Euler 2D and coupled systems: Coherent structures, solutions and stable measures

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PSPDE - V, November 28 - 30, 2016

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Persistent structures in (quasi) 2-dimensional fluid motions

Jupiter red spot

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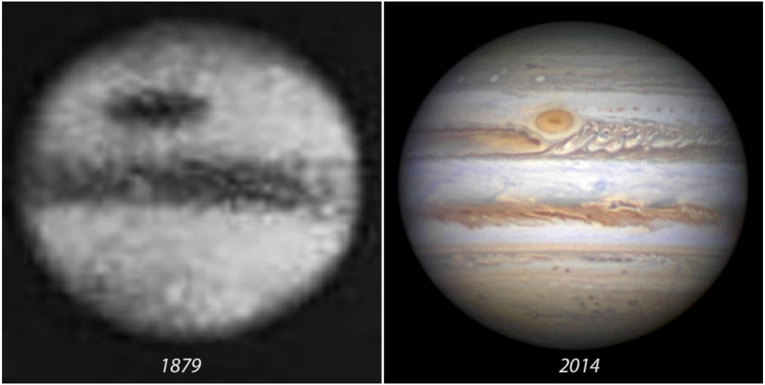
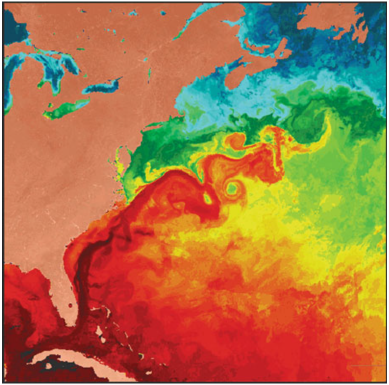


Persistent structures in (quasi) 2-dimensional fluid motions

Large-scale vortices in the oceans



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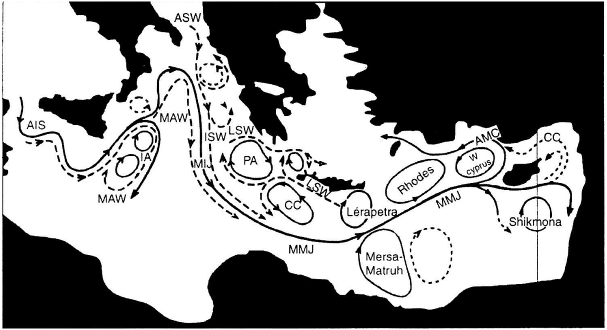

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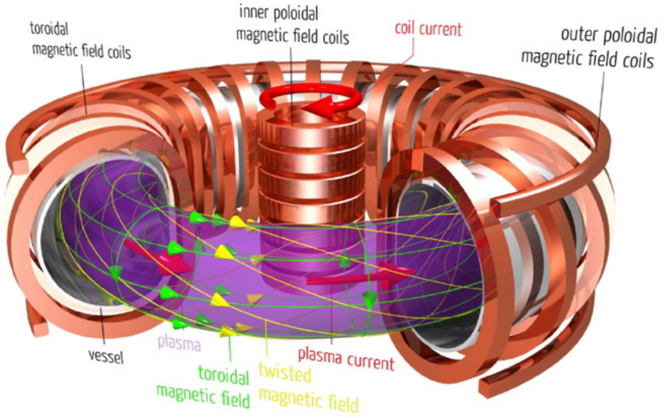
doi 10.1098/rspb.2000.1212

