X_n-n\mu}{B}$

$$P\left(S \leq x\right) \underset{n\to\infty}{\overset{\mathcal{L}}{\to}} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2}$$

converges to a Gaussian distribution. *The critical assumption is finiteness of the variances* CLT is both a powerful result and a dangerous one

Because the variances of finite samples in an experiment are always finite, one may be mislead into thinking that the superposition of many events is always Gaussian. Coupled to the Markovian hypothesis, that is, the fact that full knowledge of the present suffices to predict the future, may lead to another dangerous assumptions that stochastic processes are mostly Brownian motion (a Gaussian Markovian process).

Not Gaussian, not Brownian (still Markovian, for now)

- **Lévy processes,** $X(0) = 0$
  a) stochastic continuity $\forall t \geq 0$. $(\forall \varepsilon, \lim_{s \to t} P(|X_t - X_s| > \varepsilon) = 0)$
  b) increments are stationary and independent
  c) has a càdlàg version
- Related to **Infinitely divisible distributions**

$$X \stackrel{d}{=} X_1^{(1/n)} + \cdots + X_n^{(1/n)}$$

$$P_X = P_{X_1^{(1/n)}} * \cdots * P_{X_n^{(1/n)}}$$

$$\phi(\lambda) = \mathbb{E}\left(e^{i\lambda X}\right) = \left(\phi_{X^{(1/n)}}(\lambda)\right)^n$$

- Lévy-Kintchine

$$\phi_X(\lambda) = e^{\Psi(\lambda)} = \exp\left\{ib\lambda - \frac{\lambda^2 c}{2} + \int_R \left(e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{|x|<1}\right)\nu(dx)\right\}$$

$$\nu(\{0\}) = 0 \qquad \int_R \left(1 \wedge |x|^2\right) \nu dx < \infty$$

Not Gaussian, not Brownian (still Markovian)

- Decompose the Lévy process

$$X_t = X_1 + (X_2 - X_1) + \cdots + (X_t - X_{t-1})$$

increments being independent and stationary, X_t is infinitely divisible

$$\mathbb{E}\left(e^{i\lambda X_t}\right) = \exp\left\{t\Psi\left(\lambda\right)\right\}$$

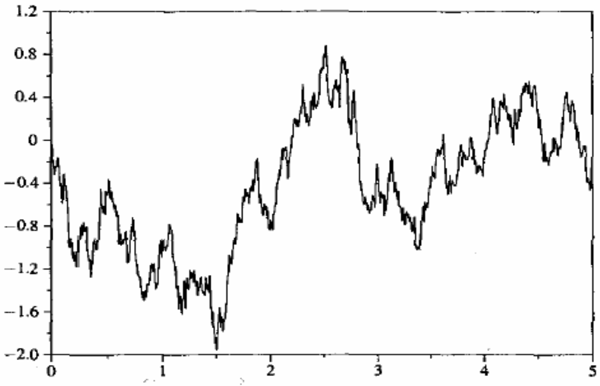
$b \longrightarrow$ **drift**, $c \longrightarrow$**diffusion**, $\nu \longrightarrow$**jump measure**

- Lévy-Itô decomposition of the Lévy process

$\Psi\left(\lambda\right) = \Psi^{(1)}\left(\lambda\right) + \Psi^{(2)}\left(\lambda\right) + \Psi^{(3)}\left(\lambda\right) + \Psi^{(4)}\left(\lambda\right)$

