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Space–times over normed division algebras, revisited

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Normed division and Clifford algebras have been extensively used in the past as a mathematical framework to accommodate the structures of the Standard Model and grand unified theories. Less discussed has been the question of why such algebraic structures appear in Nature. One possibility could be an intrinsic complex, quaternionic or octonionic nature of the space–time manifold. Then, an obvious question is why space–time appears nevertheless to be simply parametrized by the real numbers. How the real slices of an higher-dimensional space–time manifold might be almost independent from each other is discussed here. This comes about as a result of the different nature of the representations of the real kinematical groups and those of the extended spaces. Some of the internal symmetry transformations might however appear as representations on homogeneous spaces of the extended group transformations that cannot be implemented on the elementary states.

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1. Introduction

The search for order has lead many authors to frame the standard model of elementary particles as a representation of a transformation group on Hilbert spaces based on higher division algebras, in particular octonionic Hilbert spaces, although representations of exterior or Clifford algebras might work as well.[1] These ideas trace back to the works of Günaydin and Gürsey[2–5] having been, since then, further explored by many authors[6–12] and references therein. Higher division algebras also appear in supersymmetry and superstring theories[13–16] and provide some clues on how to correct the shortcomings of grand unified theories.[11,17]

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A question, that also intrigued several authors in the past, is what could be at the origin of the apparent relevance of these algebras to the structure of the matter states. Could it be that space–time itself has a complex, quaternionic of even octonionic structure? Several authors[18–25] have studied complex Minkowski spaces obtaining for example new solutions of the field equations and hints on quantized structures in space–time. Particularly interesting is the interpretation of Minkowski space as a family of lines in $\mathbb{C}P^3$.[26–29] Nevertheless a question that remains open is why the space–time of ordinary life looks like a real space instead of one with higher division algebra coordinates. Rather than the use of normed division algebras as a framework to accommodate particle states and internal quantum numbers, this is the main question that will be addressed in this paper.

From a mathematical point of view, the idea of coordinates taken from a higher division algebra might make sense. For the mathematical reconstruction of space–time, it would be natural to start from the empirical evidence for a four-dimensional structure with a pseudo-Euclidean metric. Given this *a priori* fact, what else is needed to construct a quantitative framework to describe the natural phenomena? At a minimum, we must equip each one of the four dimensions with a numerical labeling. To allow for the usual algebraic operations and the measurement of distances, the labeling must, at least, be taken from a normed division algebra. Hurwitz theorem now implies that the only labeling possibilities are the reals, the complex, the quaternions and the octonions. Could it be that space–time is indeed a manifold better labeled by an higher division algebra than by the real quantities of our everyday experience? If so, why do we not feel it?

When, for the labeling of the dimensional coordinates, an algebra other than the reals is used, consistency with the usual physical observations should of course be maintained. Fixing in the extended space four independent directions and using them as a basis for a real vector space one would obtain *a real slice of the extended space.* The consistency condition is that the symmetry group of the extended space must reduce to the real Lorentz and Poincaré groups in each real slice. Taking $\{e_\nu\}$ as an orthonormal set in one of the real slices, the set of all real slices is spanned by

$$\{e'_\mu\} = \{\varphi(\mu)\Lambda^\nu_\mu e_\nu\}\,,$$

where Λ is a real Lorentz matrix and $\varphi(\mu)$ a μ-dependent unit element of the chosen normed algebra $\mathbb{K}$, $(\varphi(\mu) \in \mathbb{K}, |\varphi(\mu)| = 1)$. In each real slice $|x_\mu| g^{\mu\nu} |x_\nu|$ is to be preserved and this implies that the Lorentz invariance group in the extended space is

$$\Lambda^\dagger G \Lambda = G\,, \tag{1}$$

G being the metric $(1, -1, -1, -1)$ and $\dagger$ the adjoint operation. This would be the $U(1, 3, \mathbb{K})$ group over the normed algebra $\mathbb{K}$. Together with the inhomogeneous translations, these are the groups that in the past, for $\mathbb{K} = \mathbb{C}$, have been called the Poincaré groups with real metric.[30–32] Notice that for $\mathbb{K} = \mathbb{C}$ this is a 16-parameter complex Lorentz group distinct from the 12-parameter group

$$\Lambda^T G \Lambda = G \tag{2}$$

used in the analytical continuation of the S-matrix and interpretation of complex angular momentum.[33,34]

In this paper the Poincaré groups with real metric will be studied over the complex numbers, the quaternions and the octonions. Taking seriously the hypothesis that the true labeling of space–time is an algebra larger than the reals, one must of course understand why the space–time of ordinary life looks like a real space, that is, why the real slices look disjoint or almost disjoint from one another. One possible (mathematical) answer is somewhat surprising. It is based on the fact that not all representations of the kinematical group of the real slices are present in the larger group and when they exist there is a superselection rule operating between them. The fact that half-integer elementary spin states cannot be "rotated" away from a real slice has as a corollary a conversion of kinematical transformations into internal symmetries on the fibers over an homogeneous space.

