Emergence as reduction

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What is emergence?

- **Some definitions:**
  ★ *Emergent entities* ‘arise’ out of more fundamental entities and yet are ‘novel’ or ‘irreducible’ with respect to them;
  ★ *Emergence* is the way complex systems and patterns arise out of a multiplicity of relatively simple interactions;
  ★ *Emergence* is the arising of novel and coherent structures, patterns and properties during the process of self-organization in complex systems;
  ★ *Emergent properties* are those which can only be distinguished by observation of the complex system, not from the properties of its parts;
  ★ An *emergent property* is one that is not a property of any component of that system, but is still a feature of the system as a whole (*"more is different"*);
  ★ *Emergence* is the concept of some new phenomenon arising in a system that was not in the system specification to start with

What is emergence?

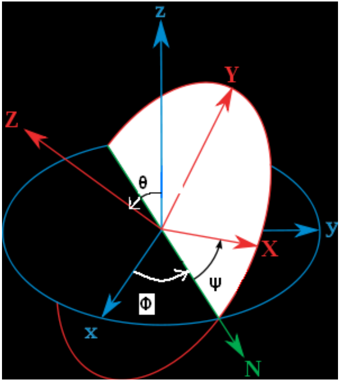
- ★ *Emergence* is not due to the failure of the microdescription as a modeling effort, since the emergent property still appears as the result of a simulation constructed using the microdescription.
- Summarizing:
  - *Novelty* as compared with the microdynamics laws
  - A property of a *collective* created by their *interactions*
  - Properties that *cannot be deduced* from the microdynamics and only become apparent when the *system as a whole* evolves.
- Emergent behaviour is hard to predict because the number of interactions between components of a system increases combinatorially with the number of components
- In practice, when a good set of emergent properties is identified, they become the intelligible set of variables that are used to deal with the system. We no longer care about the agent dynamics. In this sense, **emergence is a kind of reduction, reduction of the immense set of individual variables to a much smaller and manageable set.**

Emergence in physics: Examples and a mathematical theory

- Physics has been dealing with emergence since its very beginnings. There, however, emergence appears under other names.
- First example
  Thermodynamics: a macroscopic description of systems of many atoms (molecules) expressed in terms of *collective* variables such as *temperature, pressure, entropy, etc.*
- No individual atom "knows" its temperature or pressure, much less its entropy. These are notions without any sense at the individual level. Nevertheless the knowledge of these notions and their laws, rather than the variables of the individual atoms, is what is behind the construction of our cars and airplanes.
- Why is it so? What are the reasons and requirements for this success of the *emergent* variables? Is there a general rule?

Emergence in physics: The free rigid body

- Second example: **The free rigid body**
  First step (*reduction*, *projection*): Neglect all interparticle motions (Relative momenta = 0). Reduces from $O(10^{24})$ to 6 variables
- Euler angles θ, ϕ, and ψ, and their canonically conjugate momenta, p_θ, p_ϕ, and p_ψ



Emergence in physics: The free rigid body

- Hamiltonian

$$
\begin{aligned}
H &= \frac{p_\psi^2}{2I_3} + \frac{1}{2I_2}\left[\left(p_\phi \csc\theta - p_\psi \cot\theta\right)\cos\psi - p_\theta \sin\psi\right]^2 \\
&\quad + \frac{1}{2I_1}\left[\left(p_\phi \csc\theta - p_\psi \cot\theta\right)\sin\psi + p_\theta \cos\psi\right]^2,
\end{aligned}
$$

  I_1, I_2, and I_3 are the three diagonal elements of the inertia tensor.

- The three components of the angular momentum in the body frame

$$
\begin{aligned}
m_1 &= \left(p_\phi \csc\theta - p_\psi \cot\theta\right)\sin\psi + p_\theta \cos\psi, \\
m_2 &= \left(p_\phi \csc\theta - p_\psi \cot\theta\right)\cos\psi - p_\theta \sin\psi, \\
m_3 &= p_\psi.
\end{aligned}
$$

Emergence in physics: The free rigid body

- Verify the following relations

$$\begin{aligned} \{m_1, m_2\} &= -m_3 \\ \{m_2, m_3\} &= -m_1 \\ \{m_3, m_1\} &= -m_2. \end{aligned}$$

  The three components of the angular momentum in the body frame constitute a closed Lie subalgebra under the operation of the canonical Poisson bracket.