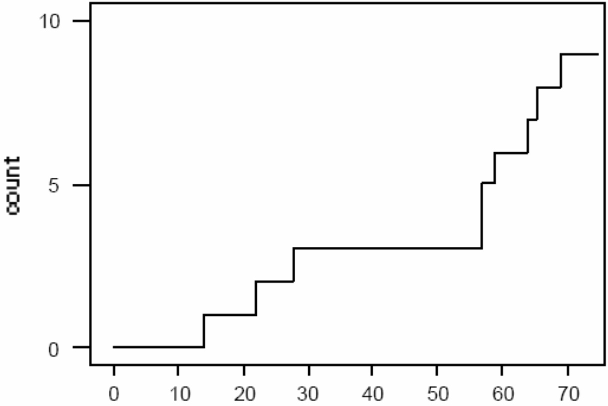
$$\begin{aligned}
\Psi^{(1)}\left(\lambda\right) &= ib\lambda \\
\Psi^{(2)}\left(\lambda\right) &= \frac{\lambda^2 c}{2} \\
\Psi^{(3)}\left(\lambda\right) &= \int_{|x|\geq 1}\left(e^{i\lambda x} - 1\right)\nu\left(dx\right) \\
\Psi^{(4)}\left(\lambda\right) &= \int_{|x|<1}\left(e^{i\lambda x} - 1 - i\lambda x\right)\nu\left(dx\right)
\end{aligned}$$