2. Complex Space−Time

2.1. *The complex Poincaré group*

The group of four-dimensional space–time with complex coordinates that, when acting inside each real slice, reduces to the Lorentz group is $U(1,3,\mathbb{C})$ satisfying

$$\Lambda^\dagger G \Lambda = G\,. \tag{3}$$

Adding the complex space–time translations one obtains the semidirect product

$$T_4 \circledS U(1,3,\mathbb{C}) \tag{4}$$

a 24-parameter group. The generators of its Lie algebra are $\{M_{\mu\nu}, N_{\mu\nu}, K_\mu, H_\mu\}$:[a]

$M_{\mu\nu} = -M_{\nu\mu}$ (6 generators) corresponding to the transformations

$$\begin{aligned} M_{ij} &: \begin{cases} x'^i = x^i \cos\theta + x^j \sin\theta\,, \\ x'^j = -x^i \sin\theta + x^j \cos\theta\,, \end{cases} \\ M_{0i} &: \begin{cases} x'^0 = x^0 \cosh u + x^i \sinh u\,, \\ x'^i = x^0 \sinh u + x^i \cosh u\,. \end{cases} \end{aligned} \tag{5}$$

[a] $\mu,\, \nu \in \{0,1,2,3,4\}$
$i,\, j \in \{1,2,3\}$.

$N_{\mu\nu} = N_{\nu\mu}$ (10 generators) corresponding to the transformations[b]

$$
\begin{aligned}
&\underset{i\neq j}{N_{ij}} : \begin{cases} x'^{\,i} = x^i \cos\theta + i x^j \sin\theta\,, \\ x'^{\,j} = i x^i \sin\theta + x^j \cos\theta\,, \end{cases} \\
&N_{0i} : \begin{cases} x'^{\,0} = x^0 \cosh u + i x^i \sinh u\,, \\ x'^{\,i} = -i x^0 \sinh u + x^i \cosh u\,, \end{cases} \\
&N_{00} : \left\{\, x'^{\,0} = e^{-i2\theta} x^0\,, \qquad N_{ii} : \left\{\, x'^{\,i} = e^{i2\theta} x^i\,. \right.\right.
\end{aligned}
\tag{6}
$$

K_μ (4 generators) corresponding to the transformations

$$
K_\mu : \ \left\{ x^{'\mu} = x^\mu + \theta\,. \right.
\tag{7}
$$

H_μ (4 generators) corresponding to the transformations

$$
H_\mu : \ \left\{ x^{'\mu} = x^\mu + i\theta\,. \right.
\tag{8}
$$

The commutation relations are

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -M_{\mu\sigma} g_{\nu\rho} - M_{\nu\rho} g_{\mu\sigma} + M_{\nu\sigma} g_{\rho\mu} + M_{\mu\rho} g_{\nu\sigma}\,, \\
[M_{\mu\nu}, N_{\rho\sigma}] &= -N_{\mu\sigma} g_{\nu\rho} + N_{\nu\rho} g_{\mu\sigma} + N_{\nu\sigma} g_{\rho\mu} - N_{\mu\rho} g_{\nu\sigma}\,, \\
[N_{\mu\nu}, N_{\rho\sigma}] &= M_{\mu\sigma} g_{\nu\rho} + M_{\nu\rho} g_{\mu\sigma} + M_{\nu\sigma} g_{\rho\mu} + M_{\mu\rho} g_{\nu\sigma}\,, \\
[M_{\mu\nu}, K_\rho] &= -g_{\nu\rho} K_\mu + g_{\mu\rho} K_\nu\,, \\
[M_{\mu\nu}, H_\rho] &= -g_{\nu\rho} H_\mu + g_{\mu\rho} H_\nu\,, \\
[N_{\mu\nu}, K_\rho] &= -g_{\nu\rho} H_\mu - g_{\mu\rho} H_\nu\,, \\
[N_{\mu\nu}, H_\rho] &= g_{\nu\rho} K_\mu + g_{\mu\rho} K_\nu\,, \qquad [K_\mu, H_\rho] = 0\,.
\end{aligned}
\tag{9}
$$

The structure of this group and some of its little groups were first studied by Barut.[30] Here one emphasizes its representations and, in particular, the relation between the representations of the full group and those of the real Poincaré group that operates on each real slice. The subgroup generated by $\{M_{\mu\nu}, K_\mu\}$ is one of the subgroups isomorphic to the real Poincaré group and $\sum_\mu N_{\mu\mu} g_{\mu\mu}$ generates an invariant subgroup that commutes with all the generators.