- Adopt the m's as generalized coordinates on a *reduced* phase space of three dimensions. The Poisson tensor on this reduced phase space is then given by $J^{\alpha\beta} = -\epsilon^{\alpha\beta\gamma} m_\gamma$, or:

$$\mathbf{J} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix},$$

Emergence in physics: The free rigid body

- It is possible to perform reduction only if the Hamiltonian is expressible in terms of the reduced coordinate set. Indeed, we have

$$H(\mathbf{m}) = \frac{m_1^2}{2I_1} + \frac{m_2^2}{2I_2} + \frac{m_3^2}{2I_3}$$

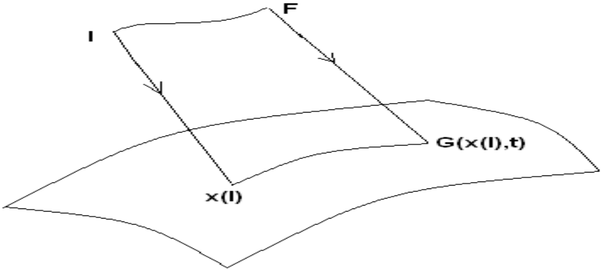
- The equations of motion are given by $\dot{\mathbf{m}} = \{\mathbf{m}, H\}$, or:

$$\begin{aligned}
\dot{m}_1 &= \left(\frac{1}{I_3} - \frac{1}{I_2}\right) m_2 m_3 \\
\dot{m}_2 &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) m_3 m_1 \\
\dot{m}_3 &= \left(\frac{1}{I_2} - \frac{1}{I_1}\right) m_1 m_2 .
\end{aligned}$$

- If the rigid body were not free (say, if it were in a gravitational field), then a potential energy term would have been present in the Hamiltonian, and that term would *not* have been expressible in terms of the m's. Thus, the reduction process would have failed.

Emergence in physics: The free rigid body

- The failure of the reduction process comes about because the gravitational field breaks the SO(3) symmetry that makes the reduction possible.
- This example already shows what is behind the "emergence" of the set of reduced variables



Emergence in physics

- First there is a projection from an immense space of variables to a small subspace (submanifold)
- Then one should check whether the reduction is consistent, that is, whether there is a dynamical evolution defined purely inside the submanifold. Chosen a point in the submanifold it should evolve there and its evolution cannot make any reference to the larger set of variables.
- In conclusion: Projection into a submanifold and selfconsistent evolution there. (**Dynamical closure of the emergent variables**)
- And behind it all are invariances and invariance groups: Here the invariance of the relative momenta and rotational invariance

Emergence in physics: The Euler equation

- Third example: **The Euler equation** for the flow of an inviscid, incompressible fluid.
- In the Lagrangian for such a fluid each fluid particle is labelled by a reference position, $\mathbf{x}_0$. The dynamical field variable is $\mathbf{x}(\mathbf{x}_0, t)$.

$$L = \sum_i \frac{m_i}{2} \dot{x}_i^2(\mathbf{x}_{i,0}, t) = \int d^3 x_0 \frac{\rho}{2} \dot{x}^2(\mathbf{x}_{i,0}, t),$$

  ρ is the constant uniform mass density.
- The canonical momentum field is then given by

$$\mathbf{p}(\mathbf{x}_0, t) = \frac{\delta L}{\delta \dot{\mathbf{x}}(\mathbf{x}_0, t)} = \rho \dot{\mathbf{x}}(\mathbf{x}_0, t),$$

  where the δ's denote functional differentiation.
- By the Legendre transformation, the system Hamiltonian is

$$H = \int d^3 x_0 \frac{1}{2\rho} \mathbf{p}^2(\mathbf{x}_0, t).$$

Emergence in physics: The Euler equation

- The canonical bracket of two functionals of $\mathbf{x}$ and $\mathbf{p}$,

$$\{A, B\} = \int d^3 x_0 \left(\frac{\delta A}{\delta \mathbf{x}(\mathbf{x}_0, t)} \cdot \frac{\delta B}{\delta \mathbf{p}(\mathbf{x}_0, t)} - \frac{\delta A}{\delta \mathbf{p}(\mathbf{x}_0, t)} \cdot \frac{\delta B}{\delta \mathbf{x}(\mathbf{x}_0, t)} \right).$$

