Fractional processes: Finite and infinite dimensions

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Nature and stochastic processes

- Mathematics is an interesting intellectual activity, but it is also a powerful tool to model the natural world
- Our knowledge of Nature can never be absolutely accurate, therefore the theory of stochastic processes appears as the most natural tool.
- What properties should the stochastic processes have to be suitable tools to deal with natural (intelligible) phenomena?
- **Stationarity**: is the idea that natural systems (after a steady drift is subtracted) fluctuate around a mean value within a well defined and stationary envelope of variability. It is this concept that allows natural scientists to take an instrumental record and establish a probability density function for various magnitudes and frequencies of events. Without some regularity, the construction of intelligible models becomes impossible.

Nature and stochastic processes

- **Selfsimilarity:** a characteristic of growing processes

The generic nature of fractional processes

- A process $\{X(t), t \geq 0\}$ is *selfsimilar* if for any a there is b such that

$$\{X(at)\} \overset{d}{=} \{bX(t)\} = \left\{a^H X(t)\right\}$$

$b = a^H$, process $H-$selfsimilar (or $H-$ss) ($H =$Hurst exponent)

- A process $X(t)$ is *stationary* if

$$\sum_j \theta_j X(t_j + h) \overset{d}{=} \sum_j \theta_j X(t_j)$$

- If $X(t)$ is H-ss, then $Y(t) = e^{-tH} X(e^t)$ is stationary. If $Y(t)$ is stationary, then $X(t) = t^H Y(\ln t)$ is H-ss
- A process has *stationary increments (si)* if any distribution of

$$\{X(t+h) - X(t), t \geq 0\}$$

is independent of $t \geq 0$

- Increments may or may not be *independent*

The generic nature of fractional processes

- **Theorem**: *If* $\{X(t), t \geq 0\}$ *is real-valued, H-ss with stationary increments and* $\mathbb{E}\left[X(1)^2\right] < \infty$, *then*

$$\mathbb{E}\left[X(t)X(s)\right] = \frac{1}{2}\left\{t^{2H} + s^{2H} - |t-s|^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

  The simplest such process is a Gaussian process called **fractional Brownian motion** (fBm), $B_H(t)$, defined to have $\mathbb{E}\left[B_H(t)\right] = 0$. fBm is the unique Gaussian $H - ss$ process with stationary increments
- The process

$$Y_t = B_H(t+1) - B_H(t)$$

  called **fractional Gaussian noise** *(fGn).* Within the stationary sequences, fractional Gaussian noise is the only Gaussian fixed point of the renormalization group $T_N : Y_t \rightarrow (T_N Y)_t = \frac{1}{N^H}\sum_{i=t}^{t+N-1} Y_i$
- If $H = \frac{1}{2}$, fBm is Brownian motion (Bm). **Bm is an isolated point in whole class of fBm's.** fBm is the generic case. What is special about Bm?

Fractional Brownian motion

- *Long-range dependence:* Let $\{X(t), t \geq 0\}$ be $H - ss$, si, $0 < H < 1$, $E\left[X(1)^2\right] < \infty$ and define

$$\xi(n) = X(n+1) - X(n)$$

$$r(n) = \mathbb{E}[\xi(0)\xi(n)] = \frac{1}{2}\left\{(n+1)^{2H} - 2n^{2H} + (n-1)^{2H}\right\}\mathbb{E}\left[X(1)^2\right]$$

- Then

$$\begin{array}{lll} r(n) & \underset{n\to\infty}{\sim} & H(2H-1)\,n^{2H-2}\mathbb{E}\left[X(1)^2\right] \quad , \quad H \neq \frac{1}{2} \\ r(n) & = & 0 \qquad\qquad\qquad\qquad\qquad\qquad H = \frac{1}{2} \end{array}$$

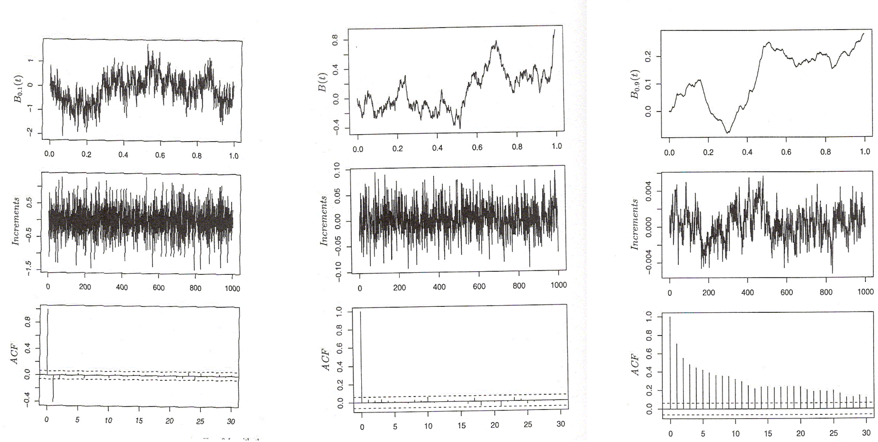
and

$$\begin{array}{ccc} 0 < H < \frac{1}{2} & , & \sum_{n=0}^{\infty} |r(n)| < \infty \\ H = \frac{1}{2} & , & \text{uncorrelated} \\ \frac{1}{2} < H < 1 & , & \sum_{n=0}^{\infty} |r(n)| = \infty \quad , \quad \text{long-range dependence} \end{array}$$