The invariants of the group are:[30]

$$
\begin{aligned}
P^2 &= K^\mu K_\mu + H^\mu H_\mu\,, \\
C_3 &= (K^\mu K_\mu + H^\mu H_\mu) N^\mu_\mu - N_{\mu\nu}(K^\mu K^\nu + H^\mu H^\nu) - 2M_{\mu\nu} K^\mu H^\nu\,,
\end{aligned}
$$

[b]For $\mathbb{K} = \mathbb{Q}$ or $\mathbb{O}$, i should be replaced by imaginary units in that algebra.

$$C_4 = \big(K^\mu K_\mu + H^\mu H_\mu\big)\big\{\big(M_{\mu\alpha}K^\alpha + N_{\mu\alpha}H^\alpha\big)\big(M^{\mu\alpha}K_\alpha + N^{\mu\alpha}H_\alpha\big) + \big(M_{\mu\alpha}H^\alpha - N_{\mu\alpha}K^\alpha\big)\big(M^{\mu\alpha}H_\alpha - N^{\mu\alpha}K_\alpha\big)\big\} - \frac{1}{2}\big\{2M_{\mu\nu}K^\mu H^\nu - N_{\mu\nu}\big(K^\mu K^\nu + H^\mu H^\nu\big)\big\}^2 - \frac{1}{2}\big(K^\mu K_\mu + H^\mu H_\mu\big)\big(M^{\mu\nu}M_{\mu\nu} + N^{\mu\nu}N_{\mu\nu}\big)\,. \tag{10}$$

This complex Lorentz group is simply connected. Therefore parity and time reversal are continuously connected to the identity.

For convenience, in the construction of the little group representations, the generators are further decomposed into

$$M_{ij} = R_k\,, \quad M_{0i} = L_i\,, \quad N_{ij} = U_k\,, \quad N_{0i} = M_i\,, \quad N_{\mu\mu} = -2g_{\mu\nu}C_\mu \tag{11}$$

with the usual permutation order being implied in the definitions of R_k and U_k. A matrix representation of these generators for the complex, the quaternionic and the octonionic cases is contained in App. A.

To study the irreducible representations of the inhomogeneous group (4) the induced representation method is used. The translation generators are diagonalized

$$P_\mu|p\rangle = \big(K_\mu + H_\mu\big)|p\rangle = (\mathrm{Re}\,p_\mu + i\,\mathrm{Im}\,p_\mu)|p\rangle \tag{12}$$

and the little groups are classified according to the values M_0^2 of the invariant $P^2 = K^2 + H^2$.[30]

(i) $\boldsymbol{M_0^2 > 0}$**.** p can be brought to the form $(p^0, 0, 0, 0)$ and the little group $G_1^{\mathbb{C}}$ is a $U(3)$ subgroup generated by $\{R_i, U_i, C_i, i = 1, 2, 3\}$.

(ii) $\boldsymbol{M_0^2 = 0,\ p^\mu \neq 0}$**.** p can be brought to the form $\big(p^0, 0, 0, p^0\big)$. The little group $G_2^{\mathbb{C}}$, generated by $\big\{R_3, U_3, C_1, C_2, L_1 + R_2, L_2 - R_1, M_1 + U_2, M_2 + U_1, M_3 + C_3 - C_0\big\}$ is a semidirect product $N \circledS U(2)$ where $U(2)$ is generated by $\big\{R_3, U_3, C_1, C_2\big\}$ and the invariant subgroup N by the remaining generators.

(iii) $\boldsymbol{M_0^2 = 0,\ p^\mu = 0}$**.** In this case the little group is the full $U(1, 3)$.

(iv) $\boldsymbol{M_0^2 < 0}$**.** p can be brought to the form $(0, 0, 0, p^3)$ and the little group is $U(1, 2)$ generated by $\{R_3, L_1, L_2, U_3, M_1, M_2, C_1, C_2, C_0\}$.

Now the representations of the first two classes will be analyzed.