- Now suppose that the fluid particles are identical. In that case, specification of $\mathbf{x}(\mathbf{x}_0, t)$ is more information than is really necessary Two configurations that differ only by swapping identical particles will have different $\mathbf{x}(\mathbf{x}_0, t)$. For a fluid of identical particles, an Eulerian description, wherein the flow velocity is given as a function of spatial position and time, say $\mathbf{v}(\boldsymbol{\xi}, t)$, suffices.
- **The reduction group is the group of identical particle interchanges**. The reduced phase space is the (smaller, though still infinite dimensional) space of all divergenceless vector fields ($\mathrm{div}\mathbf{v} = 0$ stems from the fact that we are considering incompressible flows)

$$\mathbf{v}(\boldsymbol{\xi}, t) = \frac{1}{\rho} \int d^3 x_0 \mathbf{p}(\mathbf{x}_0, t) \delta(\mathbf{x}(\mathbf{x}_0, t) - \boldsymbol{\xi}),$$

Emergence in physics: The Euler equation

- One obtains a Poisson bracket of the Eulerian field with itself using the canonical bracket.

$$\{\mathbf{v}(\boldsymbol{\xi},t),\mathbf{v}(\boldsymbol{\xi}',t)\} = \frac{1}{\rho}\left(\mathbf{v}(\boldsymbol{\xi}',t)\delta'(\boldsymbol{\xi}'-\boldsymbol{\xi}) - \delta'(\boldsymbol{\xi}-\boldsymbol{\xi}')\mathbf{v}(\boldsymbol{\xi},t)\right),$$

  So, functionals of the Eulerian field variables constitute a closed Lie subalgebra of the Lie algebra of all phase space functionals.

- The Hamiltonian is expressed in terms of the reduced variables.

$$H = \frac{\rho}{2}\int d^3\xi v^2(\boldsymbol{\xi},t),$$

- The Hamiltonian together with the bracket yields Euler's fluid equations.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p.$$

  the pressure being

$$\nabla^2 p = -\nabla\cdot(\mathbf{v}\cdot\nabla\mathbf{v}),$$

Emergence in physics

In conclusion:

- New (reduced) variables appear by projection on a smaller manifold in the full phase space of the system
- The new variables are useful if their dynamics can be written in terms of themselves without reference to the microvariables of the system (**Dynamical closure**)
- The emergence of the reduced variables is always associated to an invariance property (invariance group)
- What to do if the dynamics of the reduced variables is not closed? Find new variables until a dynamically closed set is obtained.
- Is there a general theory of the association of reduced variables and symmetry groups?
- **YES**

A mathematical theory: Symmetry reduction and momentum maps (Marsden-Weinstein reduction)

- G = Lie group, $\mathcal{G}$ = Lie algebra, $\mathcal{G}^*$ = Lie algebra dual
  $\langle \alpha, \eta \rangle : \mathcal{G}^* \otimes \mathcal{G} \to \mathbb{R}$ $\qquad \alpha \in \mathcal{G}^*, \eta \in \mathcal{G}$
- Hamiltonian system in the manifold (M, ω), with Hamiltonian function H and symplectic form ω. An Hamiltonian vector field X_H is such that

$$\omega(X_H, \bullet) = dH$$

- Let the symmetry group G with action $\phi_g : G \times M \to M$ generate symplectic (canonical) transformations

$$\phi_g^* \omega = \omega$$

- and that, associated to these symmetries, there is a momentum map. It means that the vector field X_ξ associated to each ξ in the Lie algebra $\mathcal{G}$ is Hamiltonian with Hamiltonian function J^ξ. Then the **momentum map** $J : M \to \mathcal{G}^*$ is defined by

$$\langle J(x), \xi \rangle = J^\xi(x)$$

Symmetry reduction and momentum maps

Let $\mu \in \mathcal{G}^*$ be a value of J and denote by G_μ the isotropy group of μ under the action of G on $\mathcal{G}^*$