Fractional Brownian motion

- If $0 < H < \frac{1}{2}$, $r(n) < 0$ for $n \geq 1$ (negative correlation, anti-persistent process),
- If $\frac{1}{2} < H < 1$, $r(n) > 0$ for $n \geq 1$ (positive correlation, persistent process).
- In conclusion: What makes **Brownian motion** special is the independence of increments. Therefore **fractional Gaussian noise** with $H = \frac{1}{2}$ may be appropriate as a **coordinate system in infinite dimensions**, but it is the generic case $(H \neq \frac{1}{2})$ that is expected to be more applicable **as a model of natural phenomena**.

Fractional Brownian motion



$H = 0.1$ $\qquad$ $H = 0.5$ $\qquad$ $H = 0.9$

Fractional Brownian motion. Sample path properties and integral representation

- fBm $\{B_H(t)\}$ has continuous version $(P\{X(t) = B_H(t)\} = 1)$ with sample paths Hölder continuous of order $\beta \in [0, H)$ and a. s. nowhere locally Hölder continuous of order $\gamma > H$. Sample paths of fBm have nowhere bounded variation and are not differentiable.
- fBm for $H \neq \frac{1}{2}$ is not a semimartingale
- "Time" representation as a Wiener integral, $B_H(t) \stackrel{d}{=} \frac{1}{\Gamma\left(H+\frac{1}{2}\right)} \times$

$$\left\{\int_{-\infty}^{0}\left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right) dB(u) + \int_{0}^{t} (t-u)^{H-\frac{1}{2}} dB(u)\right\}$$

- Other integral representations: finite time, spectral, Paley-Wigner
- Suggests similar "fractional" generalizations of other Lévy processes (Dreceusefond and Savy - A.I.H. Poincaré PR 42 (2006) 343-372)
  Lévy process: $X(0) = 0$, *stoch. continuity at* $t \geq 0$, *s.i. increments, sample paths right-continuous and left limits a. s.*
  Examples: linear drift, Bm, Poisson and compound Poisson, etc.

- Geometric Brownian motion $\frac{dS_t}{S_t} = \mu dt + \sigma dB(t)$ by itself is a bad mathematical model for the market
- Conjecture:

  1 - **The log-price process** $\log S_t$ **belongs to a probability space** $\Omega \otimes \Omega'$**, the first one the Wiener space and the second,** Ω'**, is a probability space to be empirically reconstructed.** ($\omega \in \Omega$, $\omega' \in \Omega'$ and $\mathcal{F}_t$ and $\mathcal{F}'_t$ the $\sigma-$algebras in Ω and Ω' generated by the processes up to t)

$$\log S_t \left(\omega, \omega'\right)$$

  2 - **For each fixed** ω'**,** $\log S_t \left(\bullet, \omega'\right)$ **is a square integrable random variable in** Ω. Then for each fixed ω',

$$\frac{dS_t}{S_t}\left(\bullet, \omega'\right) = \mu_t \left(\bullet, \omega'\right) dt + \sigma_t \left(\bullet, \omega'\right) dB(t)$$

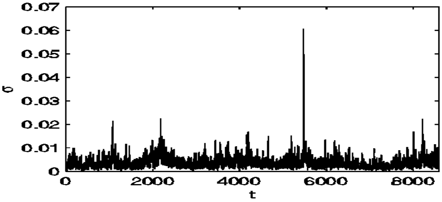
  with $\mu_t \left(\bullet, \omega'\right)$ and $\sigma_t \left(\bullet, \omega'\right)$ well-defined processes in Ω.

Fractional noise and market volatility

- If σ_t is an $\mathcal{F}_t-$adapted processes, then

$$\sigma_t^2\left(\bullet,\omega^{'}\right)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{E\left(\log S_{t+\varepsilon}-\log S_t\right)^2\right\}$$

Because each set of market data corresponds to a particular realization $\omega^{'}$, the σ_t^2 process may be reconstructed from the data.

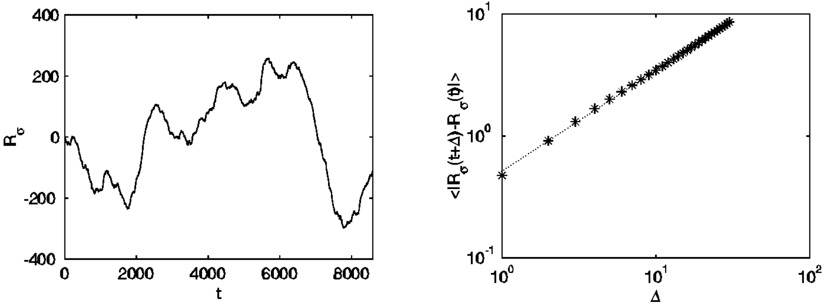


- Compute

$$\sum_{n=0}^{t/\delta}\log\sigma\left(n\delta\right)=\beta t+R_\sigma\left(t\right)$$

