The market as a mathematical object

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The market

- Benefits
  - Mobilization of savings
  - Risk capital for new technologies
- Pathologies
  - Shift of creative effort from production of goods to financial techniques
  - Market valuation more interesting than investment, enhanced by stockholders and administrator salary policies
- Many markets. Teams in a modern trading room :

  Merger arbitrage

  Index arbitrage

  Convertible bond arbitrage

  Statistical arbitrage

  ....
- Strong impact in the economy and human life

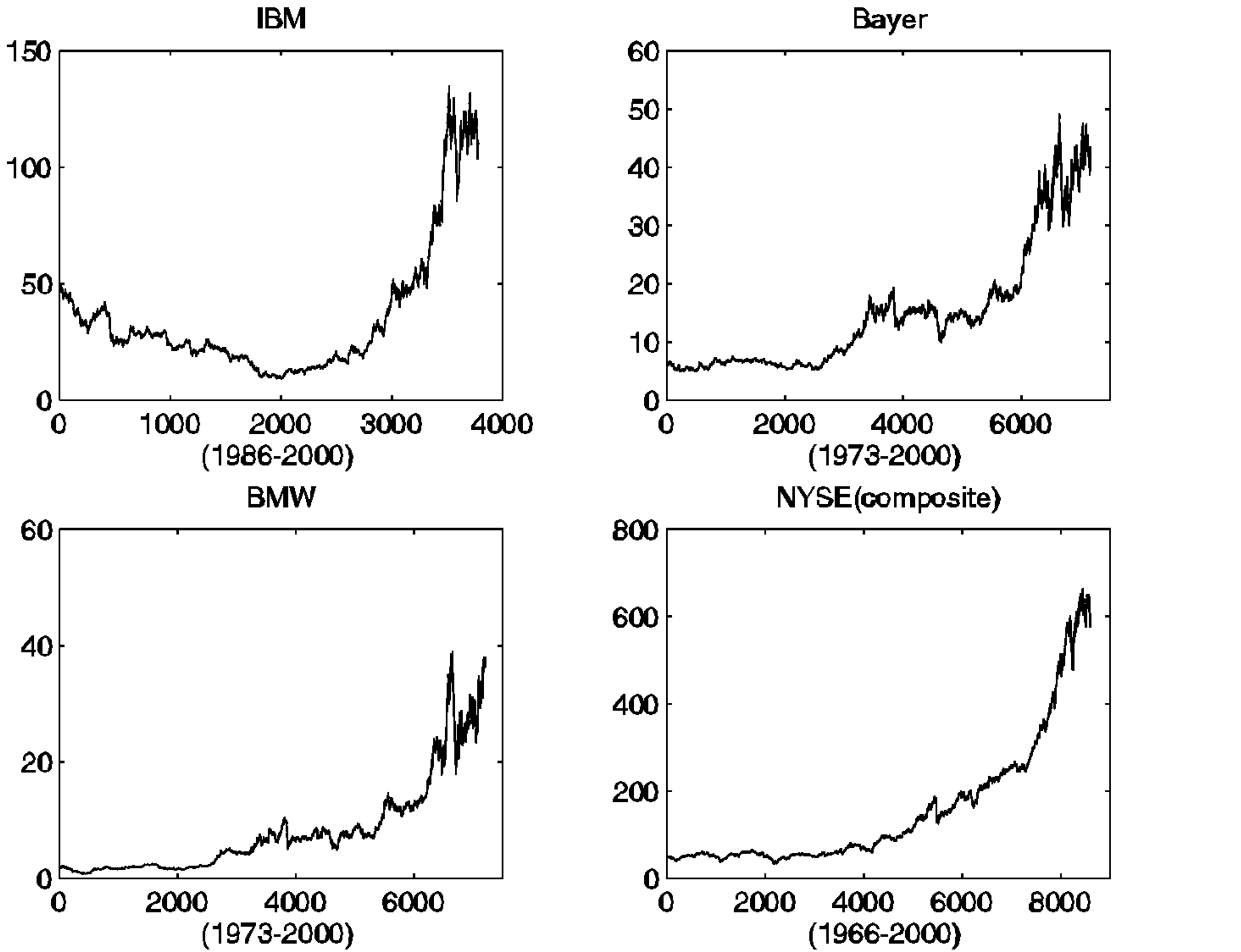
  Mathematical instruments as tools for trading

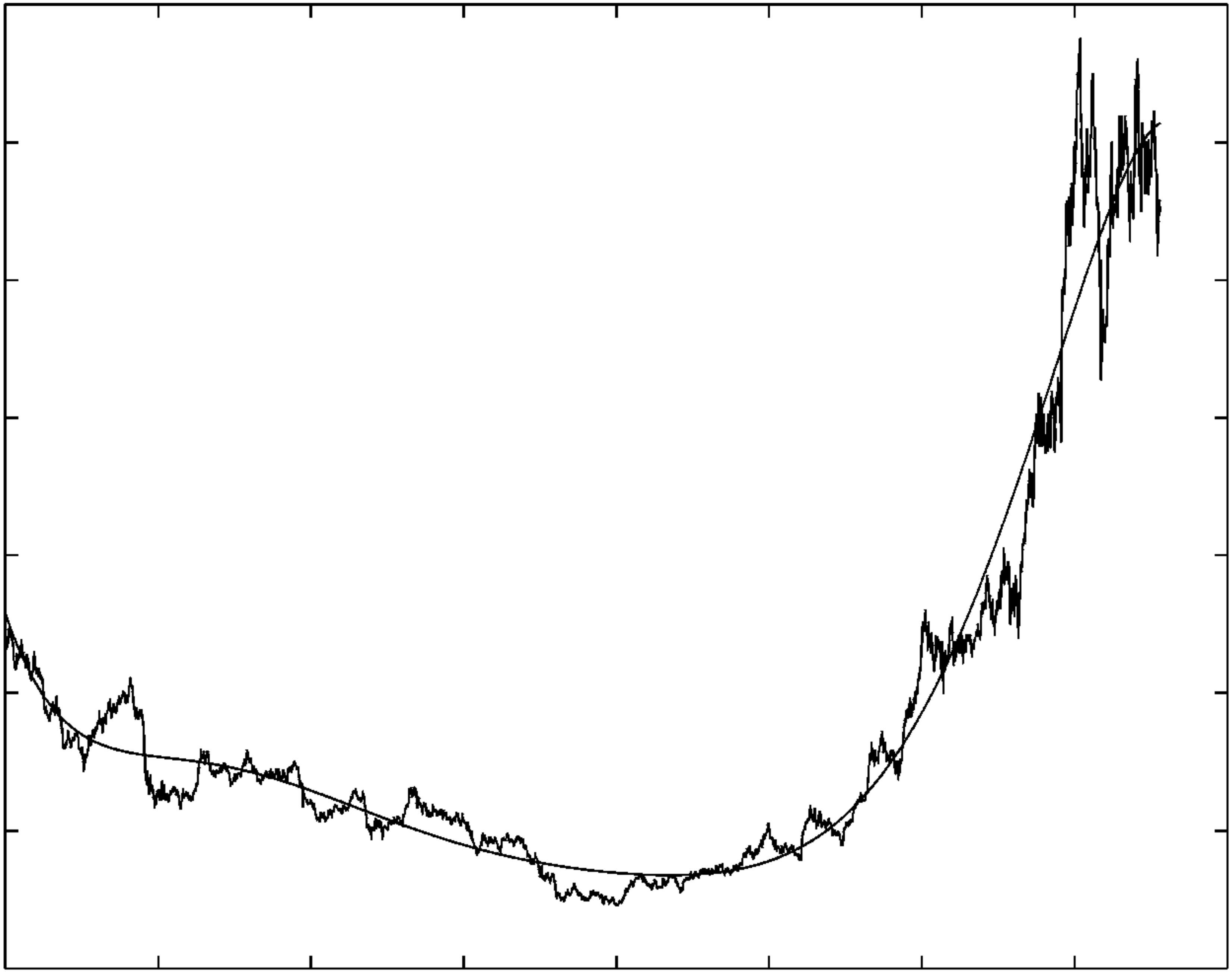
  Interesting process for mathematical modeling

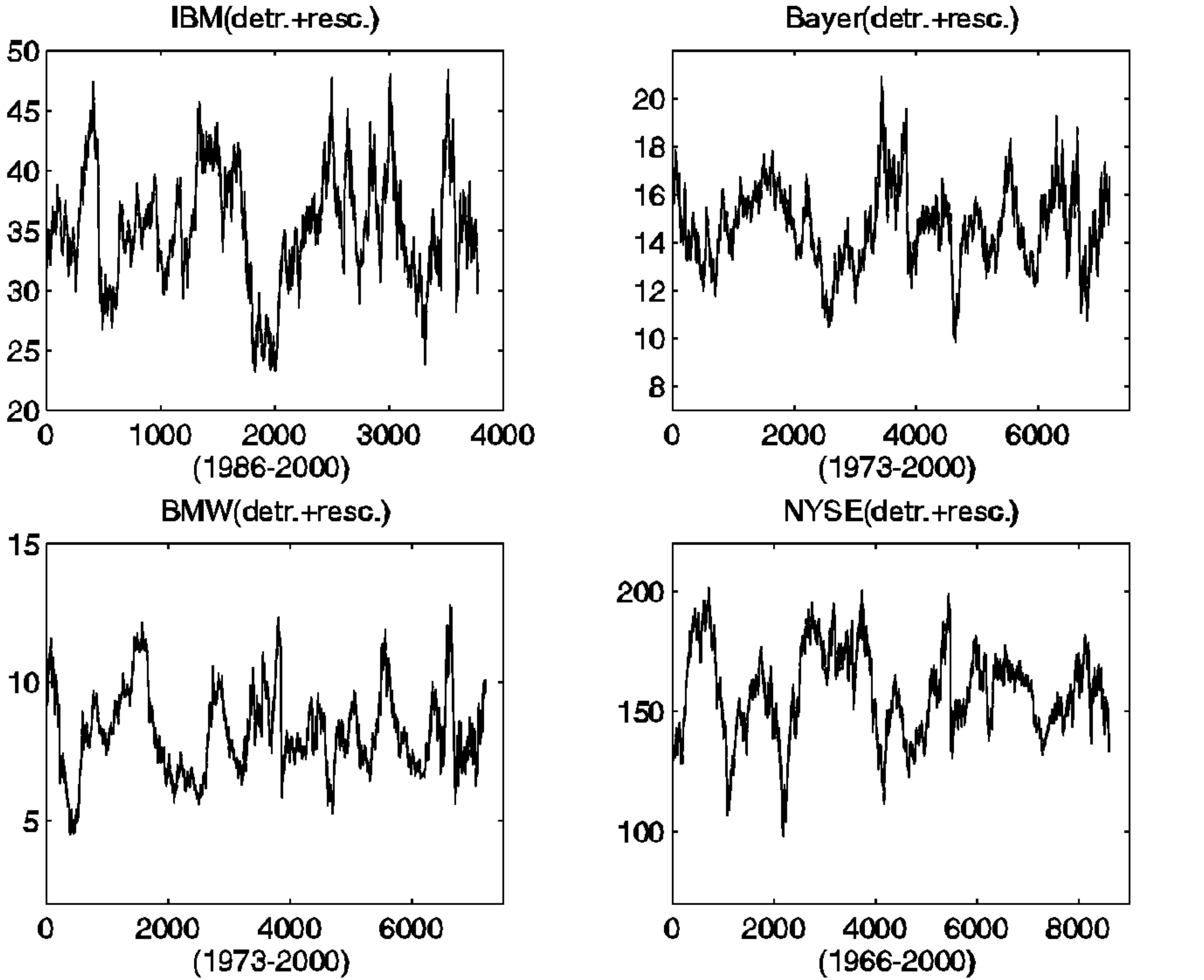
The market as a mathematical object

- 1. Extracting an asymptotic stationary process
- 2. Market fluctuations and turbulence
  Is there a Gibbs measure ?
- 3. Chains with complete connections and variable-length Markov processes
- 4. Looking for structure. The geometry of the market
- 5. Modeling volatility
  Geometric Brownian motion ?
  The induced volatility process
  Fractional noise and the coupled process
- 6. Option pricing formulae and equations