Brownian motion



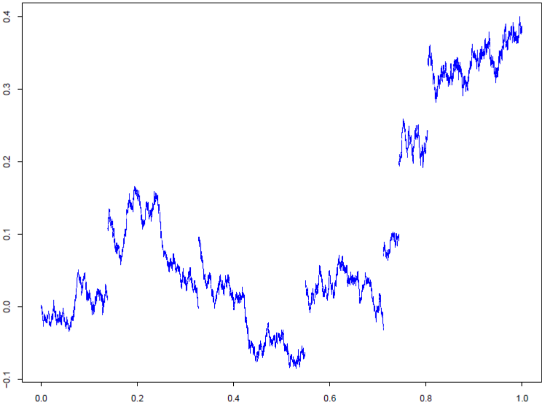
Lévy processes: examples

Poisson process



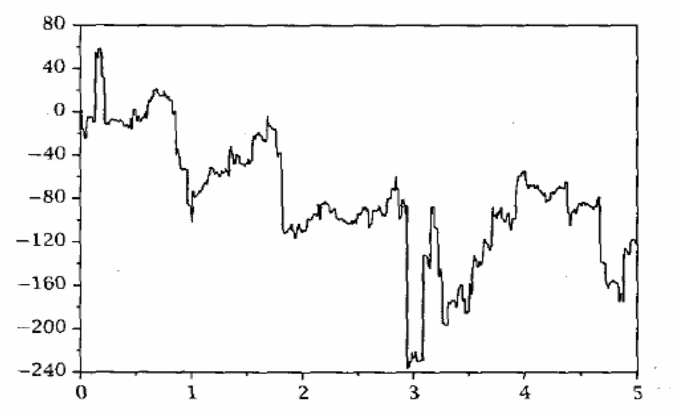
Lévy processes: examples

Jump diffusion

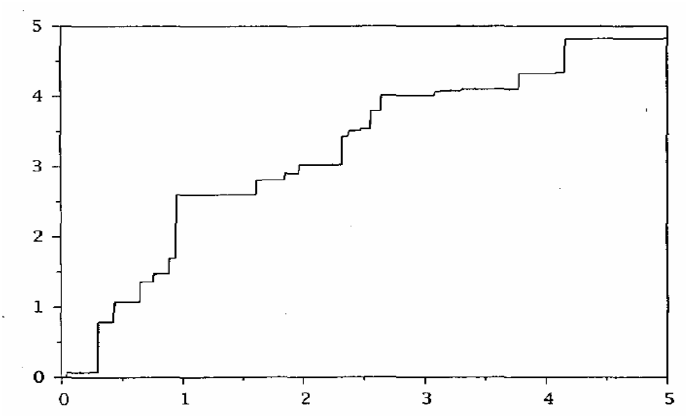


Lévy processes: examples

Cauchy process

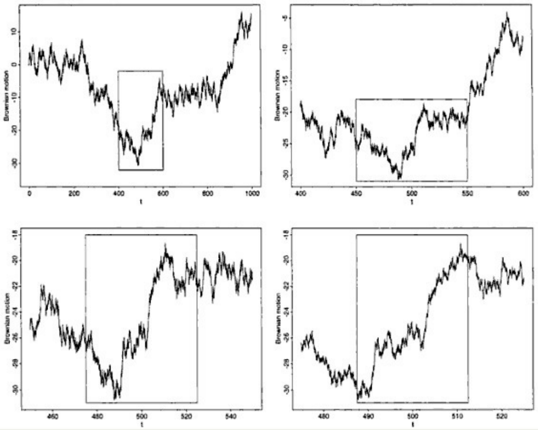


Subordinator



Processes with memory

A property of Brownian motion: selfsimilarity



$$\{X(at)\} \stackrel{d}{=} \{bX(t)\}$$

$b = a^H$, the process is $H-$selfsimilar ($H-$s.s.) - ($H =$ **Hurst exponent**)

Are there other selfsimilar Gaussian processes?

Brownian motion selfsimilar, stationary increments and covariance

$$\mathbb{E}\left[X(t)X(s)\right] = \min(t,s) = \frac{1}{2}\left\{t+s-|t-s|\right\}$$

If $\{X(t), t \geq 0\}$ *has real values, is H-s.s. with stationary increments and finite variance* ($\mathbb{E}\left[X(1)^2\right] < \infty$)*, then its covariance is*

$$\mathbb{E}\left[X(t)X(s)\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

Fractional Brownian motion $\left(\text{for } H \neq \frac{1}{2}\right)$
It has **Long-range dependance** for $H \neq \frac{1}{2}$
Define $\boxed{\xi(n) = X(n+1) - X(n)}$

$$r(n) = \mathbb{E}\left[\xi(0)\xi(n)\right] = \frac{1}{2}\left\{(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

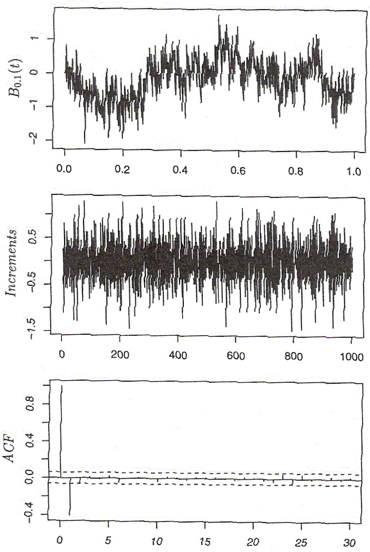
Fractional Brownian motion

$$\begin{array}{rclcl} r(n) & \underset{n\to\infty}{\sim} & 2H(2H-1)\,n^{2H-2}\mathbb{E}\left[X(1)^2\right] & , & H \neq \frac{1}{2} \\ r(n) & = & 0 & , & H = \frac{1}{2} \end{array}$$

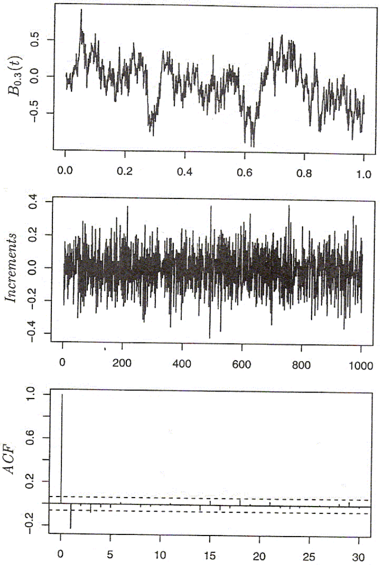
$$\boxed{\begin{array}{ccc} 0 < H < \frac{1}{2} & , & \sum_{n=0}^{\infty} |r(n)| < \infty \\ H = \frac{1}{2} & , & \text{uncorrelated} \\ \frac{1}{2} < H < 1 & , & \sum_{n=0}^{\infty} |r(n)| = \infty \end{array}}$$

$0 < H < \frac{1}{2}$, $r(n) < 0$, $n \geq 1$ (negative correlation, antipersistent process),
$\frac{1}{2} < H < 1$, $r(n) > 0$, $n \geq 1$ (positive correlation, persistent process).