Fractional noise and market volatility

- The $R_\sigma(t)$ process displays very accurate self-similar properties



- Conclusion: The fractional volatility model

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB(t) \\ \log \sigma_t &= \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} \end{aligned}$$

  δ is the observation time scale and H is in the range $0.8 - 0.9$

- The volatility (at resolution δ)

$$\sigma(t) = \theta e^{\frac{k}{\delta}\{B_H(t) - B_H(t-\delta)\} - \frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}}$$

- The integral representation of fBm gives additional insight

Fractional noise and market volatility

- Experimentally one finds in actual markets the following nonlinear correlation of the returns

$$L(\tau) = \left\langle \left| r(t+\tau) \right|^2 r(t) \right\rangle - \left\langle \left| r(t+\tau) \right|^2 \right\rangle \langle r(t) \rangle$$

This is called *leverage* or the *leverage effect* and it is found that for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus constantly decays to zero whereas for $\tau < 0$ it has almost negligible values.

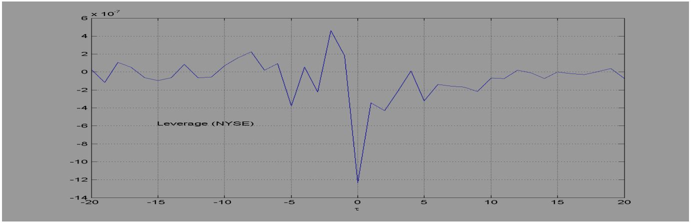
- Use in the fractional volatility model, the representation

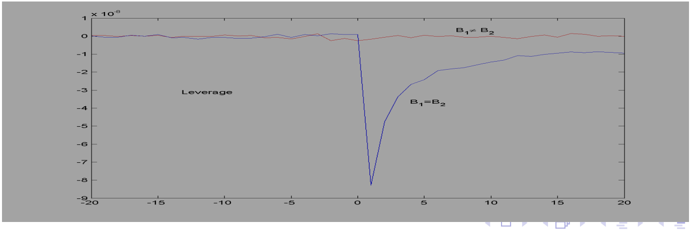
$$B_H(t) = C \left\{ \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB_s + \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \right\}$$

Then,

$$\boxed{\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dB^{(1)}(t) \\ \log \sigma_t &= \beta + k' \int_{-\infty}^{t} (t-s)^{H-\frac{3}{2}} dB^{(2)}(s) \end{aligned}}$$

$B^{(1)}(s)$ and $B^{(2)}(s)$ are Brownian processes. If $B^{(1)}(s) \neq B^{(2)}(s)$ there is no leverage effect but if $B^{(1)}(s) = B^{(2)}(s)$ one obtains a qualitatively correct leverage.





Fractional processes and differential operators

- The well-known heat equation is closely related to Brownian motion

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u(t,x) \qquad with \qquad u(0,x) = f(x)$$

$$u(t,x) = \mathbb{E}_x f(X_t)$$

  $\mathbb{E}_x$ being the expectation value, starting from x, of the Wiener process $dX_t = dB_t$
- What are the differential equations related to fractional processes?
- **Fractional calculus**

  Riemann-Liouville (right-sided) fractional integral of order α ($\alpha > 0$) for a function $f(t)$

$$I^{\alpha}_{a+} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau\,, \qquad \alpha \in \mathbb{R}$$

  is a natural generalization of a well known formula (Cauchy-Dirichlet), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. Then,

Fractional processes and differential operators

- **Riemann-Liouville fractional derivative:** $D^{\alpha} f(t) := D^m I^{m-\alpha} f(t)$

$$D^{\alpha} f(t) := \begin{cases} \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\right], & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

- **Caputo fractional derivative of order** α : $D_*^{\alpha} f(t) := I^{m-\alpha} D^m f(t)$

$$D_*^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1<\alpha<m \\ \frac{d^m}{dt^m} f(t), & \alpha = m \end{cases}$$

- **Riesz fractional derivative of order** α

$$\mathcal{F}\{D_0^{\alpha} f\}(k) := -|k|^{\alpha} \hat{f}(k)$$

- **Riesz-Feller fractional derivative of order** α **and skewness** θ

$$\mathcal{F}\{D_0^{\alpha} f\}(k) := -\psi_{\alpha}^{\theta}(k) \hat{f}(k)$$

$$\psi_{\alpha}^{\theta}(k) = |k|^{\alpha} e^{i(\text{sign}k)\theta\pi/2}, \qquad 0<\alpha\leq 2, |\theta| \leq \min\{\alpha, 2-\alpha\}$$

$-\psi_{\alpha}^{\theta}(k)$ is the log of the char. function of a Lévy stable distribution

Fractional processes and differential equations

- The integral representation of fBm may be written as a fractional integral

$$\left(I^{\alpha}1_{[a,b)}\right)(s)=\frac{1}{\Gamma(\alpha+1)}\left\{(b-s)_{+}^{\alpha}-(a-s)_{+}^{\alpha}\right\}$$

$$B_H(t)=\frac{\Gamma\left(H+\frac{1}{2}\right)}{C(H)}\int_R\left(I^{H-\frac{1}{2}}1_{[0,t)}\right)(s)\,dB(s)$$