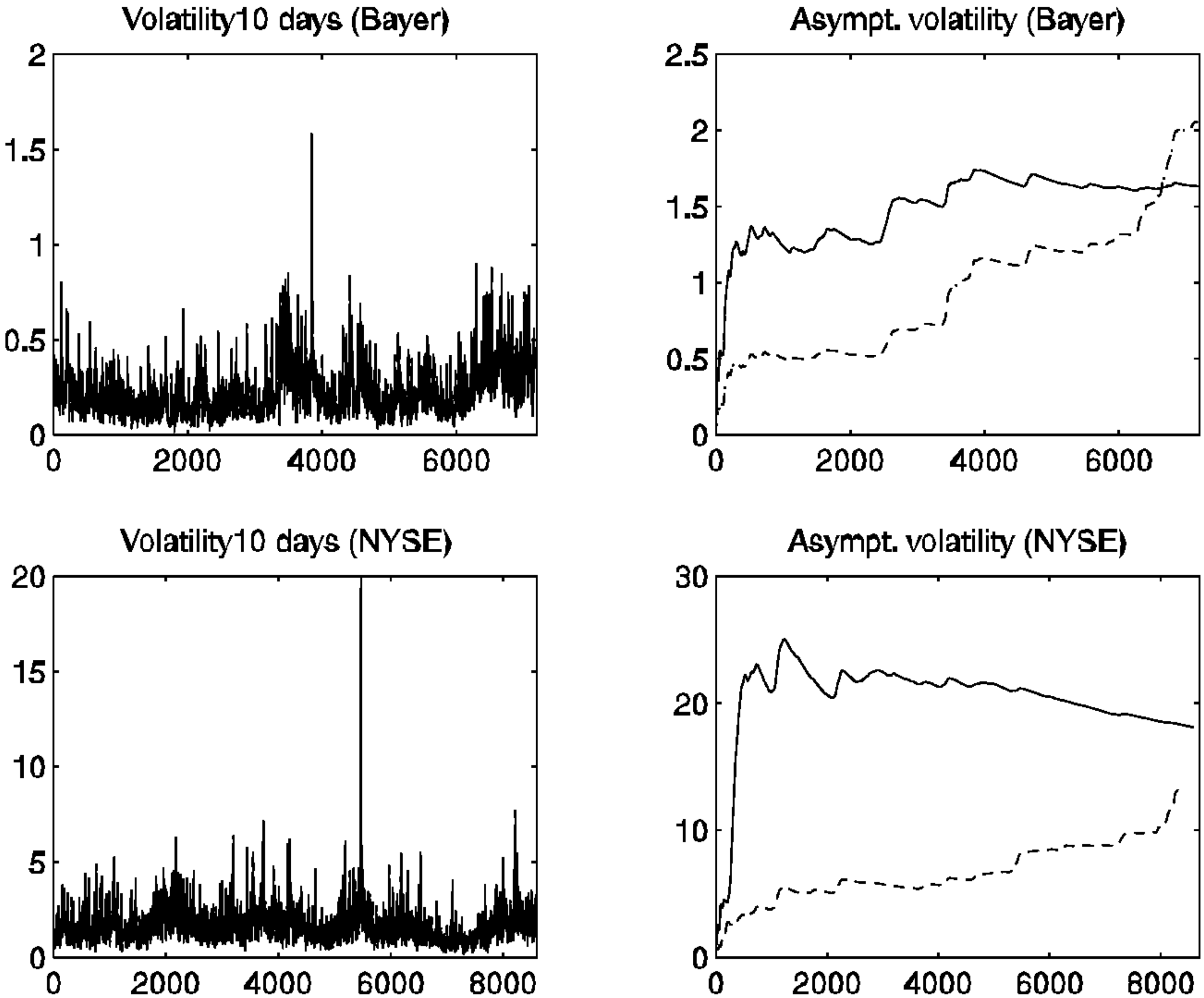
1. Data analysis and stationarity







IBM(detr.+resc.)
50
45
40
35
30
25
20
0
1000
2000
3000
4000
(1986-2000)
Bayer(detr.+resc.)
20
18
16
14
12
10
8
0
2000
4000
6000
(1973-2000)
BMW(detr.+resc.)
15
10
5
0
2000
4000
6000
(1973-2000)
NYSE(detr.+resc.)
200
150
100
0
2000
4000
6000
8000
(1966-2000)



Volatility10 days (Bayer)
Asympt. volatility (Bayer)
Volatility10 days (NYSE)
Asympt. volatility (NYSE)

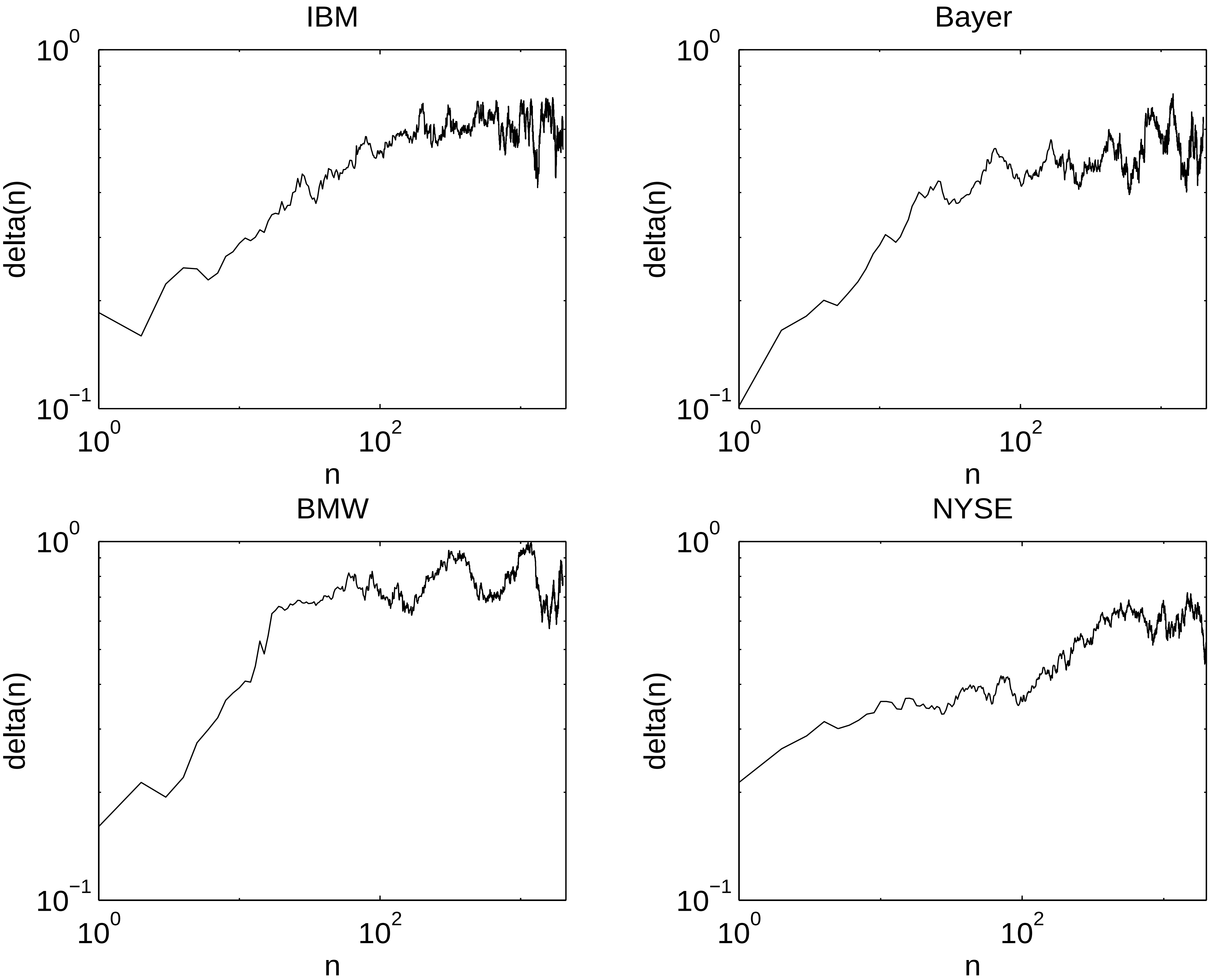
2. Comparison with hydrodynamic turbulence

