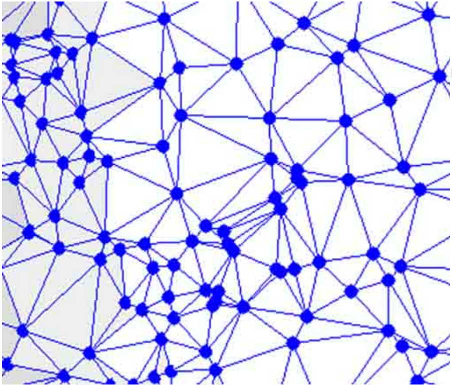
Long-range connections: Fractional networks

R. Vilela Mendes
CMAFCIO, Universidade de Lisboa
http://label2.ist.utl.pt/vilela/

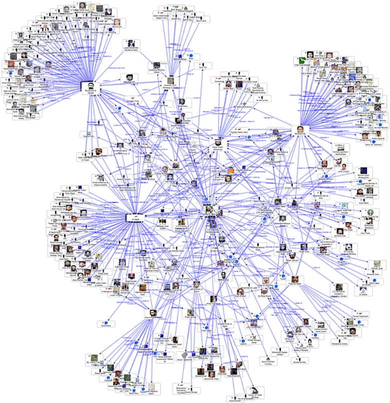
Contents

Networks and hubs

A traditional network drawing is like this



However a real life network is more like this



Networks and hubs

- Notice that in the second network there are many sparsely connected nodes and a few highly connected nodes (the **hubs**).
- The hubs are the essential nodes to address (or protect) in a network, because they control the robustness of the network and the diffusion of information. This is a fact well known by politicians, advertisement agencies and hackers.
- Several mechanisms in network growth lead to the hub formation, the better known being "preferential attachment" (rich gets richer).
- The importance of hubs has been known for a long time, even at the time of the Inquisition

Network learning and the Inquisition

In 1231 the Pope Gregory charged Dominicans and Franciscans to track down and eradicate the heresies. At the time the main heresies were the Cathars in southern France and the Waldensians in Germany and northern Italy.

Network learning and the Inquisition

- In the beginning the Inquisition used very drastic eradication methods, suppressing all agents in heresy regions regardless of being "infected" or not. For example in the massacre of Béziers it is reported that the executioners asked the abbot: "Chatolics are mixed up with heretics. What shall we do?". "Kill them all, for God knows who are his".
- It turned out that this technique was not very efficient and heresy continued to spread. However Dominicans, brutal as they were, were not stupid and Bernard Gui (a real person and not just a fictional character in Umberto Eco's "The name of the rose") wrote a inquisition instructions manual *"Practica Inquisitionis heretice pravitatis"* where he emphasized that the agents to be target were the key heretics (the spiritual elite) their town-to-town mobility and their messengers.
- This is what is now known in epidemiology as "acquaitance immunisation". They really knew about network science.

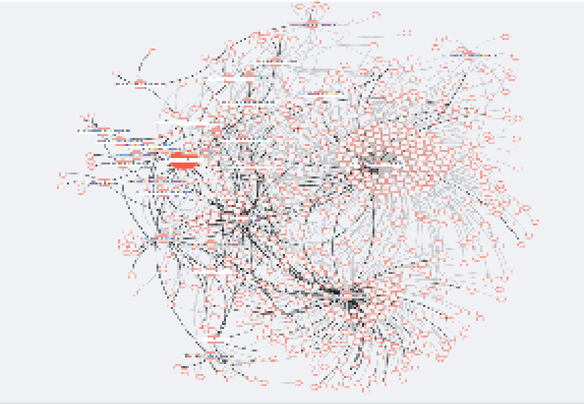
Network learning and the Inquisition

- A dominant force for several centuries the Inquisition may have been responsible for 32000 executions, mostly burnings at the stake (burned, not slain because good christians do not shed blood).
- Ocasionally the Inquisition was also used as a tool by the political rulers. Fernando II and the conversos in Spain to finance the Granada conquest. Philip IV in France to avoid paying his debts to the Templars.
- *An historical note:* The inquisition was never formally abolished by the Catholic Church. In 1542 it was named the *"Supreme Sacred Congregation of the Roman and Universal Inquisition"* and it is now called the *"Congregation for the Doctrine of the Faith".*