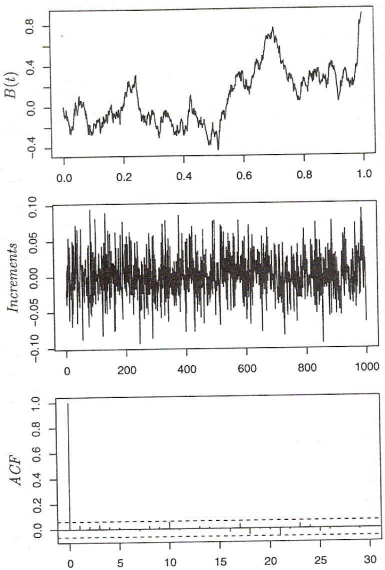
Fractional Brownian motion (H=0.1)



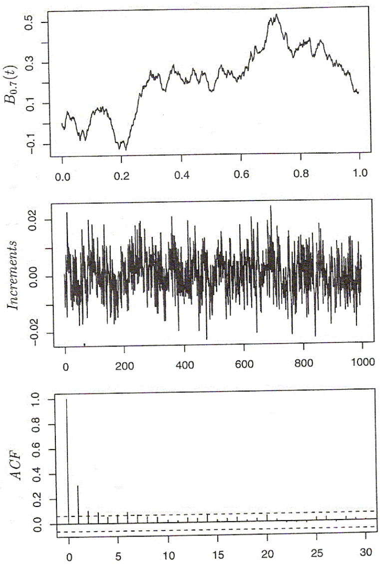
Fractional Brownian motion (H=0.3)



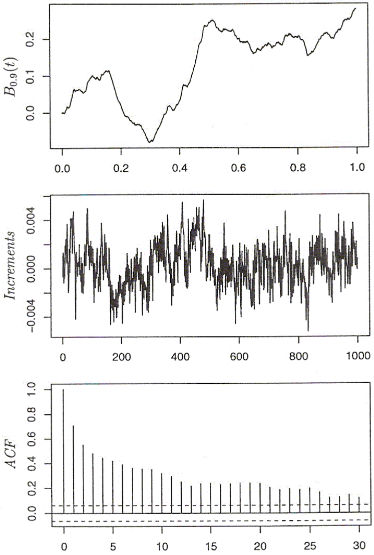
Brownian motion (H=0.5)



Fractional Brownian motion (H=0.7)



Fractional Brownian motion (H=0.9)



Fractional Lévy processes

Use an integral representation of fractional Brownian motion

$$B_H(t) \overset{d}{=} C \int_0^t K(t,s)\, dB(s)$$

$$K(t,s) = \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2}\right) s^{\frac{1}{2}-H} \int_s^t x^{H-\frac{3}{2}} (x-s)^{H-\frac{1}{2}}\, dx$$

Replace $B(s)$ by a square integrable Lévy process $L(s)$
Same covariance structure as FBM.

The dynamics of exploited fish populations

- A few years ago Niwa, studying 27 commercial fish stocks in the North Atlantic, concluded that the variability in the population growth (annual changes in the logarithm of population abundance $S(t)$)

$$r(t) = \ln\left(\frac{S(t+1)}{S(t)}\right)$$

is described by a Gaussian distribution.

- The population variability would be a geometric random walk

$$r(t) = \frac{dS(t)}{S(t)} = \sigma_r dB(t)$$

The independence of the increments of Brownian motion implying that $r(t)$ is a purely random process.

- A sobering conclusion. Natural processes that look purely random, are processes that depend on some many uncontrollable variables that any attempt to handle them is outside our reach. This would be a serious blow to, for example, the implementation of sustentability measures.