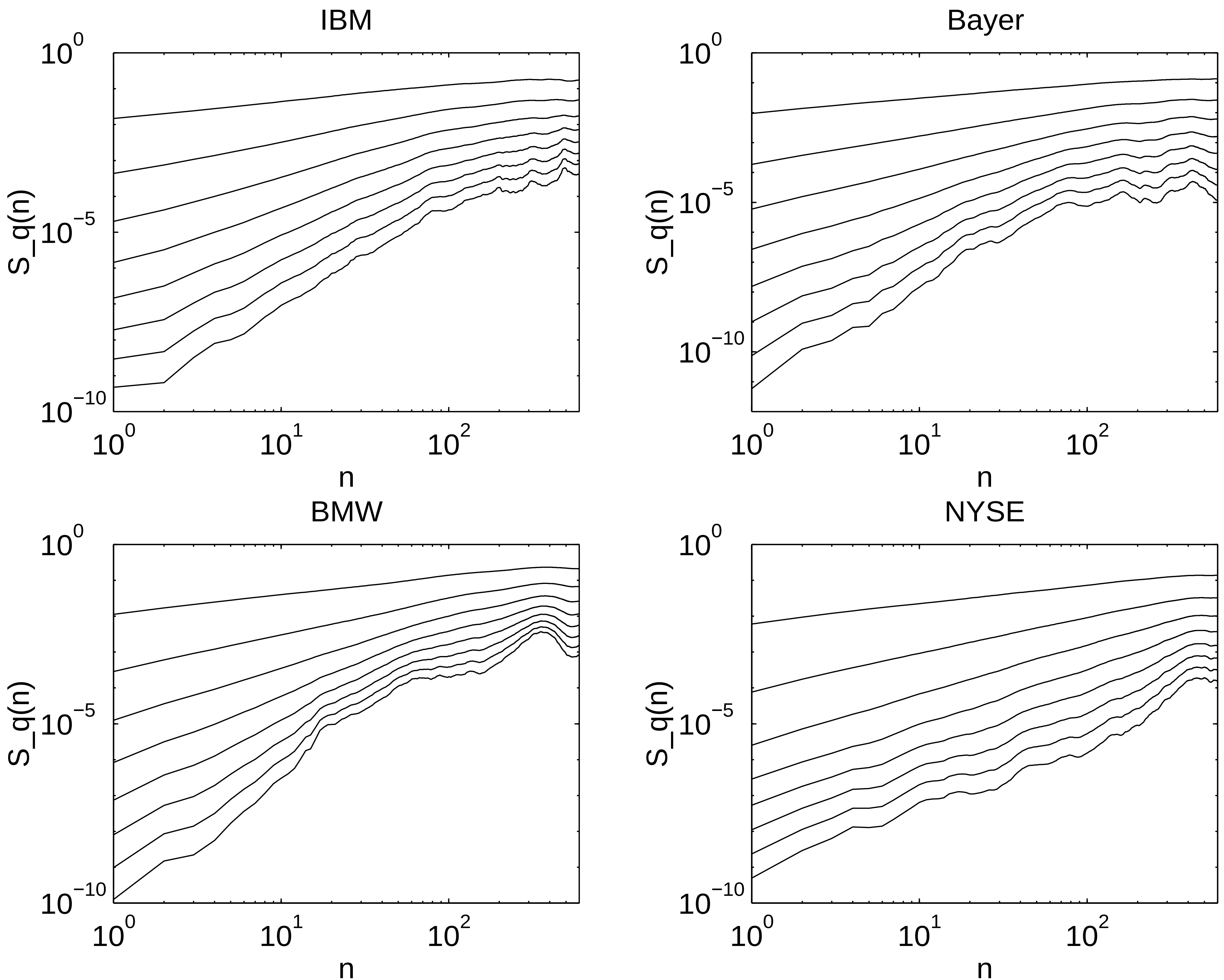
- n-days return

$$r(t,n) = \log p(t+n) - \log p(t)$$

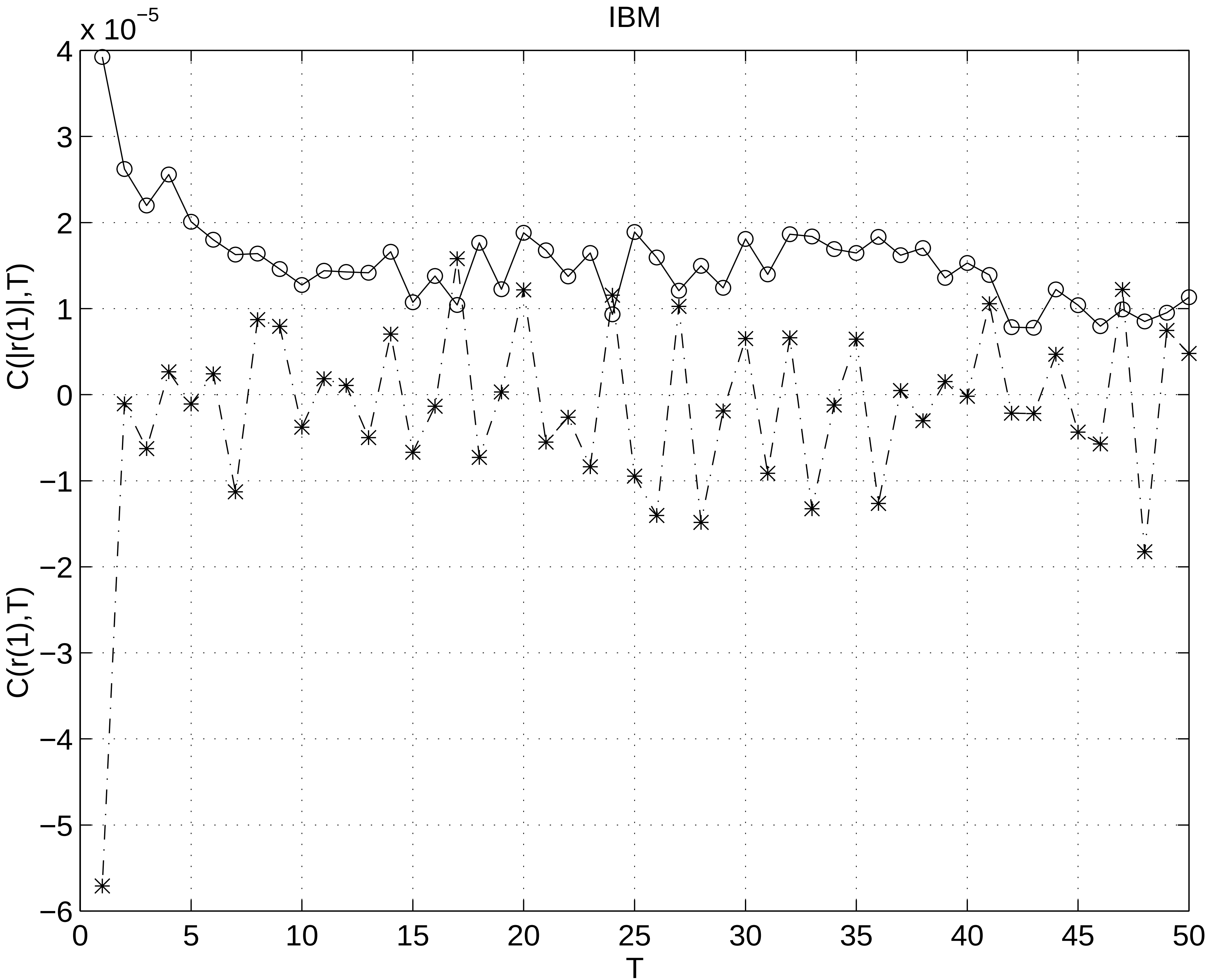
- a) $\delta(n) = \max_t \{r(t,n)\}$ log-concave, asymptotically constant
- b) $S_q(n) = \langle|r(t,n)|^q\rangle \sim n^{\chi(q)}$ (in a limited range)
- c) $\chi(q)$ increasing concave function of q
- d) $\chi(1) = 1/3$ in hydrodynamics
  $\chi(1) \approx 1/2$ in finance

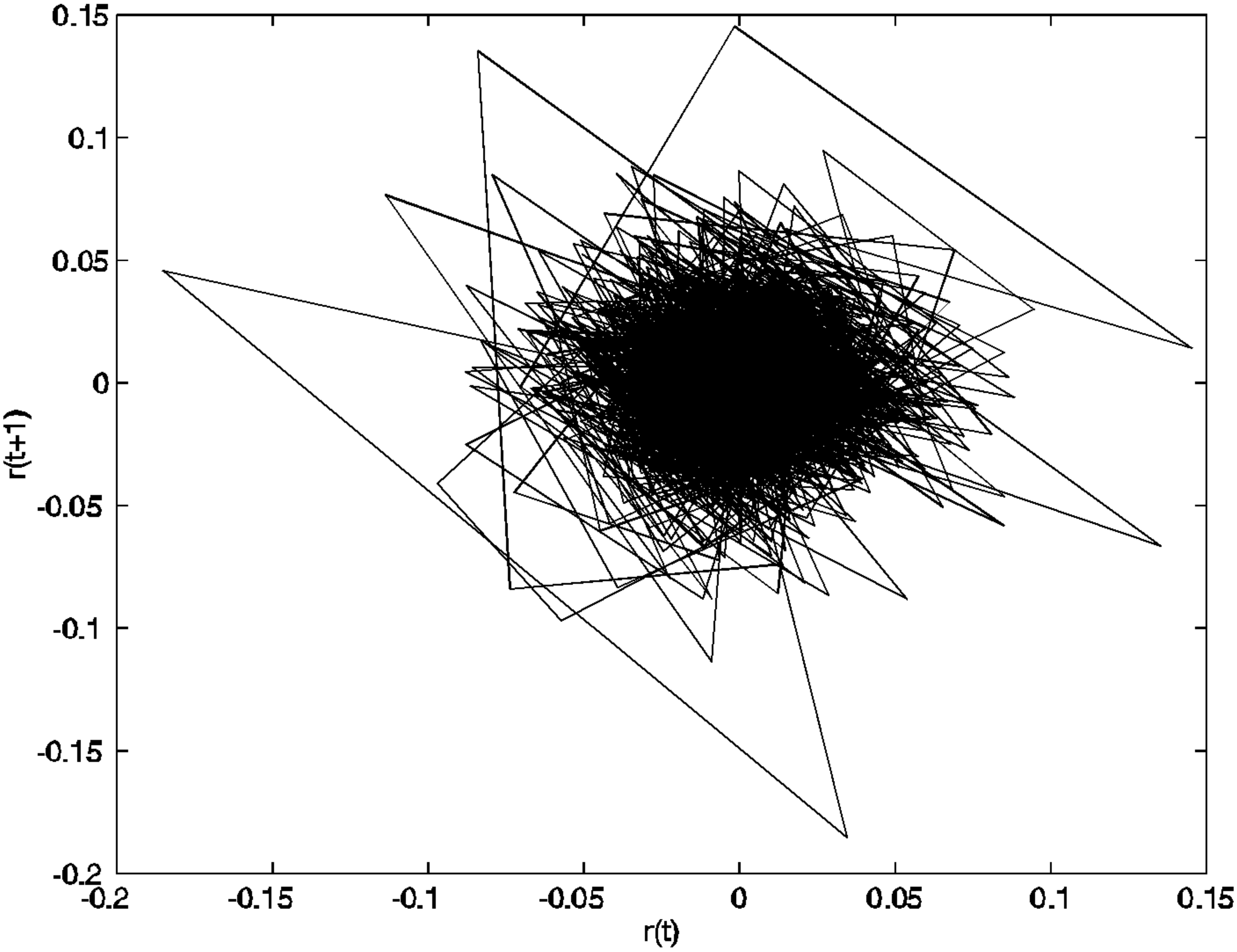
- $C(r(1),T) = \langle r(t+T,1)\, r(t,1)\rangle$


IBM
Bayer
BMW
NYSE
delta(n)
n
10^0
10^{-1}
10^0
10^2



IBM
Bayer
BMW
NYSE
S_q(n)
n
10^0
10^{-5}
10^{-10}
10^1
10^2


IBM
x 10^−5
C(|r(1)|,T)
C(r(1),T)
T
4
3
2
1
0
−1
−2
−3
−4
−5
−6
0
5
10
15
20
25
30
35
40
45
50


0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
r(t+1)
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
r(t)

Naturalness of Gibbs measures

- p_i = probability of event X_i
  1) Normalization : $\Sigma_i\, p_i = 1$
  2) Expectation value of known observables :
  $\Sigma_i\, p_i\, F_k\,(X_i) = C_k$
- Maximum entropy principle :
  $$S = -\,\Sigma_i\, p_i \log p_i + \lambda_0\, \Sigma_i\, p_i + \Sigma_k\, \lambda_k\, \Sigma_i\, p_i\, F_k(X_i)$$
  $$\partial S / \partial p_i = 0 \quad \Rightarrow \quad -\log p_i - 1 + \lambda_0 + \Sigma_k\, \lambda_k\, F_k(X_i) = 0$$
- Conclusion :

The more unbiased estimation compatible with the constraints is :

$$p_i = \exp(\, -1 + \lambda_0 + \Sigma_k\, \lambda_k\, F_k(X_i)\,)$$

with λ_0 , λ_1 , ... obtained from the constraints

Is there a Gibbs measure for the price process ?

- Given a coding

$$\mu(i_1,i_2, \dots ,i_n) = \exp \left(\Sigma_0^{n-1}\phi(\sigma^k(i_1,i_2, \dots ,i_n)) \right)$$

ϕ = (normalized) potential

- Potential range (r)

