A stochastic representation for the Poisson-Vlasov equation

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Firenze, September 2006

1 Stochastic representations of solutions of pde's. Why?

(i) Connection between functional analysis and probability theory
(ii) An intuitive characterization of the equation solutions;
(iii) A calculation tool which may replace the need for very fine integration grids;
(iv) Intrinsic characterization of the fluctuations associated to the physical system.

The stochastic principle

- Kinetic and fluid equations are obtained from the full particle dynamics in the 6N-dimensional phase-space by a chain of reductions.
- Along the way, information on the actual nature of fluctuations and turbulence may be lost. An accurate model of turbulence may exist at some intermediate (mesoscopic) level, but not in the final mean-field equation.
- A stochastic representation is a process for which the mean value is the solution of the mean-field equation. The process itself contains more information. This does not mean that the process is an accurate mesoscopic model of Nature, because we might be climbing up a path different from the one that led us down from the particle dynamics.
- But, it is a surrogate mesoscopic model from which fluctuations are easily computed. This is what we refer to as *the stochastic principle*. At the minimum, one might say that the stochastic principle provides another closure procedure.

1.1 Stochastic representation of the solutions of linear equations

A very classical field (Courant, Friedrichs and Lewy in the 1920's)

Example: Diffusion processes and elliptic operators

$$\frac{1}{2}\Delta u(x) - \lambda u(x) = -f(x)$$

$$u(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\lambda t} f(X_t)\, dt\right)$$

1.2 Stochastic representations for nonlinear equations

A developing field

McKean

Dynkin - Diffusion and branching, branching exit measures $\Delta u + u^\alpha = 0$

LeJan and Sznitman plus *Oregon school* - Navier-Stokes

2 Poisson-Vlasov equation.Stochastic representation and existence

Poisson-Vlasov equation in 3+1 space-time dimensions

$$\frac{\partial f}{\partial t} + \overrightarrow{v} \cdot \nabla_x f - \frac{e}{m} \nabla_x \Phi \cdot \nabla_v f = 0 \tag{1}$$

$$\Delta_x \Phi_f = -4\pi \left\{ e \int f\left(\overrightarrow{x}, \overrightarrow{v}, t\right) d^3 v - e \rho_B \left(\overrightarrow{x}\right) \right\}$$

$\rho_B \left(\overrightarrow{x}\right)$ = background charge density. Passing to the Fourier transform

$$F(\xi, t) = \frac{1}{(2\pi)^3} \int d^6 \eta f(\eta, t) e^{i\xi \cdot \eta}$$

with $\eta = \left(\overrightarrow{x}, \overrightarrow{v}\right)$ and $\xi = \left(\overrightarrow{\xi_1}, \overrightarrow{\xi_2}\right) \circeq (\xi_1, \xi_2)$,

$$\begin{aligned} 0 = & \frac{\partial F(\xi, t)}{\partial t} - \overrightarrow{\xi_1} \cdot \nabla_{\xi_2} F(\xi, t) \\ & + \frac{4\pi e^2}{m} \int d^3 \xi_1^{'} F\left(\xi_1 - \xi_1^{'}, \xi_2, t\right) \frac{\overrightarrow{\xi_2} \cdot \overrightarrow{\xi_1^{'}}}{\left|\xi_1^{'}\right|^2} \left\{ F\left(\xi_1^{'}, 0, t\right) - \frac{\tilde{\rho}_B \left(\xi_1^{'}\right)}{(2\pi)^{3/2}} \right\} \end{aligned} \tag{2}$$

$\tilde{\rho}_B \left(\xi_1^{'}\right)$ = Fourier transform of $\rho_B(x)$.

