# The fractional volatility model: No-arbitrage, leverage and completeness

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# Introduction

- In liquid markets the autocorrelation of price changes decays to negligible values in a few ticks. It is the basic motivation for geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB(t)$$

- Models the absence of linear correlations.
- Otherwise, does not reproduce the empirical leptokurtosis nor does it explain why nonlinear functions of the returns exhibit significant positive autocorrelation.
- Long memory effects should be represented in the process and this is not included in the geometric Brownian motion hypothesis.
- A dynamical model for volatility is needed and $\sigma$, rather than a constant, is itself a process.
- An attempt to obtain a model that is both consistent with the data and mathematically sound.

# The fractional volatility model

- The basic hypothesis for the model construction were:
- (H1) The log-price process $\log S_t$ belongs to a probability product space $(\Omega_1 \times \Omega_2, P_1 \times P_2)$ of which:
  - $(\Omega_1, P_1)$ is the Wiener space
  - $(\Omega_2, P_2)$, is a probability space to be reconstructed from the data.
  Let $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and $\mathcal{F}_{1,t}$ and $\mathcal{F}_{2,t}$ the $\sigma$-algebras in $\Omega_1$ and $\Omega_2$ generated by the processes up to $t$. A particular realization of the log-price process is denoted $\log S_t\left(\omega_1, \omega_2\right)$
- (H2) The second hypothesis is stronger, although natural. It is assumed that for each fixed $\omega_2$, $\log S_t\left(\cdot, \omega_2\right)$ is a square integrable random variable in $\Omega_1$.

$$\frac{dS_t}{S_t}\left(\cdot, \omega_2\right) = \mu_t\left(\cdot, \omega_2\right) dt + \sigma_t\left(\cdot, \omega_2\right) dB_t$$

# The fractional volatility model

- These principles and a careful analysis of the market data led, in an essentially unique way, to the following model:

$$dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dB(t)$$

$$\log \sigma_t = \beta + \frac{k}{\delta} \{B_H(t) - B_H(t-\delta)\}$$

  $B_H(t)$ being fractional Brownian motion with Hurst coefficient $H$. The data suggests values of $H$ in the range $0.8 - 0.9$.
- The model is empirically sucessful: Describes the statistics of price returns for a large $\delta$-range in different markets and leads to a new option pricing formula, with "smile" deviations from Black-Scholes
- An agent-based interpretation led to the conclusion that the statistics generated by the model was consistent with the limit order book price setting mechanism.
- The results reported here give a solid mathematical construction of the fractional volatility model, discussing existence questions, arbitrage and market completeness.

# No-arbitrage and incompleteness

- Let $(\Omega_1, \mathcal{F}_1, P_1)$ be the Wiener probability space, carrying a Brownian motion $B = (B_t)_{0 \le t < \infty}$ an a filtration $\mathbb{F}_1 = (\mathcal{F}_{1,t})_{0 \le t < \infty}$
- Let $(\Omega_2, \mathcal{F}_2, P_2)$ be another probability space associated to a fractional Brownian motion $B_H$ with Hurst parameter $H \in (0,1)$ and a filtration $\mathbb{F}_2 = (\mathcal{F}_{2,t})_{0 \le t < \infty}$ generated by $B_H$.
- Embed these two probability spaces in a product space $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}\right)$, with $\overline{\Omega}$ the Cartesian product $\Omega_1 \times \Omega_2$ and $\overline{P}$ the product measure. $\pi_1$ and $\pi_2$, are the projections of $\overline{\Omega}$ onto $\Omega_1$ and $\Omega_2$, and $\mathcal{N}$ the $\sigma-$algebra generated by all null sets from the product $\sigma-$algebra
- Let $\overline{\mathcal{F}} = (\mathcal{F}_1 \otimes \mathcal{F}_2) \vee \mathcal{N}$ and $\overline{\mathbb{F}} = \left(\overline{\mathcal{F}}_t\right)_{0 \le t < \infty}$ the filtration for $\overline{\mathcal{F}}_t = (\mathcal{F}_{1,t} \otimes \mathcal{F}_{2,t}) \vee \mathcal{N}$.
- Extend $B$ and $B_H$ to $\overline{\mathbb{F}}-$adapted processes on $\left(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}\right)$ by $\overline{B}(\omega_1, \omega_2) = (B \circ \pi_1)(\omega_1, \omega_2)$ and $\overline{B}_H(\omega_1, \omega_2) = (B_H \circ \pi_2)(\omega_1, \omega_2)$ for $(\omega_1, \omega_2) \in \overline{\Omega}$.
- Then, it is easy to prove that $\overline{B}$ and $\overline{B}_H$ are independent Brownian and fractional Brownian motions with respect to $\overline{P}$.