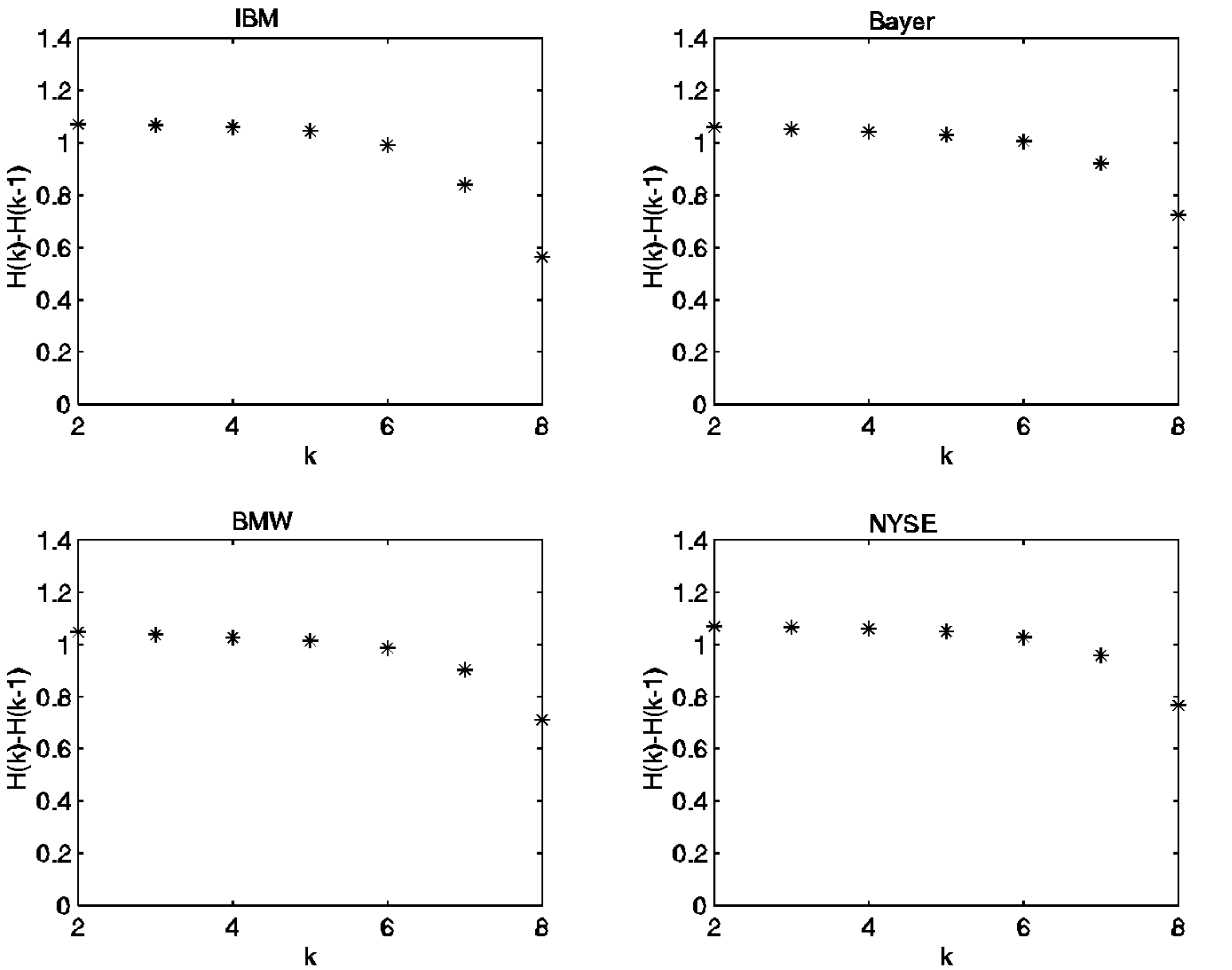
$$H_k = - \Sigma \ \mu(i_1,i_2, \dots ,i_k) \ \log \mu(i_1,i_2, \dots ,i_k)$$

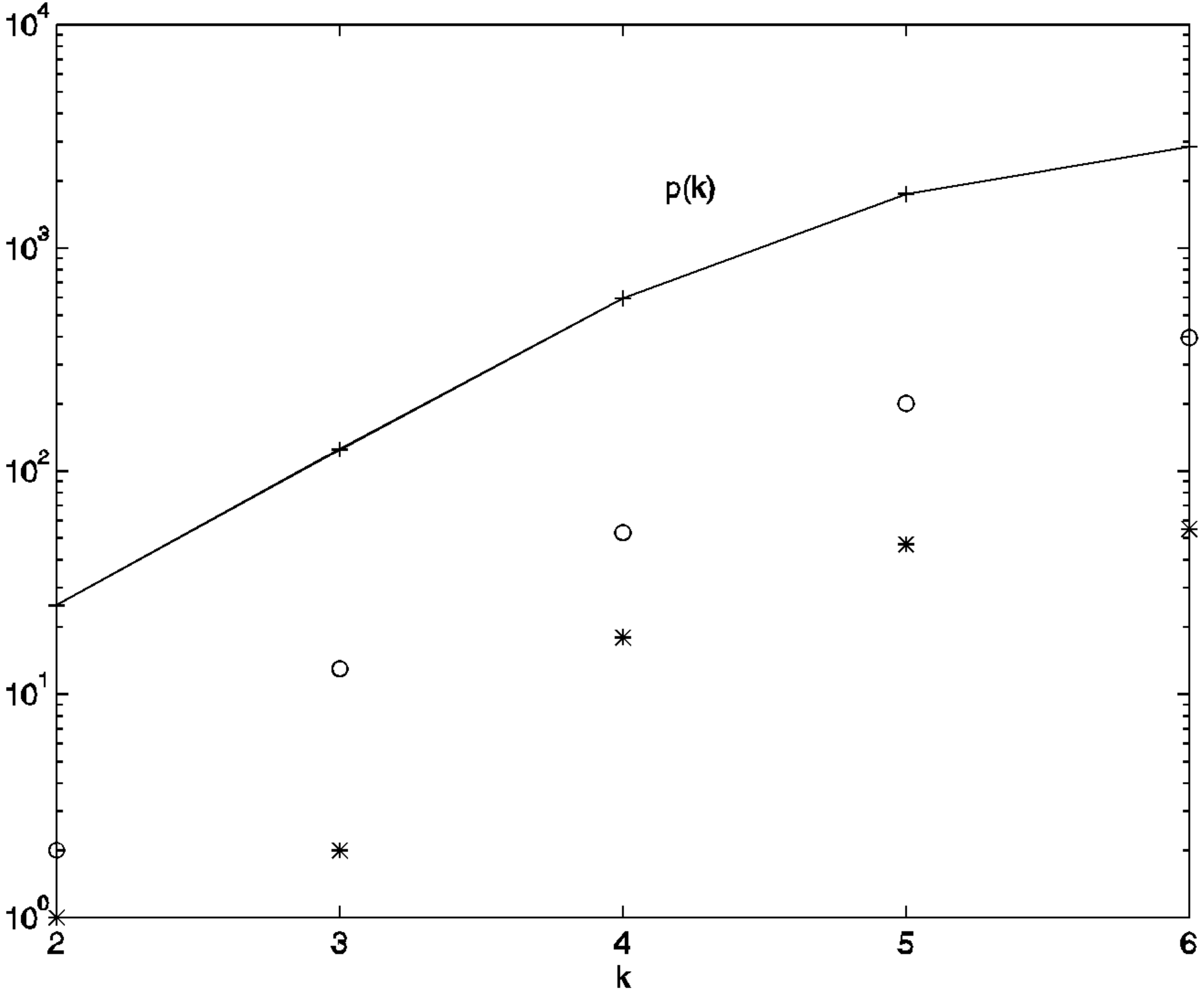
H_k = k-cylinders entropy

Range found when $H_k - H_{k-1}$ tends to a constant value

- For $k > r$

$\mu(i_1, \dots ,i_{k+1})= (\mu(i_1, \dots ,i_k) \ \mu(i_2 \dots ,i_{k+1})) / \mu(i_2, \dots ,i_k)$


IBM
Bayer
BMW
NYSE
H(k)-H(k-1)
k
1.4
1.2
1
0.8
0.6
0.4
0.2
0
2
4
6
8



p(k)
10^4
10^3
10^2
10^1
10^0
2
3
4
5
6
k

- A large number of errors in the estimation of the probabilities.
- Correspond to blocks with large returns (+ and -)
- Large deviations misrepresented by empirically constructed measure
- $\Rightarrow$ non-Gibbsian measure   or
  Gibbsian measure with long-range potential (but small statistics !)

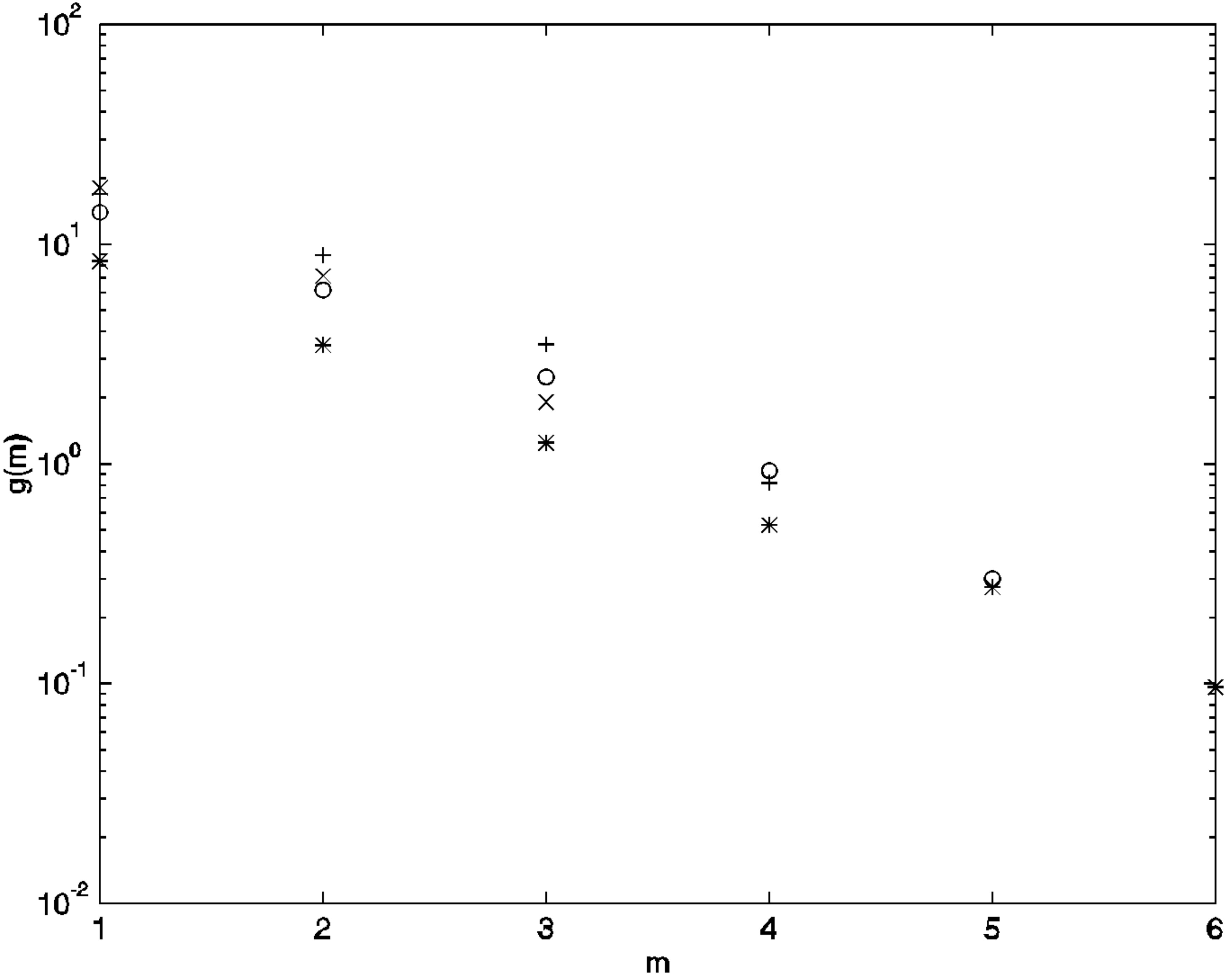
- In any case requires an approach suited to deal with long-memory processes