(more details in P. Ormerod and A. P. Roach; Physica A 339 (2004) 645-652)

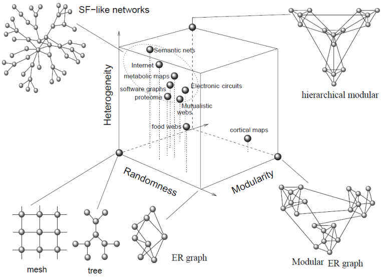
A 16th century network

The "sparse connections plus the hubs" remained for a long time the dominating structure of social netwoks.



http://www.sixdegreesoffrancisbacon.com

The old network classification paradigm

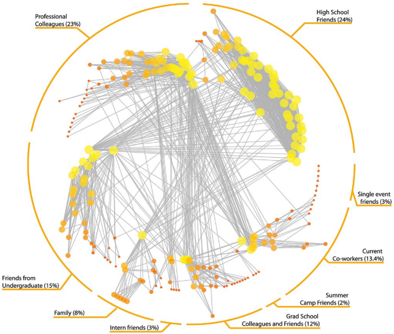


(R. V. Solé and S. Valverde; in Complex Networks,LNP650 (2004) pp. 189-207)

For social and biological networks the SF network structure has been considered the most important one. Many low connectivity nodes and a few highly connected ones (hubs). Has a power law degree distribution, high clustering, short path-length (small world). The hubs provide robustness under random disturbance or attack but are sensitive to targeted attacks. The name scale-free originates from the power-law degree distribution and power law formation probability.

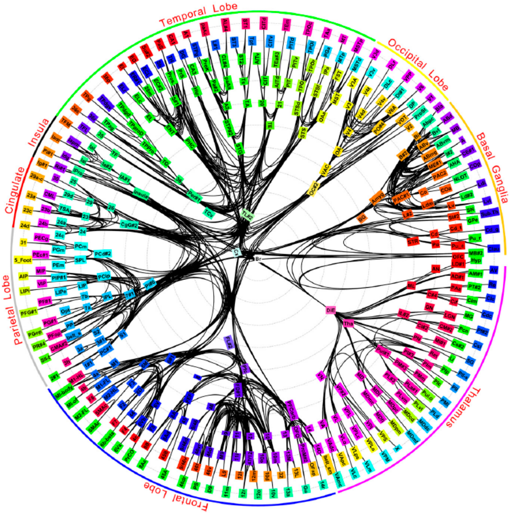
The new structure

The "*sparse connections plus the hubs*" remained for a long time the dominating structure of social netwoks. Starting in last years of the 20th century a new structure emerged. Example: A Facebook personal network



(B. Hogan in *Analyzing Social Media Networks with NodeXL*)
This network interconnected with many others leads to a global network with **many local connections and many long-range connections**.

Society imitates Nature: A brain network



(D. S. Modhaa and R. Singhb; PNAS 107 (2010) 13485–13490)

Quantitative tools for network studies

- A new structure requires either new tools or new features of the old tools
  What are the old tools?
- The **Laplacian** and the **Random-Walk** matrices and their spectra are the main tools that characterize the structure of the network and the dynamics on it.

The Laplacian matrix

$$L = G - A$$

G = degree matrix $G_{ij} = \delta_{ij} \times$ number of connections of that node
A = adjacency matrix $A_{ij} = 1$ if i and j are connected, $A_{ij} = 0$ otherwise
For a node i connected to two other nodes $i+1$ and $i-1$ the action of the matrix on a vector $\begin{pmatrix} \vdots \\ \psi(i-1) \\ \psi(i) \\ \psi(i+1) \\ \vdots \end{pmatrix}$

leads to

$$-\psi(i-1) + 2\psi(i) - \psi(i-1)$$

which is a discrete version of $-d^2$ (minus the second derivative). Hence the name "Laplacian".