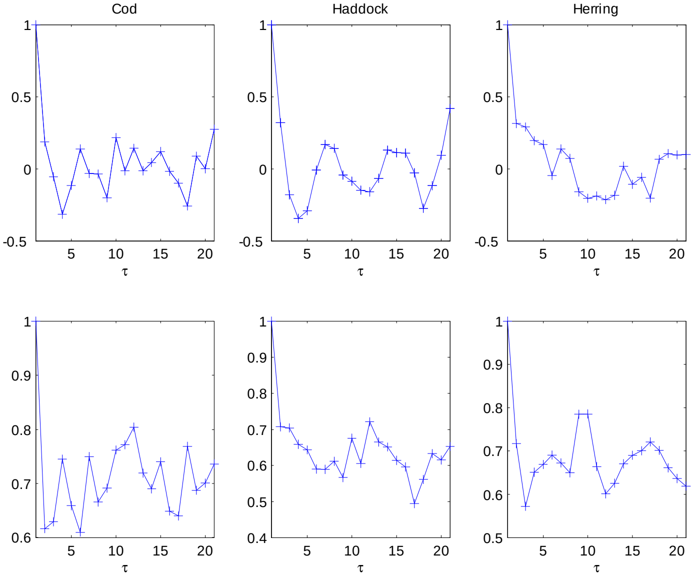
The dynamics of exploited fish populations

- Reanalyze some of the same type of data: Spawning-stock biomass (SSB) for commercial fish stocks in the North Atlantic. The SSB time-series data is derived from age-based analytical assessments estimated by the 2013 working groups of the International Council for the Exploration of the Sea (ICES), based on the compilation of data from sampling of fisheries (e.g. commercial catch-at-age) and from scientific research surveys.
- Select three North Atlantic stocks for which the annual time-series of SSB covers at least 60 years: Northeast Arctic cod (Gadus morhua), Arctic haddock (Melanogrammus aeglefinus) and the North Sea autumn-spawning herring (Clupea harengus).
- Autocorrelation functions for $r(t)$ and $|r(t)|$

$$C(r,\tau) = \frac{\mathbb{E}\{r(t)\, r(t+\tau)\}}{\sigma^2}$$

$$C(|r|,\tau) = \frac{\mathbb{E}\{|r(t)|\, |r(t+\tau)|\}}{\sigma^2}$$

The dynamics of exploited fish populations



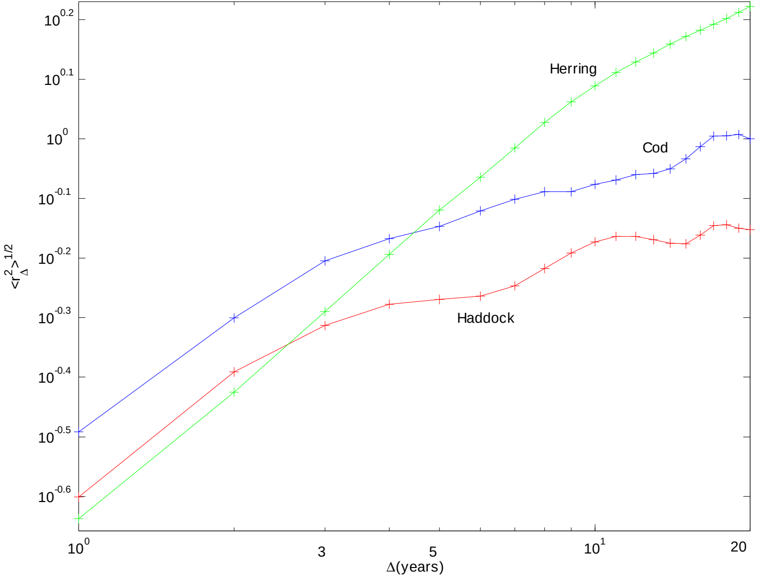
- Already for time lags of one year, autocorrelations are at noise level, suggestive of uncorrelated processes.
- However, if $S(t)$ is indeed a geometrical Brownian motion, scaling properties of $r(t)$ should be checked

$$r_{\Delta}(t) = \ln\left\{\frac{S(t+\Delta)}{S(t)}\right\} = \sum_{i=1}^{\Delta} r(t+i)$$