3. Chains with complete connections (CCC) and variable-length Markov process

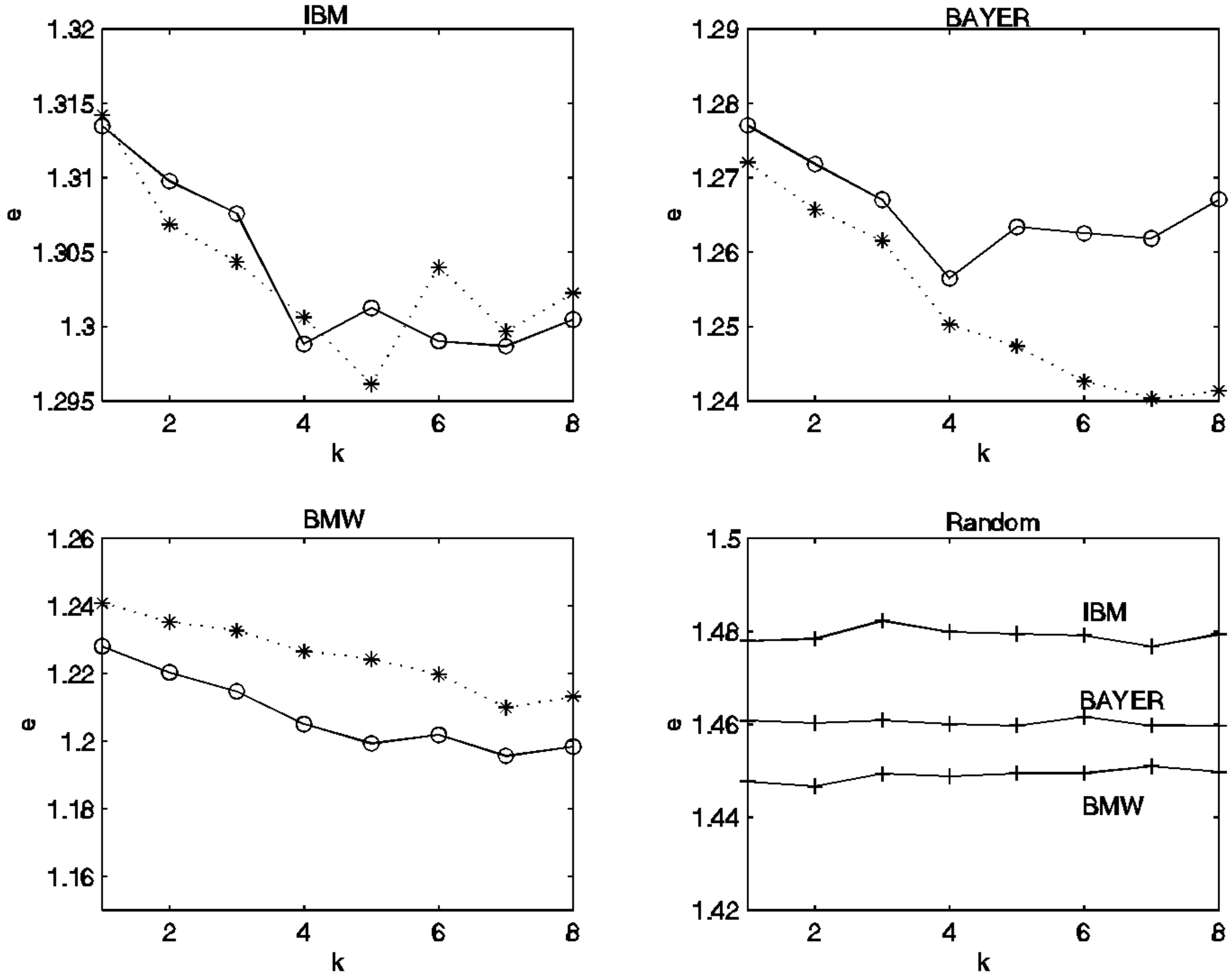
- i) $\lim_{m\to\infty} P(a_0 | a_{-1} \ldots a_{-m}[\ldots])$ exists

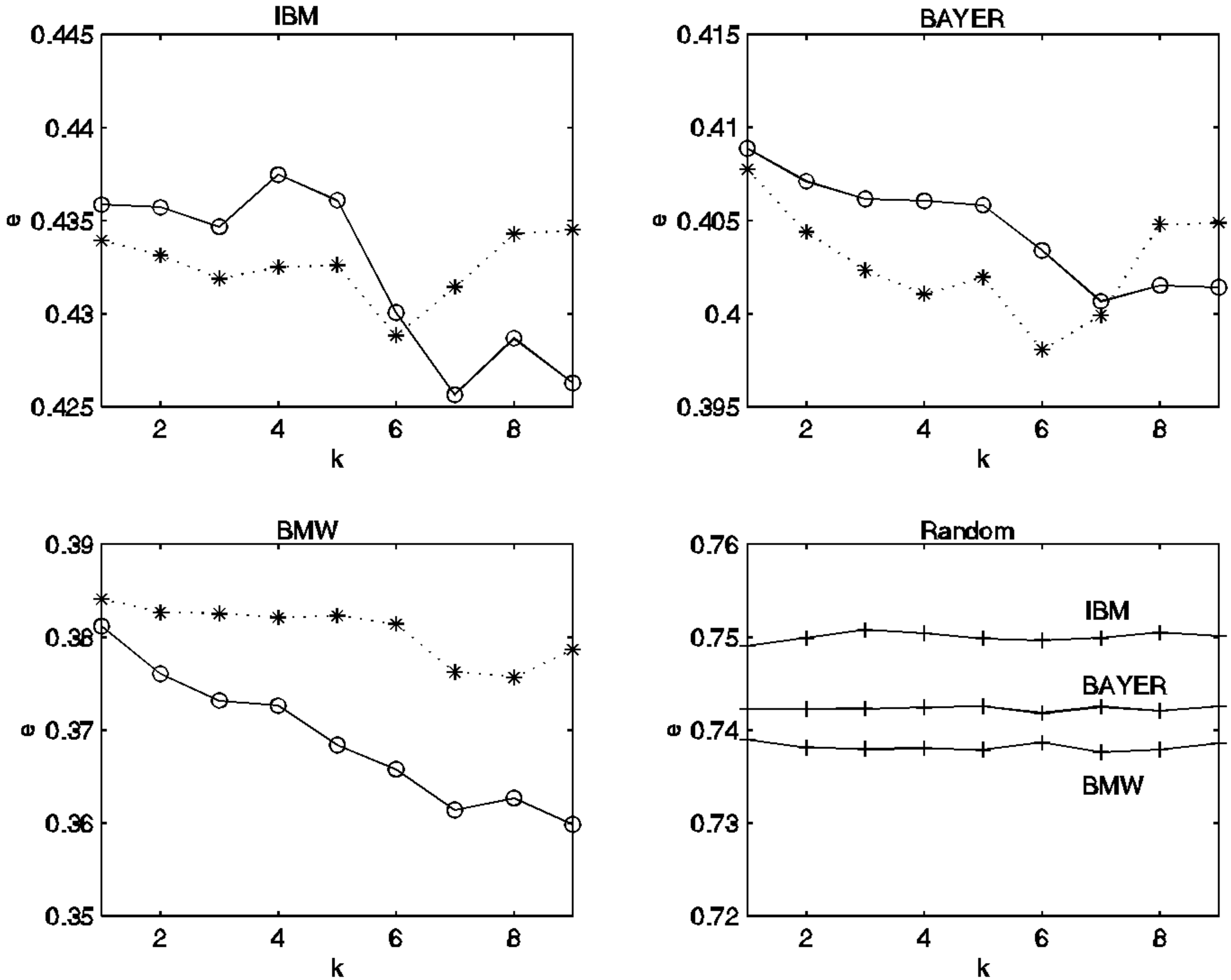
  ii) $\left| \left(\frac{P(a_0 | a_{-1} \ldots a_{-m} b_{-m-1} \ldots)}{P(a_0 | a_{-1} \ldots a_{-m} c_{-m-1} \ldots)} \right) - 1 \right| \leq \gamma_m \qquad \lim_{m\to\infty} \gamma_m = 0$

- CCC with summable decay $\sum_m \gamma_m < \infty$
- Numerical estimate of γ_m
- A CCC – process with summable decay is the d-limit of its Markov approximations of order k



g(m)
m
10^2
10^1
10^0
10^-1
10^-2
1
2
3
4
5
6


IBM
BAYER
BMW
Random
e
k
IBM
BAYER
BMW



IBM
BAYER
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Random
e
k
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BAYER
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Partial conclusions

- Small statistics for large returns reconstruction.
- Two components? Hidden Markov process?
- Variable-length Markov ("perfect") simulation of the price process at the level of transaction costs

Next :

- Look for structure. Multi-asset arbitrage
- Volatility modeling for risk assessment