# No-arbitrage and incompleteness

- Consider now the market with a risk-free asset $A_t$

$$dA_t = rA_t\, dt \quad A_0 = 1 \tag{1}$$

$$dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dB\,(t) \tag{2}$$

$$\log \sigma_t = \beta + \frac{k}{\delta}\left\{B_H\,(t) - B_H\,(t-\delta)\right\} \tag{3}$$

$\mu_t$ is a $\overline{\mathbb{F}}$-adapted process with continuous paths, $k$ the volatility intensity parameter and $\delta$ the observation time scale.

- Eq.(3) leads to $\sigma_t = \theta e^{\frac{k}{\delta}\{B_H(t)-B_H(t-\delta)\}-\frac{1}{2}\left(\frac{k}{\delta}\right)^2 \delta^{2H}}$ with $\theta = \mathbb{E}_{\overline{P}}\,[\sigma_t]$, hence $\sigma_t$ is a measurable and $\overline{\mathbb{F}}-$adapted process satisfying for all $0 \leq t < \infty_{\overline{P}}$, $\mathbb{E}_{\overline{P}}\left[\int_0^t \sigma_s^2 ds\right] = \theta^2 \exp\left\{\left(\frac{k}{\delta}\right)^2 \delta^{2H}\right\} t < \infty$
- Moreover $\int_0^t |\mu_s|\, ds$ being finite $\overline{P}-$almost surely for $0 \leq t < \infty$, an application of Itô's formula yields

$$S_t = S_0 \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dB_s\right\}$$

# No-arbitrage and incompleteness

## Lemma (1)

*Consider the measurable,* $\overline{\mathbb{F}}$*-adapted process defined by*

$$\gamma_t = \frac{r - \mu_t}{\sigma_t}, \qquad 0 \leq t < \infty$$

*with* $\mu \in L^\infty\left([0, T] \times \overline{\Omega}\right)$ *and denote by* $\eta = (\eta_t)_{0 \leq t < \infty}$ *the stochastic exponential of* $\left(\int_0^t \gamma_s dB_s\right)_{0 \leq t < \infty}$*, that is,*

$$\eta_t = \exp\left\{\int_0^t \gamma_s dB_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right\}, \qquad 0 \leq t < \infty$$

*Then,*

$$\mathbb{E}_{\overline{P}}\left[\frac{1}{2}\int_0^T \gamma_s^2 ds\right] < \infty, \qquad 0 \leq T < \infty \tag{4}$$

# No-arbitrage and incompleteness

## Proof.

Makes of use of the fact that sample paths of the fractional Brownian motion $B_H$ are Hölder continuous of any order $\alpha \geq 0$ strictly less than $H$. There is $C_\alpha > 0$ such that, $\overline{P}-$almost surely, $|B_H(t) - B_H(s)| \leq C_\alpha |t-s|^\alpha$. Then, for all $0 \leq T < \infty$