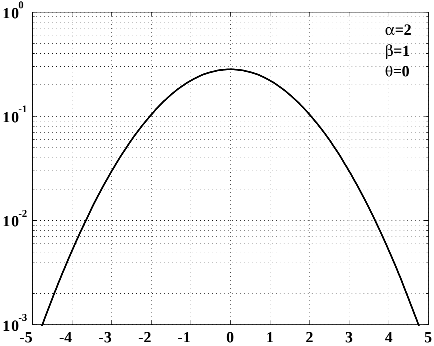
- With different choices of α,β and θ in the space-time fractional diffusion equation

$${}_tD_*^{\alpha}u(t,x)=\frac{1}{2}{}_xD_{\theta}^{\beta}u(t,x)$$

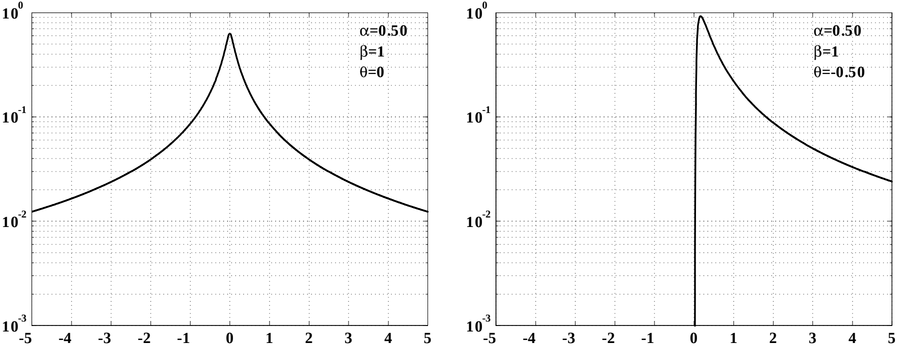
a large class of symmetric and asymmetric Green's functions are obtained. They are a powerful tool to model complex natural phenomena (see Mainardi, Luchko and Pagnini, Frac. Calc. Appl. Anal. 2 (201) 153-192)

$\{\alpha = 2\,,\ \beta = 1\}$ (*Standard diffusion*)

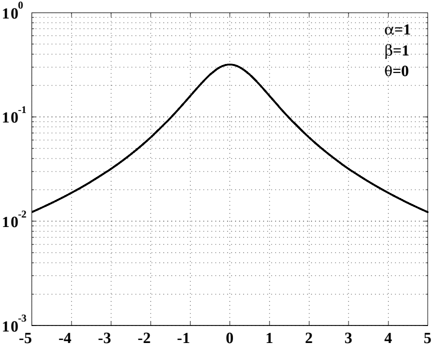
$$G^0_{2,1}(x,t) = t^{-1/2}\,\frac{1}{2\sqrt{\pi}}\,\exp[-x^2/(4t)]$$

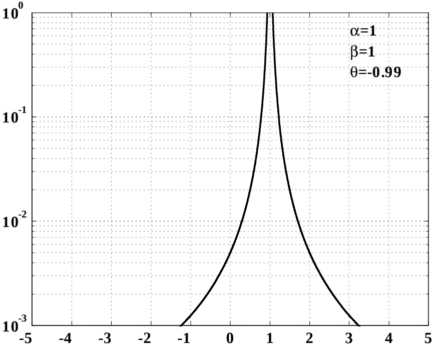


Fractional diffusion equation

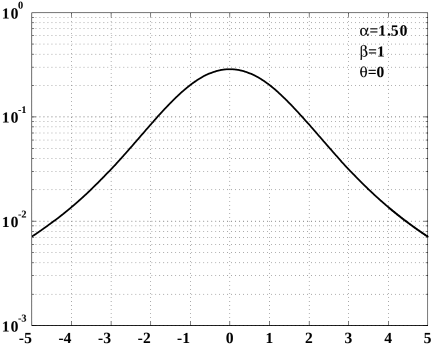


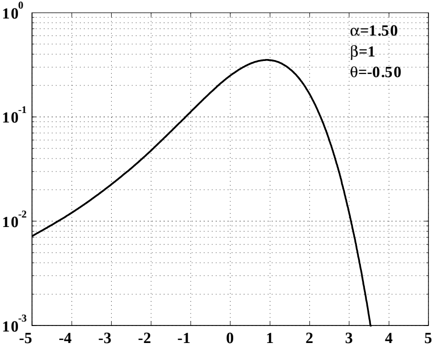
Fractional diffusion equation





Fractional diffusion equation





Fractional processes and differential equations: A nonlinear example

- The fractional KPP equation

$$\boxed{{}_tD_*^{\alpha}u\left(t,x\right)=\frac{1}{2}{}_xD_{\theta}^{\beta}u\left(t,x\right)+u^2\left(t,x\right)-u\left(t,x\right)}$$

${}_tD_*^{\alpha}$ is a Caputo derivative of order α

$${}_tD_*^{\alpha}f\left(t\right)=\begin{cases}\frac{1}{\Gamma(m-\beta)}\int_0^t\frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha+1-m}} & m-1<\alpha<m\\ \frac{d^m}{dt^m}f\left(t\right) & \alpha=m\end{cases}$$

${}_xD_{\theta}^{\beta}$ is a Riesz-Feller derivative defined through its Fourier symbol

$$\mathcal{F}\left\{{}_xD_{\theta}^{\beta}f\left(x\right)\right\}\left(k\right)=-\psi_{\beta}^{\theta}\left(k\right)\mathcal{F}\left\{f\left(x\right)\right\}\left(k\right)$$

with $\psi_{\beta}^{\theta}\left(k\right)=\left|k\right|^{\beta}e^{i(\text{sign}k)\theta\pi/2}$.