- **Adjoint and coadjoint representations (of the Lie group)**

$$Ad_g = \left(R_{g^{-1}} L_g\right)_{*e} \qquad Ad_g \eta = g\eta g^{-1} \qquad g \in G, \eta \in \mathcal{G}$$

$$\left\langle Ad_g^* \alpha, \eta \right\rangle = \langle \alpha, Ad_g \eta \rangle \qquad g \in G, \eta \in \mathcal{G}, \alpha \in \mathcal{G}^*$$

- Define

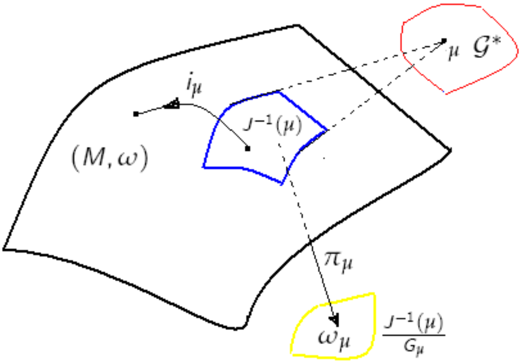
$$\begin{array}{llll} \pi_\mu : & J^{-1}(\mu) & \rightarrow & J^{-1}(\mu) / G_\mu \\ i_\mu : & J^{-1}(\mu) & \rightarrow & M \end{array}$$

- **Theorem :** (Marsden, Weinstein) There is a unique symplectic structure ω_μ in $M_\mu = \frac{J^{-1}(\mu)}{G_\mu}$ such that

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega$$

- $M_\mu = \frac{J^{-1}(\mu)}{G_\mu}$ *is the reduced phase space and the theorem means that, when there is a symmetry, the symplectic structure may be transported to the reduced phase-space.*

Symmetry reduction and momentum maps



Symmetry reduction and momentum maps

- **Mathematical recipe for reduction in case G is not known**
  1) Start from a canonical system, (for example a particle description of the whole system (q_i, p_i))
  2) Identify a set $\{z^\alpha\}$ of desired reduced variables (for example a velocity field $u(q,t)$, distribution function $f(x,v,t)$) or some collective variable that seems relevant or was revealed by simulation.
  3) Compute the Poisson brackets of the reduced variables obtaining the Poisson tensor $J^{\alpha\beta} = \{z^\alpha, z^\beta\}$.
  4) Check that the z^α's form a closed subalgebra. If not, complete the set. (probably a good recommendation for other fields, as well)
  5) Compute the Poisson bracket of arbitrary functions of the reduced variables by

$$\{F, G\} = \frac{\partial F}{\partial z^\alpha} J^{\alpha\beta} \frac{\partial F}{\partial z^\beta}$$

  6) Write the equations of motion on the reduced variables
  7) Or write the original Hamiltonian or Lagrangian in the reduced variables and obtain from them the equations of motion.

The central role of dynamical closure

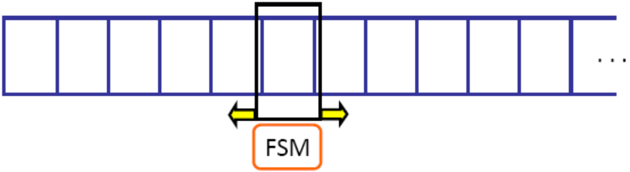
- In some philosophical discussions of emergence, the question of causation, for example downwards causation, plays an important role.
- The message from emergence in physical theories is that causation is not really the most important issue. What is central is dynamical closure, that is, the fact that once the emergent variables are chosen, their dynamics may be rigourously determined without any further reference to the underlying microsystem.
- For example in economics, the set of global variables (unemployment, gross national product, etc.) should be such that their evolution is determined solely by themselves without any reference to the microeconomic variables. If this is not the case, then either the set of global variables is "bad" or it needs to be completed.
- The same applies to the relation between neurological dynamics and the dynamics of mental states.
- Dynamical closure is central. What about **completeness?**

Is reduction all there is about emergence?