Mesoscale vortices and the paradox of the plankton

A. Bracco[1]*, A. Provenzale[1] and I. Scheuring[2]

Persistent structures in (quasi) 2-dimensional fluid motions

The SOL (scrape-off layer): Jets and blobs



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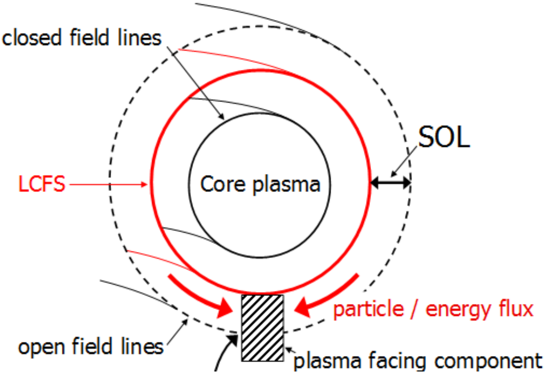
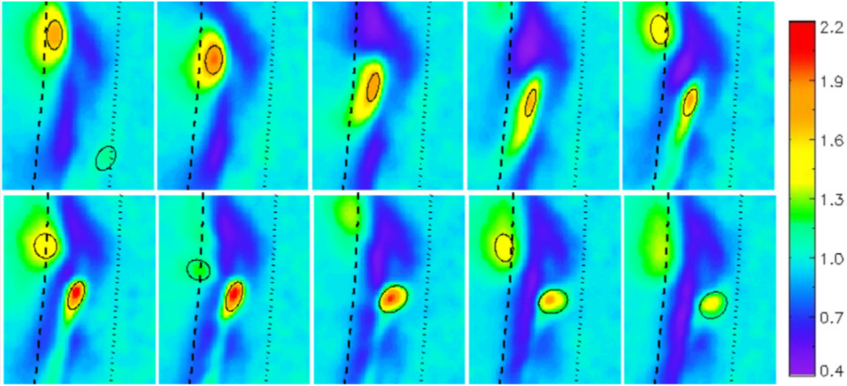


Persistent structures in (quasi) 2-dimensional fluid motions

The SOL (scrappe-off layer): Jets and blobs



The equations

- **The 2D Euler equation**

$$\begin{cases} \frac{\partial v}{\partial t} = -(v \cdot \nabla)v - \nabla p \\ div \;\; v = 0 \end{cases}$$

(*Periodic boundary conditions*)
Since $div \;\; v = 0$ there is a function $\psi(x, t)$ such that

$$v = \nabla^{\perp}\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$$

and the Euler equation becomes

$$\boxed{\partial_t \Delta\psi = -\nabla^{\perp}\psi \cdot \nabla\Delta\psi}$$

- Solutions on the 2-dimensional torus $T^2 = [0, 2\pi] \times [0, 2\pi]$ subjected to periodic boundary conditions

$$\psi(0, x_2, t) = \psi(2\pi, x_2, t), \qquad \psi(x_1, 0, t) = \psi(x_1, 2\pi, t)$$

Let $e_k(x) = \frac{1}{2\pi} e^{i\,k \cdot x}, \;\; k \in \mathbb{Z}^2$ be the eigenfunctions for $-\Delta$ with eigenvalues $k^2 = k_1^2 + k_2^2$. Are a complete set in $L^2(T^2)$.

The equations

Since ψ is a real function assume $\int_{T^2} \psi dx = 0$, then $\omega_{-k} = \overline{\omega}_k$

$$\psi(x,t) = \sum_{k \in \mathbb{Z}^2_+} \omega_k(t) e_k(x),$$

$\mathbb{Z}^2_+$ denotes the set $\{k \in \mathbb{Z}^2 : \; k_1 > 0, \; k_2 \in \mathbb{Z} \text{ or } k_1 = 0, \; k_2 > 0\}$.
$\psi = \{\omega_k\}_{k \in \mathbb{Z}^2_+}$ and introducing the operator

$$B(\omega) = \{B_k(\omega)\}_{k \in \mathbb{Z}^2_+} = \sum_k B_k(\omega) \frac{\partial}{\partial \omega_k}$$

$$B_k(\omega) = \frac{1}{2\pi k^2} \sum_{\substack{h \neq k \\ h,k \in \mathbb{Z}^2_+}} \left(k^\perp \cdot h \right) h^2 \omega_h \omega_{k-h}$$

where $k^\perp = (-k_2, k_1)$, the 2D Euler equation becomes the following infinite-dimensional ordinary differential equations system

$$\boxed{\frac{d}{dt}\omega_k = B_k(\omega) \qquad k \in \mathbb{Z}^2_+} \qquad\qquad \frac{\partial B_k}{\partial \omega_k} = 0$$