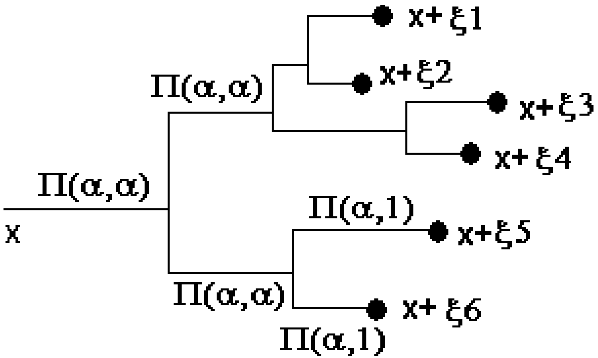
- Physically it describes a nonlinear diffusion with growing mass with memory effects in time and long range correlations in space.

Fractional processes and differential equations: A nonlinear example

The (stochastic) solution is

$$u(t,x) = \mathbb{E}_x \left(u(0^+, x+\xi_1) u(0^+, x+\xi_2) \cdots u(0^+, x+\xi_n) \right)$$

the expectation being in relation to a branching and propagation process



Fractional processes and differential equations: A nonlinear example

- **Branching process** B_α
  $E_{\alpha,1}\left(-t^\alpha\right) =$ survival probability up to time t
  $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^\alpha\right) =$ prob. density for branching at time τ
  It is fractional generalization of the Poisson process. It is this process that later will be used to develop an infinite-dimensional Poisson calculus
  $E_{\alpha,\rho}$ is the generalized Mittag-Leffler function
  $E_{\alpha,\rho}\left(-z\right) = \sum_{j=0}^{\infty} \frac{(-z)^j}{\Gamma(\alpha j+\rho)} = \int_0^\infty e^{-uz} dF\left(u\right)$
- **Propagation processes** $\Pi_{\alpha,1}^{\beta}$ and $\Pi_{\alpha,\alpha}^{\beta}$ with Green's functions $G_{\alpha,1}^{\beta}\left(t,x\right)$ and $G_{\alpha,\alpha}^{\beta}\left(t,x\right)$,

$$G_{\alpha,\rho}^{\beta}\left(t,x\right) = \frac{1}{2\pi E_{\alpha,\rho}\left(-t^\alpha\right)} \int_0^\infty dF\left(r\right) e^{-rt^\alpha} \int_{-\infty}^{\infty} dk e^{-ikx} e^{-\frac{rt^\alpha}{2}\psi_\beta^\theta(-k)}$$

Fractional white noise analysis

(Elliott, Van der Hoek, Biagini, Hu, ∅ksendal, Zhang)

- Relate fBm to classical Brownian motion by an operator defined for functions in $S(\mathbb{R})$ and extended to $L^2(\mathbb{R})$.

$$\widehat{Mf}(y) = |y|^{\frac{1}{2}-H}\,\widehat{f}(y)$$

$$\begin{aligned} Mf(x) &= C_H \int_{\mathbb{R}} \frac{f(x-t)-f(x)}{|t|^{\frac{3}{2}-H}}\,dt \quad &, \quad 0 < H < \tfrac{1}{2} \\ Mf(x) &= f(x) \quad &, \quad H = \tfrac{1}{2} \\ Mf(x) &= C_H \int_{\mathbb{R}} \frac{f(t)}{|t-x|^{\frac{3}{2}-H}}\,dt \quad &, \quad \tfrac{1}{2} < H < 1 \end{aligned}$$

$$C_H = \left\{2\Gamma\left(H-\tfrac{1}{2}\right)\cos\left(\tfrac{\pi}{2}\left(H-\tfrac{1}{2}\right)\right)\right\}^{-1}\left\{\Gamma(2H+1)\sin(\pi H)\right\}^{\frac{1}{2}}$$

- Define a space $L^2_H(\mathbb{R})$ by

$$(f,g)_{L^2_H(\mathbb{R})} = (Mf, Mg)_{L^2(\mathbb{R})}$$

and a process $\tilde{B}_H(t) := \left\langle \omega, M_{[0,t]}(\cdot) \right\rangle$ with $M_{[0,t]}(x) = M\chi_{[0,t]}(x)$