4. The geometry of the market

- Return for security k

  $r_t(k) = \log p_t(k) - \log p_{t-1}(k)$

  $C_{kl}=(<r(k)r(l)>-<r(k)><r(l)>)/(N(k)N(l))$

- Metric $d_{kl} = (2-2C_{kl})^{1/2}$

- Compute center of mass and inertial tensor

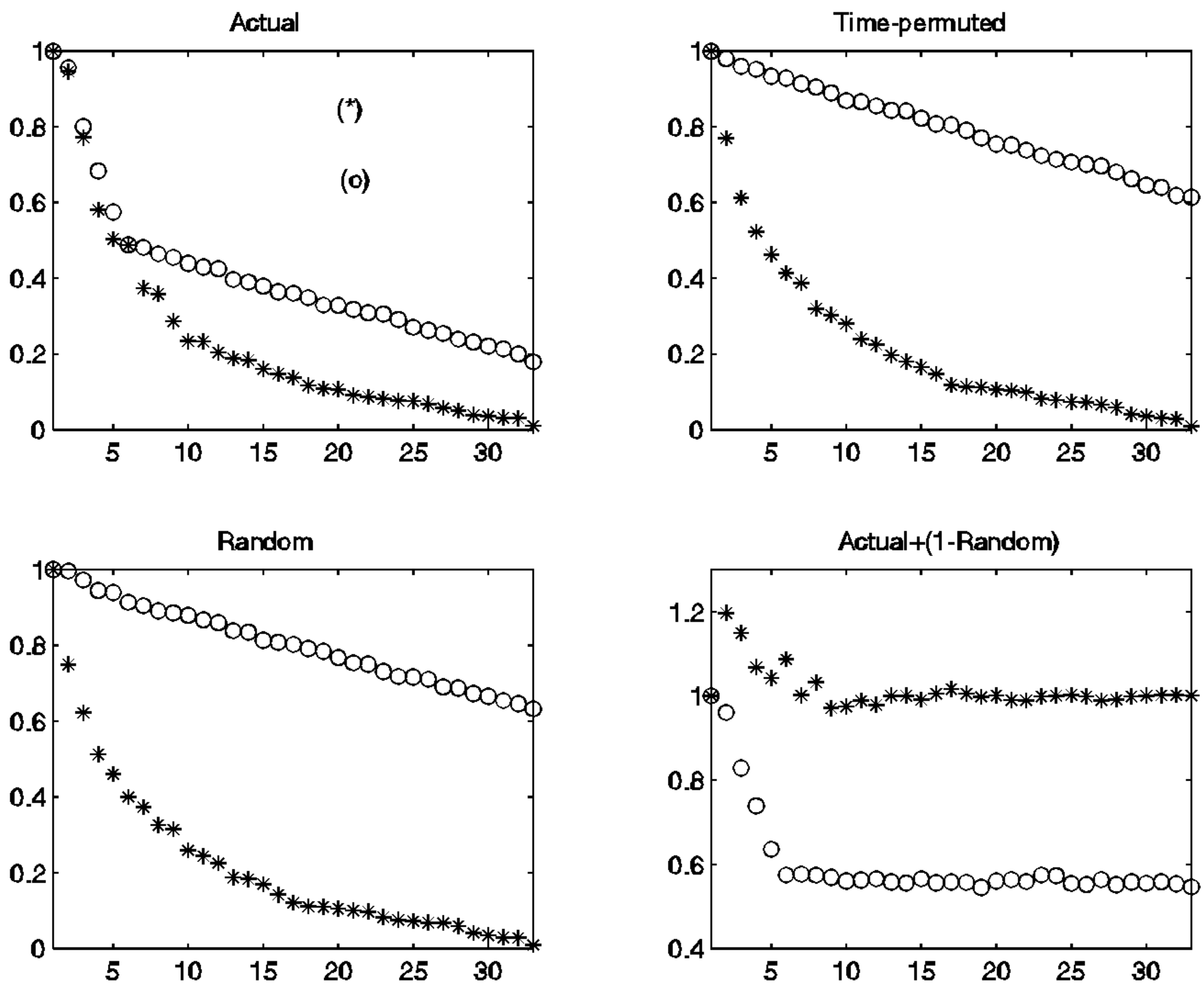
  $R = \Sigma\, m_k x(k) / \Sigma m_k$

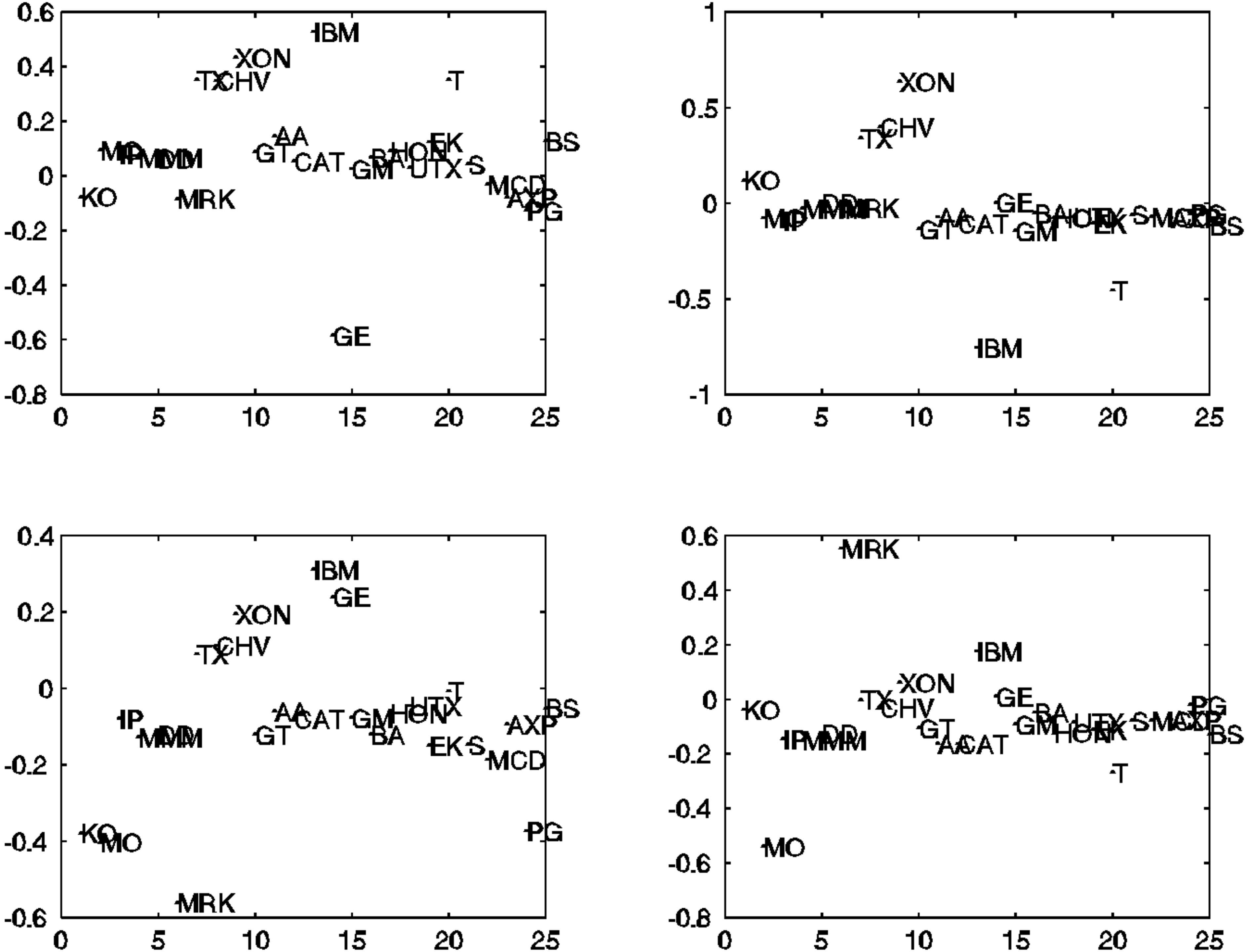
  $y(k) = x(k) - R$

  $T_{ij}= \Sigma_k\, m_k\, y_i(k)\, y_j(k)$

- Find eigenvalues and eigenvectors (λ_i, e_i)
- Find coordinates along the directions e_i

  $z_i(k) = y(k) \bullet e_i$


Actual
(*)
(o)
Time-permuted
Random
Actual+(1-Random)



IBM
XON
TX
CHV
T
AA
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CAT
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UTX
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PG
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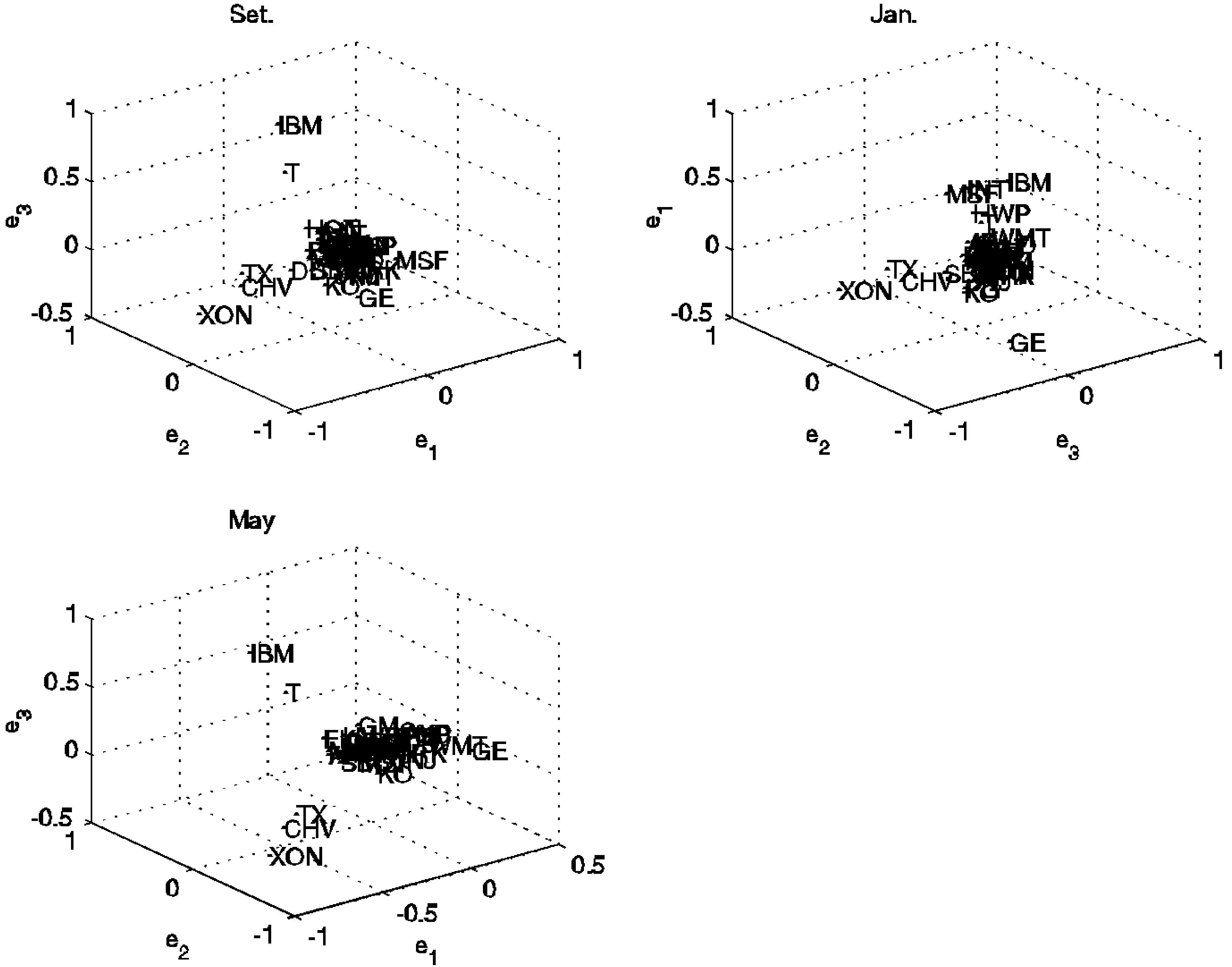
MRK
GE
MO
IP
MMM
DD


Set.
Jan.
May
IBM
T
MSF
TX
CHV
XON
KO
GE
HWP
WMT
e3
e1
e2
1
0.5
0
-0.5
-1

- Small number of relevant dimensions even for > 250 securities
- Systematic market factors
- Representative dimensions are not sectors
- Simplification of market portfolio
- Clustering and crisis

5. Modeling volatility

- Risk assessment
- Option pricing
- Independent process ? (R F Engle)
- Volatility clustering
- Long-memory
- Leverage

Geometric Brownian motion ?

- $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dB_t$?

Consequences:

$$p(\ln\frac{S_T}{S_t}) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\exp(-\frac{\left[\ln\frac{S_T}{S_t}-(\mu-\frac{\sigma^2}{2})(T-t)\right]^2}{2\sigma^2(T-t)})$$

(i) Price changes would be lognormal

$$E\left|\frac{S(t+\Delta)-S(t)}{S(t)}-\mu\Delta\right| \approx \Delta^H$$

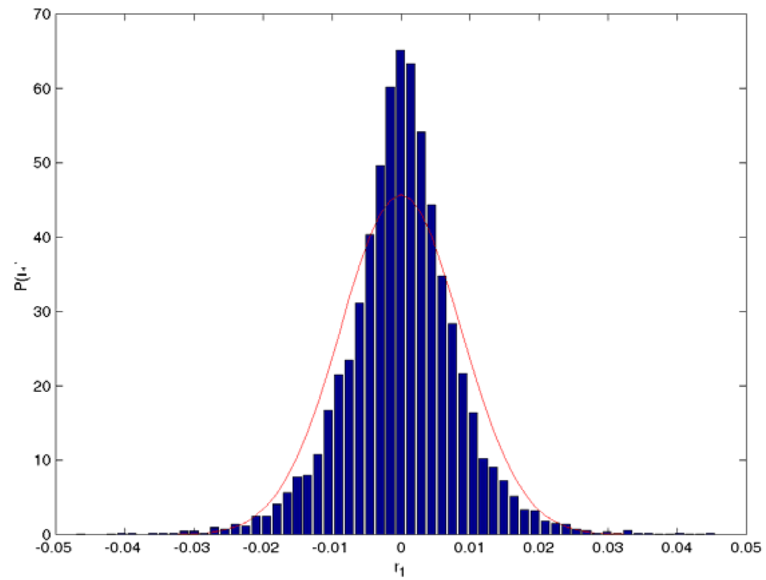
(ii) Self-similar (Law(X_{at})=Law(a^H X_t)) with Hurst coefficient = 1/2

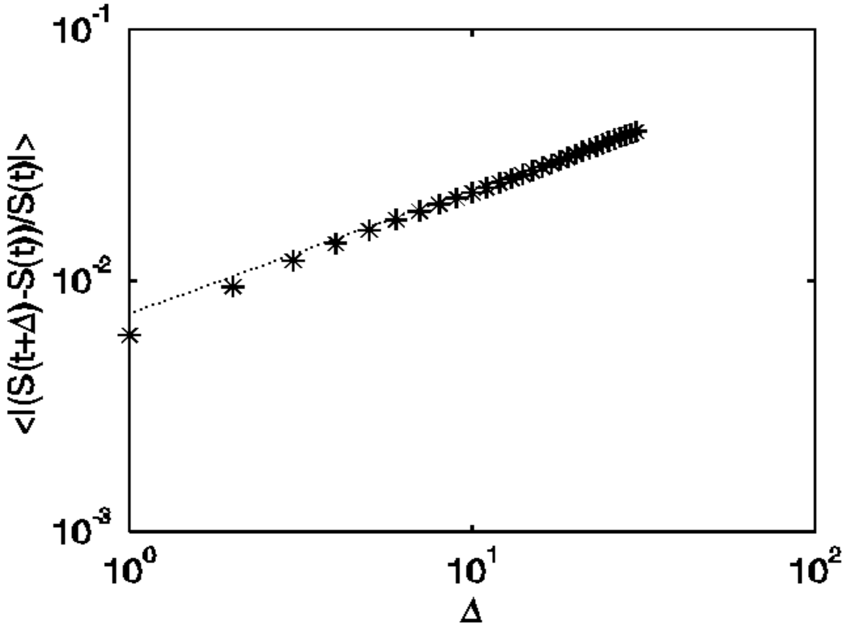
Geometric Brownian motion ?