The equations

The SOL equations (Two-field model)

$$\frac{\partial L}{\partial t} = \boxed{-\nabla\phi\cdot\nabla^{\perp}L - g\partial_2\left(L-\phi\right)} + D\left(\nabla^2 L + |\nabla L|^2\right) - \sigma_{\parallel} e^{(\Lambda-\phi}$$

$$\frac{\partial\triangle\phi}{\partial t} = \boxed{-\nabla\phi\cdot\nabla^{\perp}\triangle\phi - g\partial_2 L} + \nu\triangle^2\phi + \sigma_{\parallel}\left[1 - e^{(\Lambda-\phi)}\right],$$

Conservative component

$$\frac{\partial L}{\partial t} = -\nabla\phi\cdot\nabla^{\perp}L - g\partial_2\left(L-\phi\right)$$

$$\frac{\partial\triangle\phi}{\partial t} = -\nabla\phi\cdot\nabla^{\perp}\triangle\phi - g\partial_2 L$$

Two equations, two problems

- **2D Euler equation:** *Persistent large-scale structures*
  2-field SOL equation: *Coherent structures travelling from the core to the wall*
- *Solutions ?*
  2D Euler equation: *Stochastic stability of invariant measures*
  2-field SOL equation: *Travelling wave solutions*

Stochastic stability of invariant measures

- *Physical measure* (or SBR measure) and *stochastically stable measure* are closely related notions.
  M a state space, $f : M \longrightarrow M$ a smooth dynamical system and μ a positive Borel measure on M such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f^j(x)\right) \to \int_M \varphi d\mu$$

  for a positive measure set A of initial points x and any continuous function $\varphi : M \longrightarrow \mathbb{R}$. (*Means that time averages are given by $\mu-$spatial averages, at least for a large set of initial states x*).
- For uniformly hyperbolic systems there is a complete theory concerning existence and uniqueness of physical measures and partial results for non-uniformly hyperbolic and partially hyperbolic systems.

Stochastic stability of invariant measures

- Consider the stochastic process f_ε obtained by adding a small random noise to the deterministic system f. Under general conditions, there exists a stationary probability measure μ_ε such that, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi\left(f^j(x)\right) \to \int \varphi d\mu_\varepsilon$$

  Stochastic stability of the measure means that μ_ε converges to the physical measure μ when the noise level ε goes to zero. (uniformly hyperbolic maps, Lorenz and Hénon strange attractors and also results for partially hyperbolic systems).
- Existence and uniqueness of the invariant measure μ_ε under general conditions provides a powerful tool to obtain the physical measure of f, by randomly perturbing it and letting the noise level $\varepsilon \longrightarrow 0$.
- That this might also be useful for infinite-dimensional systems is the source of inspiration for this work. It might contribute to the understanding of the large-scale structures in geophysical phenomena.

Infinitesimally invariant measures of PDE's

- ϕ_t the flow of a partial differential equation and ϕ_t^* the push-forward semigroup acting on measures. A measure μ is invariant if
$$\phi_t^*(\mu) = \mu$$
and infinitesimally invariant if
$$\int B\varphi d\mu = 0$$
B being the generator of the flow ϕ_t. Equivalently $B^*1 = 0$.
- Let the generator B be a first or second order differential operator on a discrete set of coordinates $\omega = \{\omega_i\}$,
$$B = \sum_{i,j} u_{ij}(\omega) \frac{\partial^2}{\partial\omega_i \partial\omega_j} + \sum_i b_i(\omega) \frac{\partial}{\partial\omega_i} \qquad (1)$$
and consider a measure of the form
$$d\mu = R(\omega) \prod_i d\omega_i$$