- The geometrical Brownian motion hypothesis would imply

$$\left(\mathbb{E}\left\{r_{\Delta}^{2}\right\}\right)^{1/2} \backsim \Delta^{1/2}$$

The dynamics of exploited fish populations

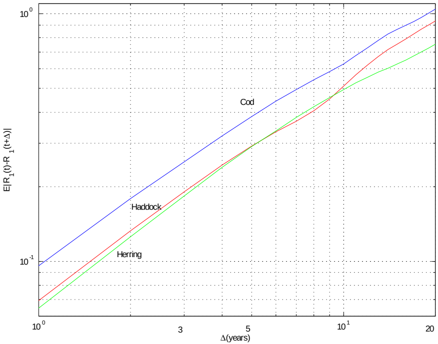


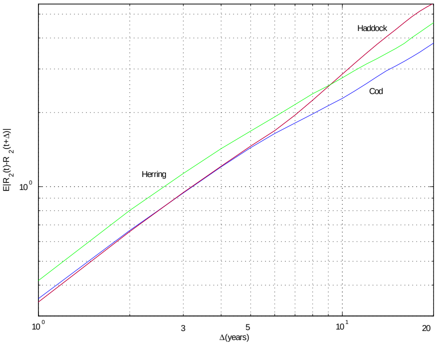
The dynamics of exploited fish populations

- At the species level the geometrical Brownian motion is not a good hypothesis. Even for Herring, where the data seems to follow a scaling law, the slope at large Δ is closer to 0.7 than to 0.5.
- Whatever is actually determining the stochastic process for each species is somehow washed out when averaging over all the 27 species as Niwa did. No surprise, recall the central limit theorem.
- Reconstruct the dynamics of $\sigma(t)$ from the data: Compute the local value of $\sigma(t)$ by the standard deviation of $r(t)$. (6−years window). The cumulative processes and scaling properties of R_1 and R_2

$$
\begin{aligned}
\sum_{i=1}^{t} \sigma(i) &= \beta_1 t + R_1(t) \\
\sum_{i=1}^{t} \ln \sigma(i) &= \beta_2 t + R_2(t)
\end{aligned}
$$

The dynamics of exploited fish populations





The dynamics of exploited fish populations

- R_1 and R_2 obey an approximate scaling law with exponents H in the range $0.8 - 0.9$. Hence R_1 and R_2 may be modelled by fractional Brownian motion implying that the fluctuations of σ and $\ln \sigma$, away from an average value, are modeled by Gaussian fractional noise.
- Alternative models for the population fluctuations

$$\begin{aligned} dS(t) &= \sigma(t) S(t) dB_t \\ \sigma(t) &= \beta_1 + \alpha_1 \left(B_{H_1}(t) - B_{H_1}(t-1) \right) \end{aligned}$$

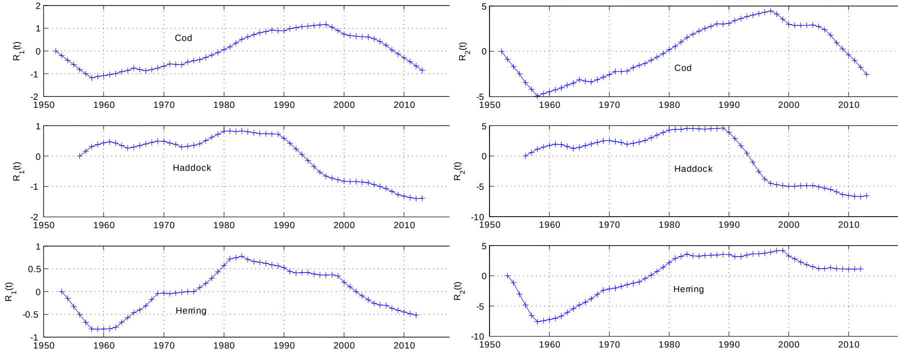
$$\begin{aligned} dS(t) &= \sigma(t) S(t) dB_t \\ \ln \sigma(t) &= \beta_2 + \alpha_2 \left(B_{H_2}(t) - B_{H_2}(t-1) \right) \end{aligned}$$

with the following values for the Hurst coefficients H_1 and H_2

	H_1	H_2
Cod	0.86	0.87
Haddock	0.89	0.9
Herring	0.93	0.87

The dynamics of exploited fish populations

- The dynamics of the fluctuations is a species-dependent long range memory process.
- The cumulative amplitude fluctuations R_1 R_2



Geometric Brownian motion as a market model?