The Laplacian matrix

Let $\psi(i)$ for each node i be the intensity of ψ across the network. It is reasonable to think that ψ diffuses from i to j proportional to $\psi(i) - \psi(j)$ if i and j are connected

$$\frac{d\psi(i)}{dt} = -k \sum_j A_{ij} (\psi(i) - \psi(j)) = -k \left(\psi(i) \sum_j A_{ij} - \sum_j A_{ij} \psi(j) \right)$$

In matrix form this is

$$\frac{d\psi}{dt} + kL\psi = 0$$

an heat equation-like equation. Therefore the Laplacian matrix controls the diffusion of quantities in the network.

The Random-Walk matrix

$$R = G^{-1}A$$

$R_{ii} = 0$
$R_{ij} = \frac{A_{ij}}{\text{degree}(i)}$ $(i \neq j)$ if i and j are connected, $R_{ij} = 0$ otherwise
The R matrix controls the random motion of a walker on the network.
The probability for a random walker to be at the node i at time t given that at time $t-1$ was at the node j is

$$p_i(t) = \sum_j \frac{A_{ij}}{\text{degree}(j)} p_j(t-1)$$

which is, in matrix form

$$p(t) = G^{-1}Ap(t-1)$$

A network with power-law connection probability

- Let the network N be embedded into an Euclidean network where distances may be defined. In the actual network the distances might mean geographical distances, separation of communities, functional separation as in a brain network, etc.
- In the network, where $A_{ij} = 0$ or 1, let the probability of links at distance d be proportional to a power of the distance

$$P_{ij} = cd_{ij}^{-\gamma} \qquad \text{with } \gamma \leq 3$$

To find the nature of the diffusion in such a network, consider a block renormalized network N^* where each set of q nearby nodes of N are mapped to a node of the N^* network. Therefore in the N^* network the connections are

$$A_{ij}^* = cqd_{ij}^{-\gamma}$$

Then

$$L^*\psi(i) = G_{ii}^*\psi(i) - cq\sum_{j\neq i} d_{ij}^{-\gamma}\psi(j)$$

Fractional Laplacian and superdiffusion

What kind of diffusion does the Laplacian matrix $L^* = G^* - A^*$ imply for the network N^*?. Consider a fractional diffusion equation

$$\frac{d\psi}{dt} = -kD^{\alpha}\psi$$

Using a symmetrized Grünwald-Letnikov representation of the fractional derivative $(a < x < b)$

$$\begin{aligned} D^{\beta}\psi(x) &= \frac{1}{2}\lim_{h\to 0}\frac{1}{h}\left\{\sum_{n=0}^{\left[\frac{x-a}{h}\right]}(-1)^n\begin{pmatrix}\beta\\ n\end{pmatrix}\psi(x-nh)\right. \\ &\quad \left. + \sum_{n=0}^{\left[\frac{b-x}{h}\right]}(-1)^n\begin{pmatrix}\beta\\ n\end{pmatrix}\psi(x+nh)\right\} \end{aligned}$$

The coefficients are

$$\left|\begin{pmatrix}\beta\\ n\end{pmatrix}\right| = \frac{\Gamma(\beta+1)\left|\sin(\pi\beta)\right|}{\pi}\frac{\Gamma(n-\beta)}{\Gamma(n+1)} \backsim_{n>>} \frac{\Gamma(\beta+1)\left|\sin(\pi\beta)\right|}{\pi}n^{-(\beta+1)}$$

Fractional Laplacian, superdiffusion and Lévy flights

and $sign \begin{pmatrix} \beta \\ n \end{pmatrix} = (-1)^{n+1}$.