Fractional white noise analysis

- Computing

$$\mathbb{E}\left[\tilde{B}_H(s)\,\tilde{B}_H(t)\right] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$$

  one concludes that the continuous version of $\tilde{B}_H(t)$ is fBm.
- An orthonormal basis for $L^2_H(\mathbb{R})$

$$\left\{e_k(x) = M^{-1}\xi_k(x), \qquad k = 1, 2, \cdots\right\}$$

  $\xi_k(x)$ is an Hermite function
- Hermite polynomials and Hermite function

$$h_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right)$$

$$\xi_k(x) = \pi^{-\frac{1}{4}}\left((n-1)!\right)^{-\frac{1}{2}} h_{n-1}\left(\sqrt{2}x\right) e^{-\frac{x^2}{2}}$$

Fractional white noise analysis

- Fractional White Noise

$$W_H(t) = \sum_{k=1}^{\infty} M\xi_k(t)\langle \omega, \xi_k \rangle \qquad \frac{dB_H(t)}{dt} = W_H(t)$$

- The extension of Ito's integral to the fractional case is

$$\int_{\mathbb{R}} Y(t,\omega)\, dB_H(t) = \int_{\mathbb{R}} Y(t,\omega) \diamond W_H(t)\, d(t)$$

- Directional derivative and fractional Malliavin derivative

$$D_\gamma^{(H)} F(\omega) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\{F(\omega + \varepsilon M\gamma) - F(\omega)\}$$

If there is $\Psi : \mathbb{R} \to (S)^*$ such that

$$D_\gamma^{(H)} F(\omega) = \int_{\mathbb{R}} M\Psi(t)\, M\gamma(t)\, dt$$

$$D_t^{(H)} F := \frac{\partial^{(H)}}{\partial \omega} F(t,\omega) = \Psi(t)$$

Fractional Poisson analysis

- Poisson measure in $\mathbb{R}$ (or $\mathbb{N}$) and characteristic function

$$\pi(A) = e^{-\sigma} \sum_{n \in A} \frac{\sigma^n}{n!} \quad ; \quad C_\pi(f) = \mathbb{E}\left(e^{if\bullet}\right) = e^{\sigma\left(e^{if}-1\right)}$$

- For $n-$tuples of Poisson variables, $C_\pi(\lambda) = e^{\sum \sigma_k \left(e^{if_k}-1\right)}$
- Probability of no event $\Psi(t) = e^{-\sigma}$ satisfies $\frac{d}{d\sigma}\Psi(\sigma) = -\Psi(\sigma)$
- Replacing $\frac{d}{d\sigma}$ by the (Caputo) fractional derivative $(0 < \alpha \leq 1)$

$$D^\alpha \Psi(\sigma) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{d\Psi(\tau)/d\tau}{(\sigma-\tau)^\alpha} = -\Psi(\sigma)$$

$$\Psi(\sigma) = E_\alpha(-\sigma^\alpha)$$

$E_\alpha(z)$ is the Mittag-Leffler function, $E_\alpha(z) = \sum_{n=o}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$

- The **fractional Poisson process** has a probability of n events, $P(X=n) = \frac{\sigma^{\alpha n}}{n!} E_\alpha^{(n)}(-\sigma^\alpha)$

$$C_\alpha(\lambda) = E_\alpha\left(\sigma^\alpha\left(e^{i\lambda}-1\right)\right)$$

Fractional Poisson analysis

The infinite-dimensional fractional Poisson measure

$$C_\alpha(f) = E_\alpha\left(\int \left(e^{if(x)} - 1\right) d\mu(x)\right)$$

$f \in \mathcal{D}$ and μ is a positive intensity measure on the underlying manifold M

Theorem

The functional $C_\alpha(f)$ is the characteristic functional of a measure on distribution space $\mathcal{D}'$.

Proof.

That C_α is continuous and $C_\alpha(0) = 1$ follows easily from the properties of the Mittag-Leffler function. To check positivity, complete monotonicity of E_α for $0 < \alpha < 1$ implies, by a simple extension of Pollard's result that

$$E_\alpha(-z) = \int_0^\infty e^{-uz} dF_\alpha(u)$$

Fractional Poisson analysis

Proof.

for $\mathrm{Re}\,(z) \geq 0$, $F_\alpha\,(u)$ being nondecreasing and bounded. Then in

$$\sum_{a,b} C_\alpha\,(f_a - f_b)\, z_a z_b = \sum_{a,b} \int_0^\infty dF_\alpha\,(u)\, e^{-u \int_M d\mu(x)\left(1-e^{f_a - f_b}\right)} z_a z_b$$

each one of the terms in the integrand is the characteristic function of a Poisson measure which we already know to be positive. Therefore the spectral integral is also positive. From the Bochner-Minlos it then follows that $C_\alpha\,(f)$ is the characteristic functional of a measure in the space $\mathcal{D}'$ of distributions in the underlying manifold M. □

Introducing the fractional Poisson measure by the above approach yields a probability measure on $\left(\mathcal{D}', C_\alpha\left(\mathcal{D}'\right)\right)$ $\mathcal{D}'$ being the dual of the test function space $\mathcal{D}\,(M)$ of real-valued $C^\infty-$functions on M with compact support.