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB(t)$$

Consequeces:

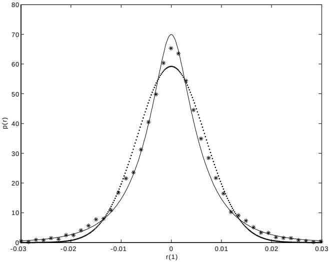
Price increments would be log-normal

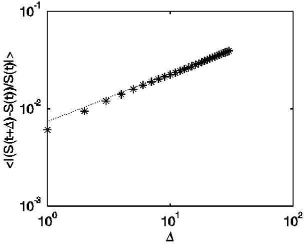
$$p\left(\ln \frac{S_T}{S_t}\right) = \frac{1}{\sqrt{2\pi\sigma^2 (T-t)}} \exp\left(-\frac{\left(\ln \frac{S_T}{S_t} - \left(\mu - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2 (T-t)}\right)$$

and selfsimilar, $Law(X(at)) = Law(a^H X(t))$ **with** $H = 1/2$

$$E\left|\frac{S(t+\Delta) - S(t)}{S(t)}\right| \approx \Delta^H$$

However





Modification: Volatility as a process

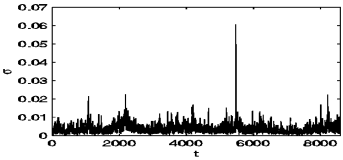
$$\frac{dS_t}{S_t}\left(\bullet,\omega^{'}\right) = \mu_t\left(\bullet,\omega^{'}\right)dt+\sigma_t\left(\bullet,\omega^{'}\right)dB\left(t\right)$$

reconstructed from market data

$$\sigma_t^2\left(\bullet,\omega^{'}\right)=\lim_{\varepsilon\rightarrow 0}\frac{1}{\varepsilon}\left\{E\left(\log S_{t+\varepsilon}-\log S_t\right)^2\right\}$$

Reconstruction of the market process

Volatility (σ)

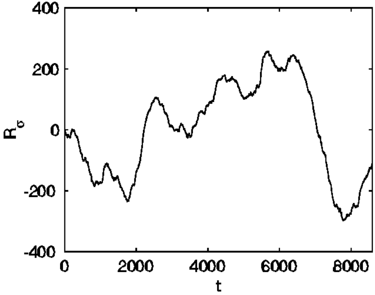


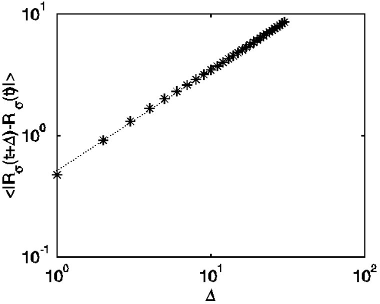
Result: The log integral of the volatility is well represented by

$$\sum_{n=0}^{t/\delta} \log \sigma \left(n\delta \right) = \beta t + R_{\sigma} \left(t \right)$$

$R_{\sigma} \left(t \right)$ has selfsimilar properties

$$E \left| R_{\sigma} \left(t + \Delta \right) - R_{\sigma} \left(t \right) \right| \sim \Delta^{H}$$





$$
\begin{array}{rcl}
dS_t & = & \mu S_t dt + \sigma_t S_t dB(t) \\
\log \sigma_t & = & \beta + \frac{k}{\delta}\{B_H(t) - B_H(t-\delta)\}
\end{array}
$$