- Empirical tests :
- $P(r_1)$ is not lognormal

  $r_1 = \log(S(t+1)/S(t))$

  Deviations from scaling
- Larger deviations for high-frequency data





Alternatives

- Use a different process for S_t

  Truncated Lévy
  Mixed diffusion - jump
  Compound normal
  Student t
  Power exponential
  Time-dependent GARCH

  (J. Töyli, M. Sysi-Aho, K. Kaski, Quant.Finance 373-382)

- Use

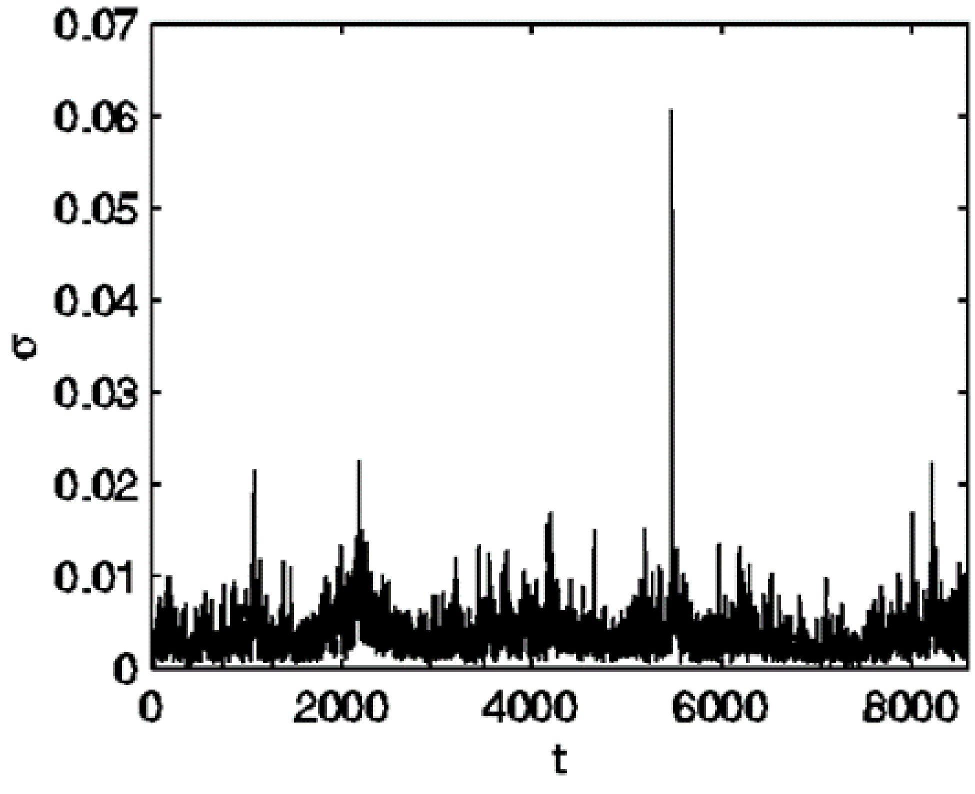
  $dS_t/S_t - \mu\, dt = \sigma_t\, dB_t$

  to define a new process

  σ_t = "Induced volatility"

  $\sigma_t^2 \approx \mathrm{var}(\log S_t)/(T_0 - T_1)$

  $(\mu=0)$



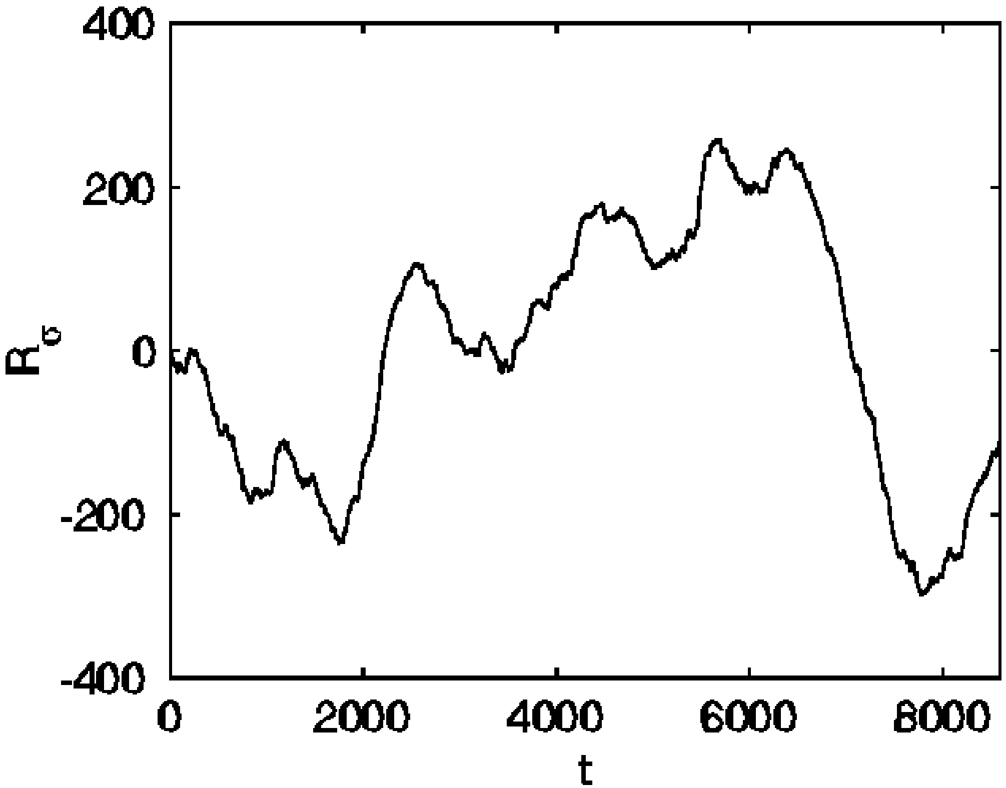
What does the data suggest for σ_t?

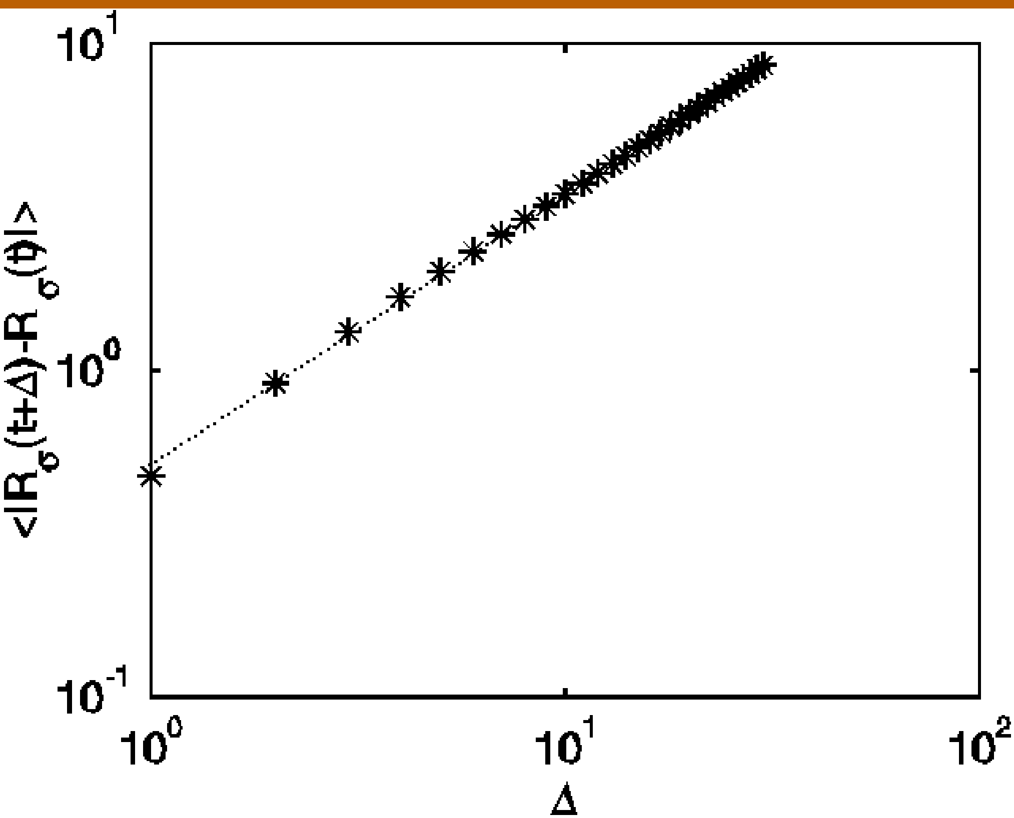
- σ_t is not self similar

$$E\left|\frac{\sigma(t+\Delta)-\sigma(t)}{\sigma(t)}\right| \neq \Delta^H$$

- However $R_\sigma(t)$ is
  $\Sigma \log \sigma(n\delta) = \beta t + R_\sigma(t)$
  $H \approx 0.8 - 0.9$

$$E\left|\frac{R_\sigma(t+\Delta)-R_\sigma(t)}{R_\sigma(t)}\right| = \Delta^H$$





What does the data suggest for σ_t ?

- Recall:
  If a process X_t has finite variance, stationary increments and is self-similar, then
  $Cov(X_s, X_t)=(|s|^{2H}+|t|^{2H}-|s-t|^{2H})E(X_1^2)$
  The simplest such process is a zero-mean Gaussian process, Fractional Brownian motion B^H_t
  with long-range dependence for H>1/2
- Conclusion :
  $$\log \sigma_t = \beta + (k/\delta) (B^H_t - B^H_{t-\delta})$$
  σ_t modeled by a stochastic exponential of fractional noise
- Two coupled processes :
  $$dS_t = \mu S_t dt + \sigma_t S_t dB_t$$
  $$\log \sigma_t = \beta + (k/\delta) (B^H_t - B^H_{t-\delta})$$

6. Option pricing. "Risk-neutral approach"

Let the value $V(S_t, \sigma_t, t)$ of an option be the expected terminal value discounted at the risk-free rate

$$V(S_t, \sigma_t, t) = e^{-r(T-t)} \int V(S_T, \sigma_T, T)\, p(S_T | S_t, \sigma_t)\, dS_T$$

$$V(S_T, \sigma_T, T) = \max[0, S - K]$$

Use of the relation between conditional probabilities of related variables,

$$p(S_T | S_t, \sigma_t) = \int p(S_T | S_t, \langle \log \sigma \rangle)\, p(\langle \log \sigma \rangle \mid \log \sigma_t)\, d(\langle \log \sigma \rangle)$$

$\langle \log \sigma \rangle$ being the random variable

$$\langle \log \sigma \rangle = \frac{1}{T-t} \int_t^T \log \sigma_s ds$$

Then

$$V\left(S_t,\sigma_t,t\right)=\int C\left(S_t,e^{\langle\log\sigma\rangle},t\right)p\left(\langle\log\sigma\rangle\mid\log\sigma_t\right)d\left(\langle\log\sigma\rangle\right)$$

$$C\left(S_t,e^{\langle\log\sigma\rangle},t\right)=\int e^{-r(T-t)}V\left(S_T,\sigma_T,T\right)p\left(S_T|S_t,\langle\log\sigma\rangle\right)dS_T$$