Fractional Poisson analysis

Next step: to find an appropriate support. Using analyticity of the Mittag-Leffler function one may rewrite

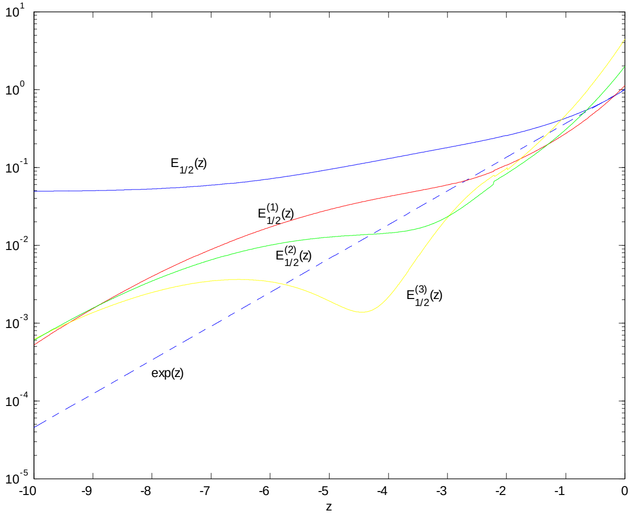
$$C_\alpha(f) = \sum_{n=0}^{\infty} \frac{E_\alpha^{(n)}\left(-\int d\mu(x)\right)}{n!}\left(\int e^{if(x)}d\mu(x)\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{E_\alpha^{(n)}\left(-\int d\mu(x)\right)}{n!}\int e^{i(f(x_1)+f(x_2)+\cdots+f(x_n))}d\mu^{\otimes n}$$

For the Poisson case ($\alpha = 1$) instead of $E_\alpha^{(n)}\left(-\int d\mu(x)\right)$ one has $\exp\left(-\int d\mu(x)\right)$ for all n, the rest being the same. One concludes that the main difference in the $\alpha \neq 1$ case is a different weight given to each $n-$particle space, but that **configuration spaces are also the natural support of the fractional Poisson measure**.

The different weights, multiplying the $n-$particle spaces measure, are physically significant in that they have decays, for large volumes, smaller than the corresponding exponential factor in the Poisson measure (*see the figure for an illustration in the* $\alpha = 1/2$ *case*).

Fractional Poisson analysis



Fractional Poisson analysis

- As in the (non-fractional) infinite-dimensional Poisson analysis define the configuration space Γ_M over a non-compact Riemannian manifold M as the set of all locally finite subsets of M, that is

$$\Gamma_M = \{\gamma \subset M : |\gamma \cap K| < \infty \text{ for all compact } K \subset M\}$$

A similar definition applies for Γ_Λ, $\Lambda \in \mathcal{B}(M)$, $\mathcal{B}(M)$ being the Borel algebra of M. In M one also defines a non-degenerate, non-atomic, infinite measure μ.

- The topology of the configuration space Γ_M is the weakest topology for which the mappings

$$\gamma \rightarrow \langle \gamma, f \rangle = \int_M f(x) \gamma(x) \, dx = \sum_{x \in \gamma} f(x)$$

are continuous for all $f \in C_0(M)$. Notice that in the elements of the configuration space are identified with the distribution

$$\gamma \rightarrow \sum_{x \in \gamma} \delta_x$$

Fractional Poisson analysis

- There is a bijective mapping between the spaces $\Gamma^{(n)}$ of $n-$point configurations and $\breve{M}^n / S_n$, $\breve{M}^n$ being the set of non-coinciding $n-$tuples $\{(x_1, x_2, \cdots, x_n), x_i \neq x_j \text{ if } i \neq j\}$ and S_n the permutation group over $\{1, 2, \cdots, n\}$. Defining the *sym* operation as

$$sym : (x_1, x_2, \cdots, x_n) \in \breve{M}^n \longrightarrow \{x_1, x_2, \cdots, x_n\} \in \Gamma^{(n)}$$

one sees that the measure μ in M induces a measure $\mu^{(n)}$ in the space $\Gamma^{(n)}$ of $n-$point configurations by

$$\mu^{(n)} := \mu^{\otimes n} \circ sym^{-1}$$

One now sees that the characteristic function $C_\alpha(f)$ may be interpreted as the characteristic function of the following probability measure in the space $\Gamma_\Lambda = \cup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$ of $n-$particle configurations in Λ

$$\lambda_\Lambda = \sum_{n=0}^{\infty} \frac{E_\alpha^{(n)}\left(-\int_\Lambda d\mu(x)\right)}{n!} \mu^{(n)}$$

Fractional Poisson analysis

- The corresponding measure λ_M in Γ_M is obtained by a standard projective limit reasoning. Therefore the characteristic functional $C_\alpha(f)$ in (**??**) defines measures both in $\mathcal{D}'(M)$ and in Γ_M.