$$
\begin{aligned}
& \mathbb{E}_{\overline{P}}\left[\frac{1}{2}\int_0^T \gamma_s^2 ds\right] \\
\leq\ & \mathbb{E}_{\overline{P}}\left[\exp\left\{\frac{e^{k^2\delta^{2H-2}}}{2\theta^2}\int_0^T (r+|\mu_s|)^2 e^{-2\frac{k}{\delta}(B(s)-B(s-\delta))} ds\right\}\right] \\
\leq\ & \mathbb{E}_{\overline{P}}\left[\exp\left\{\frac{(r+\|\mu_s\|_\infty)^2}{2\theta^2} e^{k^2\delta^{2H-2}} \int_0^T e^{-2\frac{k}{\delta}(B(s)-B(s-\delta))} ds\right\}\right] \\
\leq\ & \exp\left\{\frac{T(r+\|\mu_s\|_\infty)^2}{2\theta^2} e^{k^2\delta^{2H-2}-2C_\alpha k \delta^{\alpha-1}}\right\} < \infty
\end{aligned}
$$

# No-arbitrage and incompleteness

Additionally, assume that investors are allowed to trade only up to some fixed finite planning horizon $T > 0$.

**Theorem (1)**

*The market defined by (1), (2) and (3) is free of arbitrage*

**Proof.**

Because, by lemma 1 the process $\gamma = \frac{r-\mu_t}{\sigma_t}$ satisfies the Novikov condition (4), the nonnegative continuous supermartingale $\eta = \exp\left\{\int_0^t \gamma_s dB_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right\}$ is a true $\overline{P}-$martingale. Hence we can define for each $0 \leq T < \infty$ a new probability measure $Q_T$ on $\overline{\mathcal{F}}_T$ by

$$\frac{dQ_T}{d\overline{P}} = \eta_T, \quad \overline{P} - a.s.$$

$\square$

# No-arbitrage and incompleteness

## Proof.

Then, by the Cameron-Martin-Girsanov theorem, for fixed $T \in [0, \infty)$

$$B_t^* = B_t - \int_0^t \frac{r - \mu_s}{\sigma_s} \, ds \quad 0 \le t \le T$$

is a Brownian motion on the probability space $\left(\overline{\Omega}, \overline{\mathcal{F}}_T, Q_T\right)$. Consider now the discounted price process $Z_t = \frac{S_t}{A_t} \quad 0 \le t \le T$ . Under the new probability measure $Q_T$, equivalent to $\overline{P}$ on $\overline{\mathcal{F}}_T$, its dynamics is given by

$$Z_t = Z_0 + \int_0^t \sigma_s Z_s \, dB_s^*$$

is a martingale in $\left(\overline{\Omega}, \overline{\mathcal{F}}_T, Q_T\right)$ for the filtration $\left(\overline{\mathcal{F}}_t\right)_{0 \le t < T}$. By the fundamental theorem of asset pricing, existence of an equivalent martingale measure for $Z_t$ implies there are no arbitrages, that is, $\mathbb{E}_{Q_T}\left[Z_t | \overline{\mathcal{F}}_s\right] = Z_s$ for $0 \le s < t \le T$. $\square$

# Market (in)completeness

In this financial model, trading takes place only in the stock and in the money market and, as a consequence, volatility risk cannot be hedged. Hence, since there are more sources of risk than tradable assets, the market is expected to be incomplete.

**Theorem (2)**

*The market defined by (1),(2) and (3) is incomplete*

**Proof.**

Here one uses an integral representation for the fractional Brownian motion

$$B_H(t) = \int_0^t K_H(t,s)\, dW_s \tag{5}$$

$W_t$ being a Brownian motion with respect to $\overline{P}$, independent from $B_t$ and $K$ is a square integrable kernel □

# Market (in)completeness

## Proof.

$$K_H(t,s) = C_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du, \quad s < t$$

$(H > 1/2)$. Then the process

$$\eta_t' = \exp\left( W_t - \frac{1}{2} t \right)$$

is a square-integrable $\overline{P}-$martingale. Defining a standard bi-dimensional Brownian motion, $W_t^* = (B_t, W_t)$, the process $\eta_t^* = \eta_t \eta_t'$

$$\eta_t^* = \exp\left\{ \int_0^t \Gamma_s \bullet dW_t^* - \frac{1}{2} \int_0^t \|\Gamma_s\|^2\, ds \right\}$$

because by lemma 1, $\Gamma = (\gamma, 1)$ satisfies the Novikov condition, is also a $\overline{P}-$martingale. □