Infinitesimally invariant measures of PDE's

- To obtain the invariance condition $\int \left(B\varphi\right) R\left(\omega\right) \prod_i d\omega_i = 0$, compute the adjoint of B obtaining

$$
\begin{aligned}
B^* &= -\frac{1}{R}\left\{\sum_i \frac{\partial}{\partial\omega_i}\left(Rb_i\right) - \sum_{i,j}\frac{\partial^2}{\partial\omega_i\partial\omega_j}\left(Ru_{ij}\right)\right\} \\
&\quad + \sum_i \left\{-b_i + \frac{1}{R}\sum_j \frac{\partial}{\partial\omega_j}\left[R\left(u_{ij}+u_{ji}\right)\right]\right\}\frac{\partial}{\partial\omega_i} + \sum_{i,j} u_{ij}\frac{\partial^2}{\partial\omega_i\partial\omega_j}
\end{aligned}
$$

Therefore, to have $B^*1 = 0$, the first term should vanish leading to

Theorem

A generator B of the form in Eq.(1), u_{ij} and b_i being differentiable functions, has $d\mu = R\left(\omega\right)\prod_i d\omega_i$ as an infinitesimally invariant measure iff

$$
\sum_i \frac{\partial}{\partial\omega_i}\left(Rb_i\right) - \sum_{i,j}\frac{\partial^2}{\partial\omega_i\partial\omega_j}\left(Ru_{ij}\right) = 0
$$

Invariant measures of the 2D Euler equation

- With $u_{ij}(\omega) = 0$, the invariance condition is simply

$$\sum_i \frac{\partial}{\partial \omega_i} (RB_i) = 0$$

or

$$\sum_i B_i \frac{\partial}{\partial \omega_i} R = \sum_i \frac{d}{dt} \omega_i \frac{\partial}{\partial \omega_i} R = \frac{d}{dt} R = 0$$

- **In conclusion:** any constant of motion of the Euler equation generates an (infinitesimally) invariant measure. Among them the *energy* $E = \frac{1}{2} \sum_k k^2 \omega_k^2$ and the *enstrophy* $S = \frac{1}{2} \sum_k k^4 \omega_k^2$
- However, the Poisson structure of the Euler 2D equation being degenerate, there is an infinite set of Casimir invariants,

$$C_f = \int f(\triangle \psi)\, d^2 x$$

f being an arbitrary differentiable function. Therefore **there are infinitely many invariant measures for the 2D Euler equation**. The enstrophy is the Casimir invariant for $f(x) = x^2$.

Stochastic stability of 2D Euler invariant measures

- Consider the following infinite dimensional Ornstein-Uhlenbeck operator εQ

$$\varepsilon Qf(\omega) = \varepsilon \sum_k \left\{ a_k(\omega) \frac{\partial}{\partial \omega_k} f(\omega) + \sigma_k(\omega) \frac{\partial^2}{\partial \omega_k^2} f(\omega) \right\}$$

$$Lf(\omega) = \varepsilon Qf(\omega) + \sum_k B_k(\omega) \frac{\partial}{\partial \omega_k} f(\omega)$$

  is the infinitesimal generator for a stochastically perturbed Euler flow.
- With $W(t) = \sum_k \frac{1}{|k|} b_k(t) e_k$ a normalized brownian motion, $b_k(t)$ being independent copies of a complex brownian motion, the following perturbed Euler equation is obtained

$$\begin{aligned} X_k(t) = X_k(0) &+ \int_0^t \{B_k(X(s)) + \varepsilon a(X_k(s))\} ds \\ &+ \int_0^t \sqrt{2\varepsilon\sigma_k(X_k(s))} db_k(s), \quad \forall k \in \mathbb{Z}_+^2 \end{aligned}$$

Stochastic stability of 2D Euler invariant measures

Theorem

If $d\mu = R(\omega)\prod_i d\omega_i$ is an invariant measure for the unperturbed Euler equation, then this is also an invariant measure for the perturbed equation if $a_k(\omega)$ and $\sigma_k(\omega)$ in satisfy

$$\sum_k \left\{ \left(a_k - 2\frac{\partial \sigma_k}{\partial \omega_k} \right) \frac{\partial R}{\partial \omega_k} + R \left(\frac{\partial a_k}{\partial \omega_k} - \frac{\partial^2 \sigma_k}{\partial \omega_k^2} \right) - \sigma_k \frac{\partial^2 R}{\partial \omega_k^2} \right\} = 0$$