$C\left(S_t,e^{\langle\log\sigma\rangle},t\right)$ = Black-Scholes price for an option with average volatility $e^{\langle\log\sigma\rangle}$

$$C\left(S_t,\sigma,t\right)=S_t\left(a+b\right)N\left(a,b\right)-Ke^{-r(T-t)}\left(a-b\right)N\left(a,-b\right)$$

with

$$\begin{aligned} a &= \frac{1}{\sigma}\left(\frac{\log\frac{S}{K}}{\sqrt{T-t}}+r\sqrt{T-t}\right) \\ b &= \frac{\sigma}{2}\sqrt{T-t} \end{aligned}$$

$$N\left(a,b\right)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{\infty}dye^{-\frac{y^2}{2}(a+b)^2}$$

To compute the conditional probability $p\left(\langle\log\sigma\rangle \mid \log\sigma_t\right)$

$$\langle\log\sigma\rangle = \log\sigma_t + \frac{1}{T-t}\int_t^T \frac{k}{\delta} ds \int_t^s \left(dB_H(\tau) - dB_H(\tau-\delta)\right)$$

$$E\left\{\langle\log\sigma\rangle \mid \log\sigma_t\right\} = \log\sigma_t$$

$$\begin{aligned}\alpha^2 &= E\left\{\left(\langle\log\sigma\rangle - \log\sigma_t\right)^2\right\} \\ &= \frac{k^2}{\delta^2(T-t)}\left\{\frac{1}{2(T-t)}I_1 + I_2\right\} + k^2\delta^{2H-2}\end{aligned}$$

with

$$I_1 = \tfrac{2}{(2H+1)(2H+2)}\left\{\begin{array}{c}(T-t+\delta)^{2H+2} + (T-t-\delta)^{2H+2} \\ -2(T-t)^{2H+2} - 2\delta^{2H+2}\end{array}\right\}$$

$$I_2 = \tfrac{1}{2H+1} \left\{2(T-t)^{2H+1} - (T-t+\delta)^{2H+1} - (T-t-\delta)^{2H+1}\right\}$$

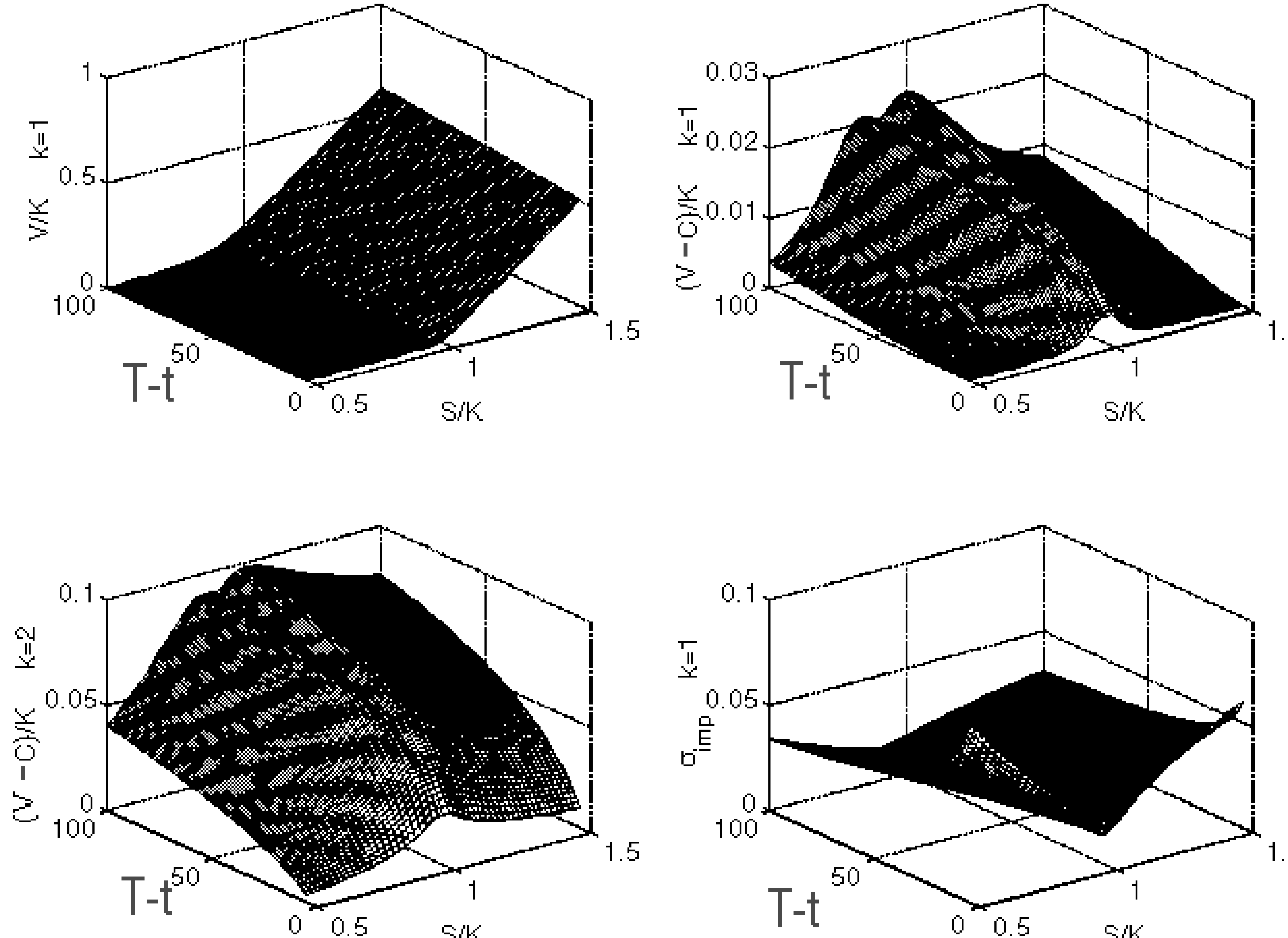
Finally

$$p\left(\langle\log\sigma\rangle \mid \log\sigma_t\right) = \frac{1}{\sqrt{2\pi}\alpha}\exp\left\{\frac{-\left(\langle\log\sigma\rangle - \log\sigma_t\right)^2}{2\alpha^2}\right\}$$

one obtains

$$\begin{aligned} V\left(S_t, \sigma_t, t\right) &= S_t\left[aM\left(\alpha, a, b\right) + bM\left(\alpha, b, a\right)\right] \\ &\quad -Ke^{-r(T-t)}\left[aM\left(\alpha, a, -b\right) - bM\left(\alpha, -b, a\right)\right] \end{aligned}$$

$$\begin{aligned} M\left(\alpha, a, b\right) &= \frac{1}{2\pi\alpha}\int_{-1}^{\infty} dy \int_0^{\infty} dx e^{-\frac{\log^2 x}{2\alpha^2}} e^{-\frac{y^2}{2}\left(ax+\frac{b}{x}\right)^2} \\ &= \frac{1}{4\alpha}\sqrt{\frac{2}{\pi}}\int_0^{\infty} dx \frac{e^{-\frac{\log^2 x}{2\alpha^2}}}{ax+\frac{b}{x}} erfc\left(-\frac{ax}{\sqrt{2}} - \frac{b}{\sqrt{2}x}\right) \end{aligned}$$



V/K k=1
1
0.5
0
100
50
T-t
0
0.5
1
1.5
S/K
(V -C)/K k=1
0.03
0.02
0.01
0
100
50
T-t
0
0.5
1
1.
S/K
(V -C)/K k=2
0.1
0.05
0
100
50
T-t
0
0.5
1
1.5
σimp k=1
0.1
0.05
0
100
50
T-t
0
0.5
1
1.

The option pricing equation

Form a portfolio

$$\Pi\left(t\right) = V\left(S,\sigma,t\right) - \Delta\left(S,\sigma,t\right)S_t$$

Choosing $\Delta\left(S,\sigma,t\right) = \frac{\partial V}{\partial S}$ we obtain

$$\begin{aligned} d\Pi\left(t\right) \; &= \; \left\{\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right\}dt \\ &+\sigma\frac{\partial V}{\partial \sigma}\frac{k}{\delta}\left(dB_H\left(t\right) - dB_H\left(t-\delta\right)\right) \\ &+\left(\sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \sigma\frac{\partial V}{\partial \sigma}\right)H\frac{k^2}{\delta^2}\delta^{2H-1}dt \end{aligned}$$

The fractional Itô formula

if $X_t = \left(X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(n)} \right)$

with $dX_t^{(i)} = c_i(t,\omega)\, dB_H^{(i)}(t)$, then

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \sum_i \frac{\partial f}{\partial X^{(i)}} dX_t^{(i)} + \sum_i \frac{\partial^2 f}{\partial X^{(i)2}} c_i(t,\omega)\, D_{i,t}^{\phi}(X_t)$$

$D_{i,t}^{\phi}(X_t)$ is the $\phi-$Malliavin derivative corresponding to the $X_t^{(i)}-$process

$$\begin{aligned} D_{i,f}^{\phi} X_t(\omega_i) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{ X \left(\omega_i + \varepsilon \int_0^{\bullet} ds \int_0^{\infty} \phi(s,u) f(u)\, du \right) \\ &\quad - X(\omega_i) \} \\ &= \int_0^{\infty} D_{i,u}^{\phi}(X_t) f(u)\, du \end{aligned}$$

and $\phi(s,u)$ the kernel

$$\phi(s,u) = H_i (2H_i - 1) |s-u|^{2H_i - 2}$$

Dealing with the stochastic term

$\sigma \frac{\partial V}{\partial \sigma} \frac{k}{\delta} \left(dB_H(t) - dB_H(t-\delta) \right)$

Volatility is not a tradable security.

Cannot be eliminated by a portfolio choice.

Instead, equate the deterministic term in $d\Pi(t)$ to

$$\left(r\Pi(t) + \nu \frac{k}{\delta} \sigma \frac{\partial V}{\partial \sigma} \right) dt$$

The second term is a measure of the market price of volatility risk

Option pricing equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \frac{k}{\delta}\left(kH\delta^{2H-2} - \nu\right)\sigma\frac{\partial V}{\partial \sigma}$$
$$+Hk^2\delta^{2H-3}\sigma^2\frac{\partial^2 V}{\partial \sigma^2} = rV$$

Solutions

$x = \log \frac{S}{K}$

$V(t, x, \sigma) = \int \int d\phi d\rho F(\phi, \rho, \sigma) e^{i(\phi t + \rho x)}$

we obtain

$$Hk^2 \delta^{2H-3} \sigma^2 \frac{\partial^2 F}{\partial \sigma^2} + \frac{k}{\delta} \left(kH\delta^{2H-2} - \nu \right) \sigma \frac{\partial F}{\partial \sigma}$$
$$+ \left(i \left(\phi + \rho r - \frac{\sigma^2 \rho}{2} \right) - \frac{\sigma^2 \rho^2}{2} - r \right) F = 0$$

Define new constants

$$\begin{aligned} \chi(\rho) &= \frac{\nu}{2Hk\delta^{2H-2}} \\ \xi^2(\rho, \phi) &= \chi^2(\rho) - \frac{r - i(\phi + \rho r)}{Hk^2 \delta^{2H-3}} \\ \zeta^2(\rho) &= -\frac{i\rho + \rho^2}{2Hk^2 \delta^{2H-3}} \end{aligned}$$

and

$F(\sigma) = \sigma^\chi Z_\xi(\zeta \sigma)$

The solution of is

$V(t, x, \sigma) = \int \int d\rho d\phi e^{i(\phi t + \rho x)} \sigma^{\chi(\rho)} Z_{\xi(\rho, \phi)}(\zeta(\rho) \sigma)$

$Z_\xi(\zeta \sigma)$ being a Bessel function.

Linear combination $Z_\xi(\zeta \sigma) = c_1 J_\xi(\zeta \sigma) + c_2 N_\xi(\zeta \sigma)$

Coefficients c_1 and c_2 fixed by the boundary conditions,

(call options) $V(T, x, \sigma) = \max(x, 0)$

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