Comparing with the expression for $L^* \psi(i)$, the conclusion is that diffusion in the N^* network is fractional diffusion of exponent $\beta = \gamma - 1$

$\beta = 2$ is normal diffusion, all $\beta < 2$ are superdiffusions.

$$\langle x^2(t) \rangle \backsim t^{\frac{2}{\beta}}$$

On the other hand, analyzing the structure of the random walks controlled by $G^{-1}A$ the conclusion is that whereas for normal diffusion the jumps are of one step, for $\gamma < 3$ arbitrarily large large jumps occur with a power law (Lévy flights)

The general conclusion is that in these netwoks both mobility and diffusion of information occur at a very fast rate. Therefore it may considered as a new structure distinct from SF networks, a Fractional Network (FR).

Strenghts and weaknesses of fractional networks

- The new structure has wide implications for the control of the networks. The hubs are no longer the controllers.
- In a SF network, the hubs are both the strength and the weakness of the network. They insure global connectivity even if a large number of links are destroyed. But when directly targeted the network is deeply affected (targeted structural weakness).
  Propagation of ideas, oppinions, fads (memes) are most effective if introduced to the hubs. However fast global establishment of a trend requires its introduction at many hubs.
- A FR network is structurally very stable and resilient to attack. It is pointless or too expensive to disrupt the network. The network itself is the HUB.
  Superdiffusion is both the strength and the weakness of the network. Well crafted memes propagate very fast. But also do counter-memes. In SF networks the memes are most efficiently introduced at the hubs. Here they might be introduced anywhere.

Some references

- A.-L. Barabási and R. Albert; *Emergence of scaling in random networks*, Science 286 (1999) 509-512.
- P. Ormerod and A. P. Roach; *The Medieval inquisition: scale-free networks and the suppression of heresy*, Physica A 339 (2004) 645-652.
- A. Clauset, C. R. Shalizi and M. E. J. Newman; *Power-law distributions in empirical data*, SIAM REVIEW 51 (2009) 661–703.
- B. Hogan; *Visualizing and Interpreting Facebook Networks* in "Analysing Social Media Networks with NodeXL", D. L. Hansen et al. (Eds.), pp. 165-179, Elsevier 2011.
- C. J. Billedo, P. Kerkhof and C. Finkenauer; *The use of social networking sites for relationship maintenance in long-distance and geographically close romantic relationships*, Cyberpsychol Behav. Soc. Netw. 18 (2015) 152-157.
- R. Carvalho and G. Iori; *Socioeconomic networks with long-range interactions*, Phys. Rev. E 78 (2008) 016110.

Some references

- N. T. Markov et al.; *The role of long-range connections on the specificity of the macaque interareal cortical network*, PNAS 110 (2013) 5187–5192.
- D. S. Modha and R. Singh; *Network architecture of the long-distance pathways in the macaque brain*, PNAS 107 (2010) 13485–13490.
- T. R. Knösche and M. Tittgemeyer; *The role of long-range connectivity for the characterization of the functional–anatomical organization of the cortex*, Frontiers in Systems Neuroscience 5 (2011) Article 58.
- P. Barttfeld et al.; *Organization of brain networks governed by long-range connections index autistic traits in the general population*, Journal of Neurodevelopmental Disorders 2013, 5:16.
- K. B. Gustafson, B. S. Bayati and P. A. Eckhoff; *Fractional Diffusion Emulates a Human Mobility Network during a Simulated Disease Outbreak*, Frontiers in Ecology and Evolution 5 (2017) Article 35.

Some references

- A. P. Riascos and J. L. Mateos; *Fractional dynamics on networks: Emergence of anomalous diffusion and Lévy flights*, Phys. Rev. E 90 (2014) 032809.
- R. Gorenflo and F. Mainardi; *Essentials of fractional calculus*, www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k
- RVM; *Fractional networks, the new structure*, https://arxiv.org/abs/1804.10605