## Proof.

Then, by the Cameron-Martin-Girsanov theorem, the process $\widetilde{W}_t^* = \left(\widetilde{W}_t^{*(1)}, \widetilde{W}_t^{*(2)}\right)$ with

$$\begin{aligned} \widetilde{W}_t^{*(1)} &= B_t - \int_0^t \gamma_s ds \\ \widetilde{W}_t^{*(2)} &= W_t - t \end{aligned}$$

is a bi-dimensional Brownian motion on the probability space $\left(\overline{\Omega}, \overline{\mathcal{F}}_T, Q_T^*\right)$, where $Q_T^*$ is the probability measure

$$\frac{dQ_T^*}{d\overline{P}} = \eta_T^* = \eta_T \eta_T^{'} \tag{6}$$

Moreover, the discounted price process $Z$ remains a martingale with respect to the new measure $Q_T^*$. $Q_T^*$ being an equivalent martingale measure distinct from $Q_T$, the market is incomplete. □

# Leverage and completeness

- **Leverage as a motivation:**
  The following nonlinear correlation of the returns
  $$L(\tau) = \left\langle \left| r(t+\tau) \right|^2 r(t) \right\rangle - \left\langle \left| r(t+\tau) \right|^2 \right\rangle \langle r(t) \rangle$$
  is called *leverage* and the *leverage effect* is the fact that, for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus decays to zero whereas for $\tau < 0$ it has almost negligible values.
- In the market model discussed before the volatility process $\sigma_t$ affects the log-price, but is not affected by it. Therefore, in this form the fractional volatility model contains no leverage effect.
- Leverage may, however, be implemented in the model in a modified model

$$\begin{aligned} dS_t &= \mu_t S_t dt + \sigma_t S_t dW_t \\ \log \sigma_t &= \beta + k' \int_{-\infty}^{t} (t-u)^{H-\frac{3}{2}} \, dW_u \end{aligned} \tag{7}$$

when the two Brownian motions $W_t$ and $W_u$ are identified.

# Leverage and completeness

**Theorem (3)**

*The market defined by (7) and (1) is free of arbitrage and complete.*

**Proof.**

The proof of the first part of the proposition is analogous to that of theorem 1. A similar argument to Lemma 1 yields that

$$\eta_t = \exp\left\{ \int_0^t \frac{r-\mu_s}{\sigma_s} dW_s - \frac{1}{2}\int_o^t \left(\frac{r-\mu_s}{\sigma_s}\right)^2 ds \right\}$$

is a $P_1$-martingale with respect to $(\mathcal{F}_{1,t})_{0\leq t<T}$ and the probability measure $Q_T$, defined by $\frac{dQ_T}{dP_1} = \eta_T$ is an equivalent martingale measure. Now that we have shown that the set of equivalent local martingale measures for the market is non-empty, let $Q^*$ be an element in this set. $\square$

# Leverage and completeness

## Proof.

By the Girsanov converse there is a $(\mathcal{F}_{1,t})_{0\leq t<T}$ progressively measurable $\mathbb{R}$-valued process $\phi$ such that the Radon-Nikodym density of $Q^*$ with respect to $P_1$ equals $\frac{dQ^*}{dP_1} = \exp\left\{\int_0^T \phi_s dW_s - \frac{1}{2}\int_0^T \phi_s^2 ds\right\}$
Moreover the process $W_t^* = W_t - \int_0^t \phi_s ds$ is a standard $Q^*-$Brownian motion and the discounted price process $Z$ satisfies the stochastic differential equation $dZ_t = (\mu_t - r + \sigma_t\phi_t)Z_t dt + \sigma_t Z_t dW_t^*$
Because $Z_t$ is a $Q^*-$martingale, then it must be hold $\mu(t,\omega) - r + \sigma(t,\omega)\phi(t,\omega) = 0$ almost everywhere w.r.t. $meas \times P$ in $[0,T]\times\Omega$, *meas* being the Lebesgue measure on the line. It implies

$$\phi(t,\omega) = \frac{r-\mu(t,\omega)}{\sigma(t,\omega)}$$

a. e. $(t,\omega)\in[0,T]\times\Omega_1$. Hence $Q^* = Q$, that is, $Q$ is the unique equivalent martingale measure. This market model is complete. □