To proceed with the construction of the infinite-dimensional fractional Poisson analysis it is convenient to define an orthogonal basis. It is well known that the Charlier polynomials $C_n(x)$ defined by the generating function

$$e_\pi(\lambda, x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(x)$$

form an orthogonal basis for the (one-dimensional) Poisson measure π

$$(C_m, C_n)_{L^2(\pi)} = n! \sigma^n \delta_{m,n}$$

Let $g_\alpha(x)$ be the function

$$g_\alpha(x) = \sum_{n=0}^{\infty} \frac{e^{-\frac{\sigma}{2}}}{E_\alpha^{(n)\frac{1}{2}}(-\sigma^\alpha)} \sigma^{\frac{n}{2}(1-\alpha)} \frac{\sin \pi(x-n)}{\pi(x-n)}$$

Then the set $\left\{ F_n^{(\alpha)}(x) = g_\alpha(x)\, C_n(x) \right\}$ is an orthogonal set for the fractional Poisson measure $\pi_\alpha(\sigma)$

$$\left(F_m^{(\alpha)}, F_n^{(\alpha)} \right)_{L^2(\pi_\alpha)} = n! \sigma^n \delta_{m,n}$$

Extending this procedure to infinite dimensions develop the fractional Poisson analysis by reducing it to the usual Poisson analysis, etc.
A more direct approach. Consider the exponential

$$\exp\left(\langle \omega, \ln(1+\varphi) \rangle \right)$$

with $\omega \in \mathcal{D}'$, $\varphi \in \mathcal{U}$, $\mathcal{U}$ a neighborhood of zero in the complexified $\mathcal{D}_{\mathbb{C}}$.
Standard arguments imply the existence of a decomposition

$$\exp\left(\langle \omega, \ln(1+\varphi) \rangle \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle C_n^{\mu,\alpha}(\omega), \varphi^{\otimes n} \right\rangle$$

the kernel $C_n^{\mu,\alpha}(\omega)$ being a unique distribution in $\mathcal{D}'^{\hat{\otimes} n}$. We now prove:

Fractional Poisson analysis

Proposition

Restricted to functions φ such that $\langle\varphi\rangle_\mu = \int \varphi d\mu = 0$, the kernels $C_n^{\mu,\alpha}(\omega)$ are an orthogonal set with orthogonality relation

$$\left(\left\langle C_n^{\mu,\alpha}, \varphi^{(n)}\right\rangle \left\langle C_m^{\mu,\alpha}, \phi^{(m)}\right\rangle\right)_{L^2\left(\pi_\mu^\alpha\right)} = \delta_{n,m} \frac{(n!)^2}{\Gamma(\alpha n + 1)} \left(\varphi^{(n)}, \phi^{(m)}\right)_{L^2(\mu^{\otimes n})} \quad (1)$$

Fractional Poisson analysis

Proof.

$$
\begin{aligned}
&\int \exp\left(\langle \omega, \ln\left(1+z_1\varphi\right)+\ln\left(1+z_2\phi\right)\rangle\right) d\pi_\mu^\alpha\left(\omega\right) \\
&= E_\alpha\left(\int_M \left(z_1\varphi+z_2\phi+z_1 z_2\varphi\phi\right) d\mu\right) \\
&= \sum_{n=0}^{\infty} \frac{z_1^n z_2^n}{\Gamma\left(\alpha n+1\right)} \left(\varphi^{(n)}, \phi^{(n)}\right)_{L^2\left(\mu^{\otimes n}\right)}
\end{aligned}
$$

Comparing with

$$
\begin{aligned}
&\int \exp\left(\langle \omega, \ln\left(1+z_1\varphi\right)+\ln\left(1+z_2\phi\right)\rangle\right) d\pi_\mu^\alpha \\
&= \sum_{n,m=0}^{\infty} \frac{z_1^n z_2^m}{n!m!} \int_{\mathcal{D}'} \left\langle C_n^{\mu,\alpha}\left(\omega\right), \varphi^{(n)}\right\rangle \left\langle C_m^{\mu,\alpha}\left(\omega\right), \phi^{(m)}\right\rangle d\pi_\mu^\alpha\left(\omega\right)
\end{aligned}
$$

one obtains (1). □

Fractional Poisson analysis

Now for arbitrary functions we define the kernels $\overset{\backsim}{C}_m^{\mu,\alpha}$ by

$$\left(\overset{\backsim}{C}_n^{\mu,\alpha}(\omega), \varphi^{(n)}\right) := \left(C_n^{\mu,\alpha}(\omega), \left(\varphi - \langle\varphi\rangle_\mu\right)^{(n)}\right)$$

These new kernels are also orthogonal with orthogonality relation

$$\left(\left\langle \overset{\backsim}{C}_n^{\mu,\alpha}, \varphi^{(n)}\right\rangle \left\langle \overset{\backsim}{C}_m^{\mu,\alpha}, \phi^{(m)}\right\rangle\right)_{L^2\left(\pi_\mu^\alpha\right)}$$

$$= \frac{\delta_{n,m}(n!)^2}{\Gamma(\alpha n+1)} \left((\varphi - \langle\varphi\rangle)^{(n)}, (\phi - \langle\phi\rangle)^{(n)}\right)_{L^2(\mu^{\otimes n})}$$

With these $\overset{\backsim}{C}_n^{\mu,\alpha}$ kernels one may write a chaos expansion for distributions F in the fractional Poisson space

$$F = \sum_{n=0}^{\infty} \left(\overset{\backsim}{C}_n^{\mu,\alpha}, f^{(n)}\right)$$

establish the isomorphism with Fock space and from it construct creation and annihilation operator, differential operators, etc., etc.

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