Rescaling the time

$$\tau = \gamma\left(|\xi_2|\right) t$$

$\gamma\left(|\xi_2|\right)$ a positive continuous function

$$\begin{array}{ll} \gamma\left(|\xi_2|\right) = 1 & if \quad |\xi_2| < 1 \\ \gamma\left(|\xi_2|\right) \geq |\xi_2| & if \quad |\xi_2| \geq 1 \end{array}$$

leads to

$$\begin{aligned} \frac{\partial F\left(\xi,\tau\right)}{\partial \tau} &= \frac{\overrightarrow{\xi_1}}{\gamma\left(|\xi_2|\right)} \cdot \nabla_{\xi_2} F\left(\xi,\tau\right) - \frac{4\pi e^2}{m} \int d^3\xi_1^{'} F\left(\xi_1 - \xi_1^{'}, \xi_2, \tau\right) \\ &\quad \times \frac{\overrightarrow{\xi_2} \cdot \hat{\xi}_1^{'}}{\gamma\left(|\xi_2|\right)\left|\xi_1^{'}\right|} \left\{ F\left(\xi_1^{'}, 0, \tau\right) - \frac{\tilde{\rho}_B\left(\xi_1^{'}\right)}{\left(2\pi\right)^{3/2}} \right\} \end{aligned}$$

with $\hat{\xi}_1 = \frac{\overrightarrow{\xi_1}}{|\xi_1|}$

Or, in integral form

$$\begin{aligned} F\left(\xi,\tau\right) &= e^{\tau \frac{\overrightarrow{\xi_1}}{\gamma\left(|\xi_2|\right)} \cdot \nabla_{\xi_2}} F\left(\xi_1,\xi_2,0\right) - \frac{4\pi e^2}{m} \int_0^\tau ds e^{(\tau - s)\frac{\overrightarrow{\xi_1}}{\gamma\left(|\xi_2|\right)} \cdot \nabla_{\xi_2}} \\ &\quad \times \int d^3\xi_1^{'} F\left(\xi_1 - \xi_1^{'}, \xi_2, s\right) \frac{\overrightarrow{\xi_2} \cdot \hat{\xi}_1^{'}}{\gamma\left(|\xi_2|\right)\left|\xi_1^{'}\right|} \left\{ F\left(\xi_1^{'}, 0, s\right) - \frac{\tilde{\rho}_B\left(\xi_1^{'}\right)}{\left(2\pi\right)^{3/2}} \right\} \end{aligned} \quad (3)$$

We obtain a stochastic representation for the following function

$$\chi(\xi_1,\xi_2,\tau) = e^{-\lambda\tau}\frac{F(\xi_1,\xi_2,\tau)}{h(\xi_1)}$$

with λ a constant and $h(\xi_1)$ a positive function to be specified later on. The integral equation for $\chi(\xi_1,\xi_2,\tau)$ is

$$\begin{aligned}\chi(\xi_1,\xi_2,\tau) \;=\; & e^{-\lambda\tau}\chi\left(\xi_1,\xi_2+\tau\frac{\xi_1}{\gamma(|\xi_2|)},0\right)-\frac{8\pi e^2}{m\lambda}\frac{\left(|\xi_1|^{-1}h*h\right)(\xi_1)}{h(\xi_1)}\int_0^\tau ds\lambda e^{-\lambda s} \\ & \times\int d^3\xi_1' p\left(\xi_1,\xi_1'\right)\chi\left(\xi_1-\xi_1',\xi_2+s\frac{\xi_1}{\gamma(|\xi_2|)},\tau-s\right) \\ & \times\frac{\vec{\xi_2}\cdot\hat{\xi_1'}}{\gamma(|\xi_2|)}\left\{\frac{1}{2}e^{\lambda(\tau-s)}\chi\left(\xi_1',0,\tau-s\right)-\frac{1}{2}\frac{\tilde{\rho}_B\left(\xi_1'\right)}{(2\pi)^{3/2}h\left(\xi_1'\right)}\right\}\end{aligned} \quad (4)$$

with

$$\left(|\xi_1|^{-1}h*h\right)=\int d^3\xi_1'\left|\xi_1'\right|^{-1}h\left(\xi_1-\xi_1'\right)h\left(\xi_1'\right)$$

and

$$p\left(\xi_1,\xi_1'\right)=\frac{\left|\xi_1'\right|^{-1}h\left(\xi_1-\xi_1'\right)h\left(\xi_1'\right)}{\left(|\xi_1|^{-1}h*h\right)}$$

Consider:

1) An exponential process with parameter λ (and a time shift in the second variable)
$e^{-\lambda\tau}$ is the survival probability during time τ,
$\lambda e^{-\lambda s}$ is the decay probability in the interval $(s, s+ds)$,

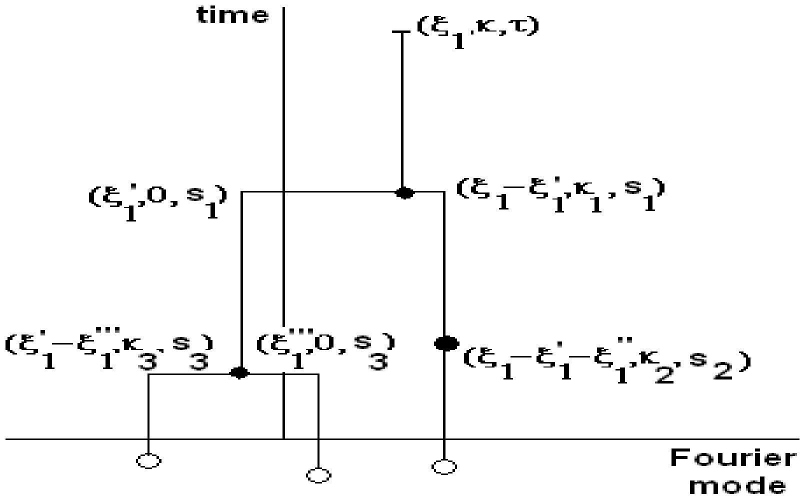
2) A branching process with probability density $p\left(\xi_1, \xi_1^{'}\right) d^3\xi_1^{'}$

3) A Bernoulli process (probabilities $\frac{1}{2}, \frac{1}{2}$)

Stochastic interpretation of the Eq.(4)

- Starting at (ξ_1, ξ_2, τ), a particle lives for an exponentially distributed time s up to time $\tau - s$.
- At its death a coin l_s (probabilities $\frac{1}{2}, \frac{1}{2}$) is tossed.
- If $l_s = 0$ two new particles are born at time $\tau - s$ with Fourier modes $\left(\xi_1 - \xi_1^{'}, \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right)$ and $\left(\xi_1^{'}, 0\right)$ with probability density $p\left(\xi_1, \xi_1^{'}\right)$.
- If $l_s = 1$ only the $\left(\xi_1 - \xi_1^{'}, \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}\right)$ particle is born and the process also samples the background charge at $\tilde{\rho}_B\left(\xi_1^{'}\right)$.

Each one of the newborn particles continues its backward-in-time evolution, following the same death and birth laws. When one of the particles of this tree reaches time zero it samples the initial condition.



1.

The process $X\left(\xi_1, \xi_2, \tau\right)$, is obtained as the limit of the following iteration

$$\begin{aligned} X^{(k+1)}\left(\xi_1, \xi_2, \tau\right) = \; & \chi\left(\xi_1, \xi_2+\tau \frac{\xi_1}{\gamma\left(\left|\xi_2\right|\right)}, 0\right) \mathbf{1}_{[s>\tau]}+g_2\left(\xi_1, \xi_1^{'}, s\right) \\ & \times X^{(k)}\left(\xi_1-\xi_1^{'}, \xi_2+s \frac{\xi_1}{\gamma\left(\left|\xi_2\right|\right)}, \tau-s\right) X^{(k)}\left(\xi_1^{'}, 0, \tau-s\right) \mathbf{1}_{[s<\tau]} \mathbf{1}_{\left[l_s=0\right]} \\ & +g_1\left(\xi_1, \xi_1^{'}\right) X^{(k)}\left(\xi_1^{'}, 0, \tau-s\right) \mathbf{1}_{[s<\tau]} \mathbf{1}_{\left[l_s=1\right]} \end{aligned}$$

- At each branching point where two particles are born, the coupling constant is

$$g_2\left(\xi_1,\xi_1^{'},s\right)=-e^{\lambda(\tau-s)}\frac{8\pi e^2}{m\lambda}\frac{\left(\left|\xi_1\right|^{-1}h*h\right)\left(\xi_1\right)}{h\left(\xi_1\right)}\frac{\vec{\xi_2}\cdot\hat{\xi_1^{'}}}{\gamma\left(\left|\xi_2\right|\right)}$$