# Conclusions

- Reconstructed from empirical data, the fractional volatility model describes well the statistics of returns. Once the parameters are adjusted by the data for a particular observation time scale $\delta$, it describes well different time lags.
- Specific trader strategies and psychology play a role on market crisis and bubbles. However, the fact that in the fractional volatility model the same set of parameters would describe very different markets seems to imply that the market statistical behavior (in normal days) is more influenced by the nature of the financial institutions (the double auction process) than by the traders strategies. Therefore some kind of universality of the statistical behavior of the bulk data throughout different markets would not be surprising.
- This conclusion is borne out by agent-based simulation models.

# Conclusions

- The identification of the Brownian process of the log-price with the one that generates the fractional noise driving the volatility, introduces an asymmetric coupling between $\sigma_t$ and $S_t$ that is also exhibited by the market data.
- In this paper, mathematical consistency of, both versions, of the fractional volatility model has been established. This and its better consistency with the experimental data, makes it a candidate to replace geometrical Brownian as the standard market model.
- Both models, studied in this paper, satisfy the no-arbitrage property in the usual sense of having an equivalent martingale measure for the price process. An interesting question, deserving to be explored in the future, is its behavior in a scenario of volatility trading.

# References


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# The statistics of price returns

$$P_{\delta}\left(r\left(\Delta\right)\right) = \frac{1}{4\pi\theta k\delta^{H-1}\sqrt{\Delta}}\frac{1}{\sqrt{\lambda}}\left(e^{-\frac{1}{C}\left(\log\lambda-\frac{d}{dz}\right)^{2}}\Gamma\left(z\right)\right)\Bigg|_{z=\frac{1}{2}}$$

with asymptotic behavior, for large returns

$$P_{\delta}\left(r\left(\Delta\right)\right) \sim \frac{1}{\sqrt{\Delta\lambda}}e^{-\frac{1}{C}\log^{2}\lambda}$$

$$r\left(\Delta\right) = \log S_{t+\Delta} - \log S_{t}\ ,\ \theta = e^{\beta},\ \lambda = \frac{\left(r\left(\Delta\right) - r_{0}\right)^{2}}{2\Delta\theta^{2}}$$

# The statistics of price returns

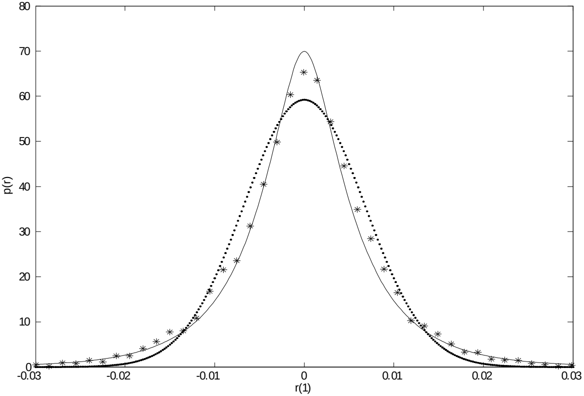


Figure: One-day NYSE returns compared with the model predictions and the lognormal