As an example, the Gaussian measure constructed from the enstrophy $d\mu_S = e^{-\frac{1}{2}\sum_k k^4 \omega_k^2} \prod_j d\omega_j$ remains invariant if $a_k = -k^2 \omega_k$ and $\sigma_k = \frac{1}{k^2}$
However, we are not only adding noise but also modifying the deterministic part, actually adding noise to a Navier-Stokes equation

$$\begin{aligned} \partial_t \Delta\psi &= -\nabla^{\perp}\psi \cdot \nabla\Delta\psi + \varepsilon\triangle^2\psi \\ \frac{\partial v}{\partial t} &= -(v\cdot\nabla)v + \varepsilon\triangle v - \nabla p \end{aligned}$$

These are NOT the *stochastically stable measures of 2D Euler*.

Stochastic stability of 2D Euler invariant measures

Instead, consider noise perturbations without changing the deterministic part ($a_k(\omega) = 0$). Use Galerkin approximations of arbitrary order N

$$B_k^N(\omega) = \frac{1}{2\pi k^2} \sum_{\substack{0<|h|\leq N \\ 0<|k-h|\leq N}} \left(k^\perp \cdot h\right) h^2 \omega_h \omega_{k-h}$$

The equation for the density $R(\omega)$ of the invariant measure becomes

$$\sum_k B_k^N(\omega) \frac{\partial}{\partial \omega_k} R - \varepsilon \sigma_k \frac{\partial^2}{\partial \omega_k^2} R = 0$$

Two cases are of physical interest, $\sigma_k = 1$ and $\sigma_k = \frac{1}{k^2}$. However, by the change of variables $z_k = |k| \omega_k$ and $B_k^{N'}(\omega) = |k| B_k^N(\omega)$ the second case becomes identical to the first one and we have to deal with

$$\sum_k B_k^{N'}(z) \frac{\partial}{\partial z_k} R - \varepsilon \frac{\partial^2}{\partial z_k^2} R = 0$$

Stochastic stability of 2D Euler invariant measures

This is an elliptic regularization of a first order Hamilton-Jacobi equation $(\varepsilon = 0)$ which has at least as many solutions as the number of constants of motion of the $N-$Galerkin approximation of the Euler equation. Hence, existence and uniqueness of a stochastically-stable solution for R is equivalent to the establishment of a viscosity solution for this Hamilton-Jacobi problem, in its vanishing viscosity modality.

Associated to the uniformly elliptic equation there is a diffusion process X_t with diffusion coefficient $\sqrt{\varepsilon}$ and drift $B_k^{N'}(z)$. In each bounded domain D of $z-$space $(\sim \mathbb{R}^N)$ the drift, being a quadratic polynomial, is uniformly Lipschitz continuous. Therefore the Dirichlet problem has a unique solution with stochastic representation

$$R(z)|_D = \mathbb{E}_z \{f(X(\tau))\}$$

f being the boundary condition at ∂D and τ the first exit time from D

For the unbounded $z-$space we may consider a sequence of nesting open domains $\{D_n\}$, with boundary functions $f_n|_{\partial D_n}$. Then in each $D_n \backslash D_{n-1}$ domain one has a unique solution.