- The role and requirements for the collective (emergent) variables seems well understood (at least in physics)
- Is it really? Provisionally I will say yes, later I will say "no" or "maybe not"
- Are indeed all emergent variables (and notions) just projections of the microscopic variables and dynamics?
- Or, instead, are there macroscopic laws that are fundamental statements about nature which do not follow from the lower level?
- Einstein once said: *"Subtle is the Lord, but malicious he is not"*
  I would like to paraphrase this sentence as applied to mathematics
- *"Mathematics is subtle, but malicious it also can be"*

Are there undecidable questions in complex systems?

Turing Machine Formal Description



7-tuple: $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$

Q: finite set of states

Σ: input alphabet (cannot include blank symbol, _)

Γ: tape alphabet, includes Σ and _

δ: transition function: $Q \times \Gamma \rightarrow Q \times \Gamma \times \{\mathbf{L}, \mathbf{R}\}$

q_0: start state, $q_0 \in Q$

q_{accept}: accepting state, $q_{\text{accept}} \in Q$

q_{reject}: rejecting state, $q_{\text{reject}} \in Q$

(Sipser's notation)

Are there undecidable questions in complex systems?

- The Church-Turing thesis:
  Everything that is calculable can be computed by a Turing machine
  A Turing machine can do everything a real computer can do
- Things real computers can do and Turing machines cannot

Generate Heat

Stop a Door

Provide an adequate habitat for fish

Are there undecidable questions in complex systems?

- Mathematics, by providing a consistent framework to derive logical consequences of a set of principles, became a powerful tool to reason about Nature. However, in mathematics, there are internal statements that cannot be proven to be true or false. They are undecidable questions.
- Some undecidable questions (problems)
  A *problem* = set of instances and their (yes-no) answers
  The number of instances is at most countable when expressed by strings of a finite alphabet
- The number of problems is uncountable

	Instance 1	Instance 2	Instance 3	$\cdots$
Problem 1	[yes]	no	yes	$\cdots$
Problem 2	no	[no]	no	$\cdots$
Problem 3	yes	no	[yes]	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

- $\Longrightarrow$ Because the number of different Turing machines is countable, **there are undecidable problems**

Are there undecidable questions in complex systems?

- The halting problem:
  Is there a Turing machine that given any Turing machine and its input can decide whether it halts or not?
- Suppose there is one, T_H. Enumerate all Turing machines and their (stop-nostop) results for all instances. Now, each element of the following matrix is an instance of T_H

	Input 1	Input 2	Input 3	···
Turing machine 1	[stop]	nostop	stop	···
Turing machine 2	nostop	[nostop]	nostop	···
Turing machine 3	stop	nostop	[stop]	···
...	...	...	...	...

  By taking the diagonal elements and flipping them one obtains a new (different) Turing machine not in the domain of T_H
- Therefore no such T_H can exist.

Are there undecidable questions in complex systems?

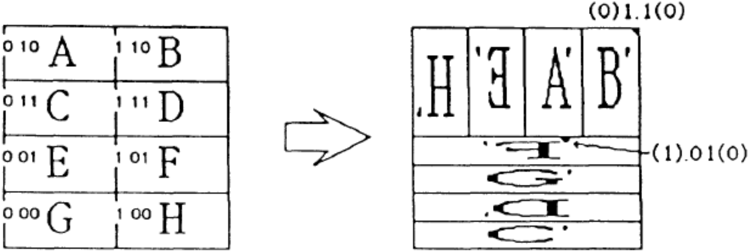
- *Undecidability and incompleteness of formal systems*
  Formal system = Axioms+rules $\Longrightarrow$ proof of statements
  A proof implies the existence of an algorithm to find it
- $\Longrightarrow$ For any formal system powerful enough to represent Turing machines, there must be statements representable in the formal system for which there is no proof (yes or no) in the formal system.
- Example: The number of valid proofs in a formal system is countable. Construct the following procedure
- Procedure INCOM

```
i=1
while (TRUE)
    if the i-th valid proof shows that INCOM does not terminate
    then terminate
    i=i+1
```

- If the procedure terminates is because there is a formal proof showing that it does not terminate. Therefore it cannot terminate. But we know that it does not terminate. However there is no formal proof.