δ is the temporal observation scale and H has values in the range $0.8 - 0.9$ (*volatility clustering*)

$$
\sigma(t) = \theta e^{\frac{k}{\delta}\{B_H(t)-B_H(t-\delta)\}-\frac{1}{2}\left(\frac{k}{\delta}\right)^2\delta^{2H}}
$$

Long-range correlation vs. roughness

$p-$variation of a process $X(t)$

$$V_p(0,T) = \sup_{\text{partitions}} \sum_{k=1}^{n} \left| X(t_k) - X(t_{k-1}) \right|^p$$

$p-$variation index I

$$I(X, |0,T]) = \inf \{ p > 0; V_p(0,T) < \infty \}$$

Hölder regularity (roughness)

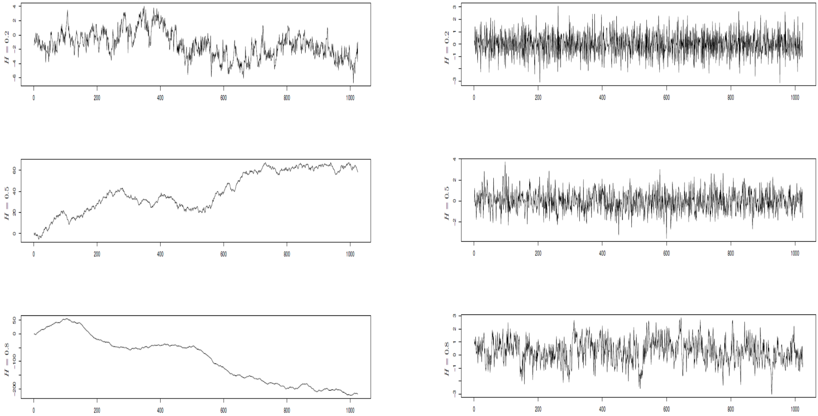
$$H_r = \frac{1}{I}$$

For fractional Brownian motion $H = H_r$

Long memory or rough volatility?

- Recent work (Gatheral) examining the roughness of high-frequency data suggests $H < 0.5$. for the volatility. Contradiction with volatility bclustering. How can it be consistent with long-memory in case volatility is described by fBM.
- Possible explanation: For high frequency data a path for which the *realized* high-frequency roughness volatility is $< \frac{1}{2}$ may have *spot* volatility $> \frac{1}{2}$. The realized volatility at high frequency is strongly afected by discretization (*microstructure noise*).
- Alternatively one might have a process, not fBM, with $H \neq H_r$
- Or even simpler: volatility driven by fractional Gaussian noise, not fBM as in the model reconstructed from the data

Long memory or rough volatility? fBM vs. fGN



Interpretation and prediction

- In conclusion: Mathematics, and in particular the mathematics of stochastic processes, provides a framework to interpret the many types of randomness and uncertainty that we face in the natural and social phenomena.
- Sometimes it may also provide a means to predict the future or the outcome of certains actions. Rarely a sure prediction, but at least a means to assign different degrees of probability to possible futures.

Prediction yes, we need prediction, but beware of fortune tellers

Caravaggio

References

R. Cont and P. Tankov; *Financial Modelling with Jump Processes*, Chapman and Hall 2009.
P. Embrechts e M. Maejima; *Selfsimilar processes*, Princeton Univ. Press 2002.
H. C. Mendes, A. Murta and RVM; *Long range dependence and the dynamics of exploited fish populations*, Advances in Complex Systems 18 (2015) 1550017.
RVM and M. J. Oliveira; *A Data-Reconstructed Fractional Volatility Model*; E-Economics DP 2008-22.
RVM, M.J. Oliveira, A.M. Rodrigues; *No-arbitrage, leverage and completeness in a fractional volatility model*, Physica A419 (2015) 470
J. Gatheral, T. Jaisson and M. Rosenbaum; *Volatility is rough*, Quantitative Finance 18 (2018) 933-949
R. Cont and P. Das; *Rough volatility: fact or artefact?*, arXiv:2203.13820, April 2022.
RVM; *The fractional volatility model and rough volatility*, arXiv:2206.02205, June 2022.