$H = 0.83,\ k = 0.59,\ \beta = -5,\ \delta = 1$

# The statistics of price returns

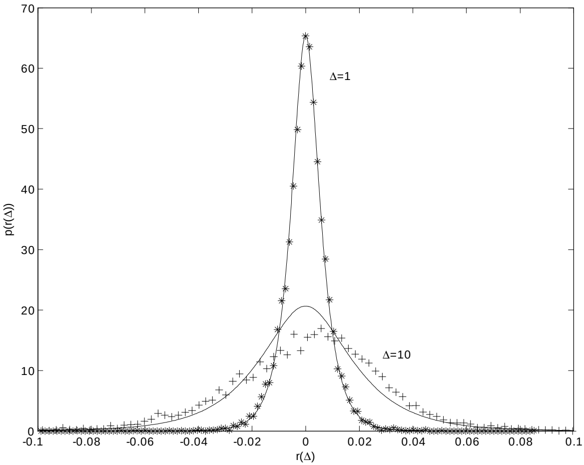


Figure: One and ten-days NYSE returns compared with the model predictions

# The statistics of price returns

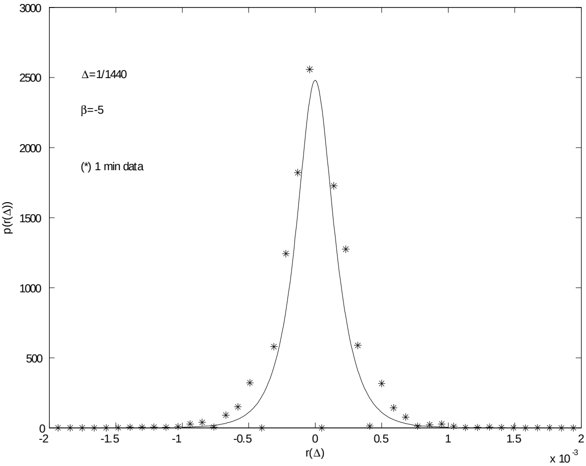


Figure: One-minute USD-Euro returns compared with the model predictions, with parameters obtained from one-day NYSE data

# Option pricing

A new option price formula

$$\begin{aligned} V(S_t, \sigma_t, t) &= S_t \left[ aM(\alpha, a, b) + bM(\alpha, b, a) \right] \\ &\quad - Ke^{-r(T-t)} \left[ aM(\alpha, a, -b) - bM(\alpha, -b, a) \right] \end{aligned}$$

$$\begin{aligned} M(\alpha, a, b) &= \frac{1}{2\pi\alpha} \int_{-1}^{\infty} dy \int_0^{\infty} dx e^{-\frac{\log^2 x}{2\alpha^2}} e^{-\frac{y^2}{2}\left(ax + \frac{b}{x}\right)^2} \\ &= \frac{1}{4\alpha} \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx \frac{e^{-\frac{\log^2 x}{2\alpha^2}}}{ax + \frac{b}{x}} erfc\left( -\frac{ax}{\sqrt{2}} - \frac{b}{\sqrt{2}x} \right) \end{aligned}$$

erfc is the complementary error function and

$$\begin{aligned} a &= \frac{1}{\sigma} \left( \frac{\log \frac{S_t}{K}}{\sqrt{T-t}} + r\sqrt{T-t} \right) \\ b &= \frac{\sigma}{2} \sqrt{T-t} \end{aligned}$$

# Option pricing

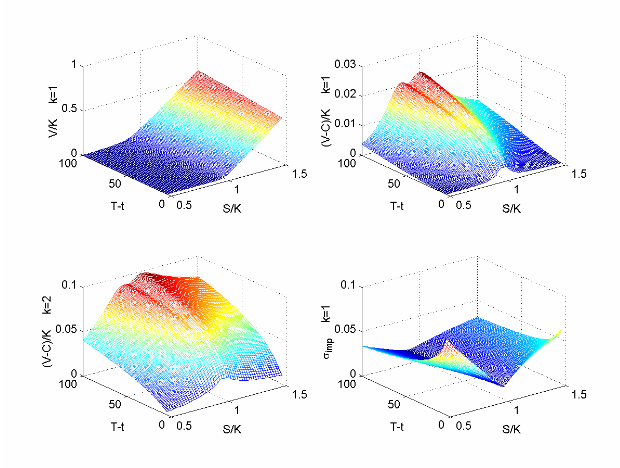


Figure: Option price and equivalent implied volatility in the "risk-neutral" approach to the stochastic volatility model