Are there undecidable questions in complex systems?

- And here is the *malicious face of mathematics*. On the one hand it gives the tools to formulate and solve the problems. On the other hand it says that some problems cannot be solved.
- *What does it all have to do with physical complex systems?*
- Several people have shown that some complex systems can represent universal Turing machines.
- Two examples:
  Particles and mirrors (Moore, Phys. Rev. Lett. 64 (1990) 2354)
  Piecewise linear maps of the plane



Are there undecidable questions in complex systems?

- Generalized shift map

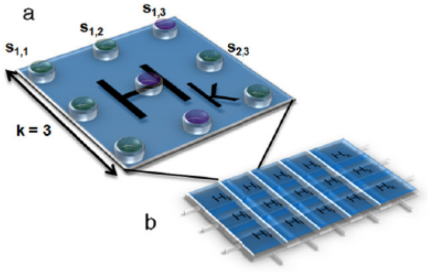
$$\Phi : a \longrightarrow \sigma^{F(a)} \left(a \oplus G(a) \right)$$

$a_{-1}.a_0 a_{+1}$	F	G
0.00	−1	0.11
0.01	+1	1.01
0.10	+1	1.11
0.11	−1	0.00
1.00	+1	0.01
1.01	−1	0.10
1.10	+1	0.11
1.11	−1	0.01

- The conclusion is that this map, which can be implemented with particles and mirrors (billiards), can act like a Turing machine.
- Being an universal Turing machine there should be undecidable questions about it.
- The construction is nice. The conclusion a bit hasty.

Are there undecidable questions in complex systems?

- Gu et al. (Physica D 238 (2009) 835-839) came closer to a more complete statement with their construction of an arbitrary number of different Turing machines in tesselated infinite Ising networks.



$$H = \sum c_{x,y} S_x S_y + \sum M_x S_x$$

- If there are undecidable questions about complex systems, then perhaps there are also emergent variables (statements) that cannot be obtained from the microdynamics.

The role of infinity

- Independently of whether there are undecidability questions about complex physical systems, there is one thing that *emerges* from the previous discussion about undecidability
- Irreducible emergent properties are associated to infinite number of degrees of freedom or $N \to \infty$ limit (thermodynamics, phase transitions, velocity fields, etc.)
- Even if (at least) some emergent variables can be obtained by reduction, in fact they are only precisely defined in the infinite limit: Thermodynamic variables, phase transitions, velocity fields, etc. In this sense they are proxys of something that only exists in the realm of infinity.
- Therefore what the mathematical mechanism does is to approximate something that "exists" outside of the finitary framework that we start with. In this sense the emergent variables might be considered as novel entities, not as simple reductions but as *shadows of entities of the infinite universe.*

The role of infinity

- And in the infinite limit, emergent variables are **complete**, in the sense that they are all that can be said about the system
- But what kind of infinity are we speaking about? A simple one in fact. In mathematics the notion of infinity appears in several forms. Here we are concerned with cardinal infinity. And with the first in an infinite chain of infinitary transitions

  $$0, 1, 2, 3, \cdots, n, \cdots; \aleph_0, \aleph_1, \aleph_2, \cdots, \aleph_\alpha, \cdots.$$

  Continuous hypothesis

  $$\mathbf{C} = 2^{\aleph_0} = \aleph_1$$

- The undecidability-incompleteness problems that we have been speaking about lie at the boundary of two cardinalities *aleph*0 and *C*.
- What about other mathematical phenomena (and emergent properties) at the other boundaries in the chain of cardinal numbers?