- When only one particle is born and the process samples the background charge, the coupling is

$$g_1\left(\xi_1,\xi_1^{'}\right)=\frac{8\pi e^2}{m\lambda}\frac{\left(\left|\xi_1\right|^{-1}h*h\right)\left(\xi_1\right)}{h\left(\xi_1\right)}\frac{\tilde{\rho}_B\left(\xi_1^{'}\right)}{\left(2\pi\right)^{3/2}h\left(\xi_1^{'}\right)}\frac{\vec{\xi_2}\cdot\hat{\xi_1^{'}}}{\gamma\left(\left|\xi_2\right|\right)}$$

- When one particle reaches time zero and samples the initial condition the coupling is

$$g_0\left(\xi_1,\xi_2\right)=\frac{F\left(\xi_1,\xi_2,0\right)}{h\left(\xi_1\right)}$$

The solution is the expectation value of a multiplicative functional that is the product of all these couplings for each realization of the process $X\left(\xi_1,\xi_2,\tau\right)$

$$\chi\left(\xi_1,\xi_2,\tau\right)=\mathbb{E}\left\{\Pi\left(g_0g_0^{'}\cdots\right)\left(g_1g_1^{'}\cdots\right)\left(g_2g_2^{'}\cdots\right)\right\}$$

Convergence of the multiplicative functional hinges on the fulfilling of the following conditions:

(A) $\left|\frac{F(\xi_1,\xi_2,0)}{h(\xi_1)}\right| \leq 1$

(B) $\left|\frac{\tilde{\rho}_B(\xi_1)}{(2\pi)^{3/2}h(\xi_1)}\right| \leq 1$

(C) $\left(|\xi_1|^{-1}\, h * h\right) \leq h\left(\xi_1\right)$

Condition (C) is satisfied, for example, for

$$h\left(\xi_1\right) = \frac{c}{\left(1+|\xi_1|^2\right)^2} \qquad and \qquad c \leq \frac{1}{4\pi}$$

Indeed computing $\frac{1}{h(\xi_1)}\left(|\xi_1|^{-1}\, h * h\right)$ one obtains

$$\frac{1}{h\left(\xi_1\right)}\left(|\xi_1|^{-1}\, h * h\right) = 4\pi c \int_0^\infty dr \frac{r}{(1+r^2)^2} \frac{\left(1+|\xi_1|^2\right)^2}{\left(1+(|\xi_1|-r)^2\right)\left(1+(|\xi_1|+r)^2\right)}$$

This integral is bounded by a constant for all $|\xi_1|$, therefore, choosing c sufficiently small, condition (C) is satisfied.

Once $h\left(\xi_1\right)$ consistent with (C) is found, conditions (A) and (B) only put restrictions on the initial conditions and the background charge. With the conditions (A) and (B), choosing $\lambda = \frac{8\pi e^2}{m}$ and $c \leq e^{-\lambda\tau}\frac{1}{4\pi}$, the absolute value of all coupling constants is bounded by one.

The branching process, being identical to a Galton-Watson process, terminates with probability one and the number of inputs to the functional is finite (with probability one). With the bounds on the coupling constants, the multiplicative functional is bounded by one in absolute value almost surely.

Once a stochastic representation is obtained for $\chi(\xi_1, \xi_2, \tau)$, one also has a stochastic representation for the solution of the Fourier-transformed Poisson-Vlasov equation.

The results are summarized in:

Theorem - *There is a stochastic representation for the Fourier-transformed solution of the Poisson-Vlasov equation* $F(\xi_1, \xi_2, t)$ *for any arbitrary finite value of the arguments, provided the initial conditions at time zero and the background charge satisfy the boundedness conditions (A) and (B).*

Corollary - *An existence result for (arbitrarily large) finite time.*

Notice that existence by the stochastic representation method requires only boundedness conditions on the initial conditions and background charge and not any strict smoothness properties.