Stochastic stability of 2D Euler invariant measures

For a bounded smooth boundary condition the solution R_ε is bounded and continuous on compact subsets of D. Then, when $\varepsilon \longrightarrow 0$ R_ε converges locally uniformly to a function R. This function is not necessarily a classical solution of $\sum_k B_k^{N'}(z) \frac{\partial}{\partial z_k} R = 0$, but a standard construction shows that it is a viscosity solution, in the sense that, given a $\mathbb{C}^\infty$ function g, if $R - g$ has a local maximum at a point z_0 then $\sum_k B_k^N(z_0) \frac{\partial}{\partial z_k} g(z_0) \leq 0$ and if it is a local minimum $\sum_k B_k^N(z_0) \frac{\partial}{\partial z_k} g(z_0) \geq 0$. Hence

Theorem

*For each choice of boundary conditions in z− space and noise level (ε), one has a unique invariant measure density $R_\varepsilon(z)$, solution of (**??**). Furthermore, in the $\varepsilon \longrightarrow 0$ limit, R_ε converges to a viscosity solution of $\sum_k B_k^{N'}(z) \frac{\partial}{\partial z_k} R = 0$.*

Stochastic stability of 2D Euler invariant measures

So far we have dealt with N-dimensional Galerkin approximations of the 2D Euler equation. When $N \to \infty$ several modifications are needed. It makes no sense to define $R(\omega)$ as a density of the non-existent flat measure in infinite dimensions. Instead, $R(\omega)$ should be defined as the Radon-Nykodim derivative for some other measure, for example the Gaussian enstrophy measure. Then the equation for the density $R(\omega)$ would be

$$\sum_k \left\{ B_k(\omega) \frac{\partial}{\partial \omega_k} - k^4 \omega_k B_k(\omega) \right\} R(\omega) = 0$$

an Hamilton-Jacobi equation in infinite dimensions. Such equations have been extensively studied and given the appropriate boundary condition, for example $R(\omega) \to 1$ when $|\omega| \to \infty$, the construction of the density as a limiting viscosity solution of

$$\sum_k \left\{ B_k(\omega) \frac{\partial}{\partial \omega_k} - k^4 \omega_k B_k(\omega) - \varepsilon \frac{\partial^2}{\partial \omega_k^2} \right\} R(\omega) = 0$$

would follow similar steps as in the finite dimensional case.

Travelling wave solutions of the SOL equations

- With the transformation

$$\begin{aligned} L(x_1, x_2, t) &= \widetilde{L}(x_1, x_2, t) - gx_1 \\ \phi(x_1, x_2, t) &= \widetilde{\phi}(x_1, x_2, t) - gx_1 \end{aligned}$$

becomes

$$\begin{aligned} \frac{\partial \widetilde{L}}{\partial t} &= -\nabla\widetilde{\phi} \cdot \nabla^{\perp}\widetilde{L} \\ \frac{\partial \triangle\widetilde{\phi}}{\partial t} &= -\nabla\widetilde{\phi} \cdot \nabla^{\perp}\triangle\widetilde{\phi} - g\partial_2\widetilde{L} + g\partial_2\triangle\widetilde{\phi}. \end{aligned}$$

- Look for travelling-wave solutions to this system

$$\begin{aligned} \widetilde{L}(x_1, x_2, t) &= \widetilde{L}(x_1 - v_1 t, x_2 - v_2 t) \\ \widetilde{\phi}(x_1, x_2, t) &= \widetilde{\phi}(x_1 - v_1 t, x_2 - v_2 t) \end{aligned}$$

Travelling wave solutions of the SOL equations

Several classes of travelling wave solutions were constructed:

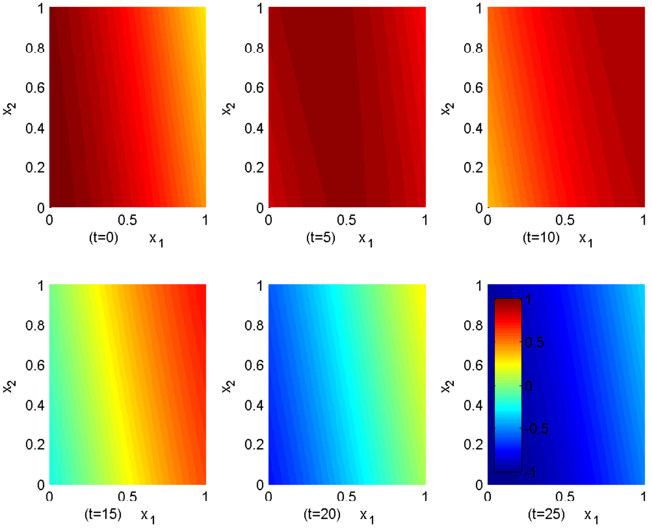
- $$\begin{aligned} L(x_1,x_2,t) &= \alpha\left[\phi(x_1,x_2,t) - v_2(x_1 - v_1 t) + v_1(x_2 - v_2 t)\right] \\ &\quad + (\alpha-1)\,\tilde{g}x_1 \\ \phi(x_1,x_2,t) &= \left(v_2 + \tilde{g} + \frac{\alpha g}{k^2}\right)(x_1 - v_1 t) - v_1(x_2 - v_2 t) - \tilde{g}x_1 \\ &\quad + A\cos\left[k_1(x_1 - v_1 t) + k_2(x_2 - v_2 t)\right] \\ &\quad + B\sin\left[k_1(x_1 - v_1 t) + k_2(x_2 - v_2 t)\right] \\ &\quad + C_0 + C_1\cos\left[k(x_1 - v_1 t)\right] \\ &\quad + C_2\sin\left[k(x_1 - v_1 t)\right]. \end{aligned}$$

Travelling wave solutions of the SOL equations

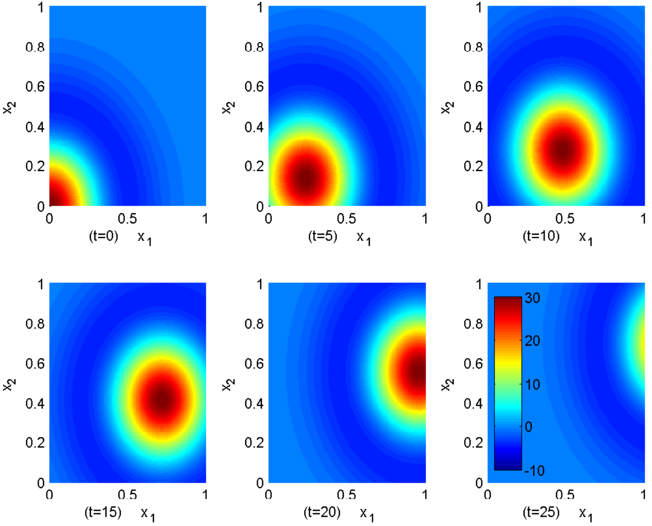
- $$\begin{aligned} L(x_1,x_2,t) &= \alpha\left[\phi(x_1,x_2,t) - v_2(x_1 - v_1 t) + v_1(x_2 - v_2 t)\right] \\ &\quad + (\alpha - 1)\tilde{g}x_1 \\ \phi(x_1,x_2,t) &= \left(v_2 + \tilde{g} - \frac{\alpha g}{k^2}\right)(x_1 - v_1 t) - v_1(x_2 - v_2 t) - \tilde{g}x_1 \\ &\quad + Ae^{[k_1(x_1 - v_1 t) + k_2(x_2 - v_2 t)]} + C_0 \\ &\quad + C_1 e^{k(x_1 - v_1 t)} + C_2 e^{-k(x_1 - v_1 t)} \end{aligned}$$

- $$\begin{aligned} L(x_1,x_2,t) &= -\gamma A e^{-\gamma[(x_1 - v_1 t)^2 + (x_2 - v_2 t)^2]/2} \\ &\quad \times \left\{2 - \gamma\left[(x_1 - v_1 t)^2 + (x_2 - v_2 t)^2\right]\right\} - gx_1 \\ \phi(x_1,x_2,t) &= A e^{-\gamma[(x_1 - v_1 t)^2 + (x_2 - v_2 t)^2]/2} + v_2(x_1 - v_1 t) \\ &\quad - v_1(x_2 - v_2 t) - gx_1 \end{aligned}$$

Travelling wave solutions of the SOL equations



Travelling wave solutions of the SOL equations



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