2.2. $\boldsymbol{M_0^2 > 0}$ *representations*

The little group that classifies the representations in this case is $U(3)$ with Hermitian generators

$$-iR_i\,;\ -iU_i\,;\ -i\big(C_1 - C_2\big)\,;\ -i\big(C_1 + C_2 - 2C_3\big) \tag{13}$$

generating $SU(3)$ and a $U(1)$ generator $-i\big(C_1 + C_2 + C_3\big)$. Notice that in this $u(3)$ algebra the $\{R_i\}$-subalgebra is the ordinary space rotation algebra.

In the 4×4 matrices representing R_i, U_i and C_i (App. A), by suppressing the zero-valued, first line and first column one obtains the following correspondence to the usual λ matrices of $su(3)$

$$\begin{gathered} -iR_1 \to \lambda_7\,, \quad -iR_2 \to -\lambda_5\,, \quad -iR_3 \to \lambda_2\,, \\ -iU_1 \to \lambda_6\,, \quad -iU_2 \to \lambda_4\,, \quad -iU_3 \to \lambda_1\,, \\ -i(C_1 - C_2) \to \lambda_3\,, \quad -i(C_1 + C_2 - 2C_3) \to \sqrt{3}\lambda_8\,, \\ -i(C_1 + C_2 + C_3) \to \mathbf{1}\,. \end{gathered} \tag{14}$$

All the irreducible representations of $SU(3)$ are obtained by tensor products of $\mathbf{3}$ and $\bar{\mathbf{3}}$. Hence the spin operator acting on this states is

$$\begin{aligned} J^2 &= -R_1^2 - R_2^2 - R_3^2 = \lambda_7^2 + \lambda_5^2 + \lambda_2^2 \\ &= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = (1 \times (1+1))\mathbf{1}_3\,, \end{aligned} \tag{15}$$

which operating on $\mathbf{3}$ or $\bar{\mathbf{3}}$ yields spin 1. Therefore the conclusion is that only integer spins occur in the reduction of $SU(3)$ with respect to the rotation subgroup.

The extension of the kinematical symmetry group from the real to the complex Poincaré group does not contain massive half-integer spin elementary states. A phase generated by $-i\big(C_1 + C_2 + C_3\big)$, a $U(1)$ quantum number, appears naturally. Kinematic group transformations are the tools that, when applied to a state, exhibit all its aspects. In particular, general complex Lorentz transformations would take states from one real slice to another. The fact that massive half-integer spin states are not representations of the complex Poincaré group means that these elementary states cannot be taken out of the real slice where they are representations of the real Poincaré group. As an illustration let us try to implement the complex Lorentz group $\{M_{\mu\nu}, N_{\mu\nu}\}$ on a Dirac spinor.

The tensor structure of the Dirac spinor being coded in the Dirac equation, the representation matrices S of $M_{\mu\nu}$ and $N_{\mu\nu}$ must satisfy

$$\Lambda_\mu^\nu \gamma^\mu = S^{-1}\gamma^\nu S \tag{16}$$

with, for an infinitesimal transformation

$$\Lambda_\mu^\nu = g_\mu^\nu + \Delta\omega_\mu^\nu \tag{17}$$

and

$$\Delta\omega^{\nu\mu *} = -\Delta\omega^{\mu\nu} \tag{18}$$

following from (3), which implies $\Delta\omega^{\nu\mu} = -\Delta\omega^{\mu\nu}$ for $M_{\mu\nu}$ and $\Delta\omega^{\nu\mu} = \Delta\omega^{\mu\nu}$ for $N_{\mu\nu}$.

Therefore, to implement the 6 real Lorentz group transformations generated by R_i and L_i ($M_{\mu\nu}$) in the Dirac equation one has to find a functional $\Gamma_{\alpha\beta}$ of the gamma matrices satisfying

$$2i\left(g_\alpha^\nu \gamma_\beta - g_\beta^\nu \gamma_\alpha\right) = [\gamma^\nu, \Gamma_{\alpha\beta}]$$

which has the solution

$$\Gamma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$$

To implement the remaining 10 generators ($N_{\mu\nu}$) the corresponding equation would be

$$2i\left(g_\alpha^\nu \gamma_\beta + g_\beta^\nu \gamma_\alpha\right) = \left[\gamma^\nu, N_{\alpha\beta}\right]$$

with $N_{\alpha\beta}$ a symmetric functional of the gamma matrices. But because the only such symmetric functional is $g_{\alpha\beta}$, the equation has no solution.

The fact that half-integer spin states cannot be elementary states of the complex Poincaré group, only elementary states of the real group, implies that matter composed of half-integer elementary blocks cannot communicate between different real slices, in the sense that they cannot be rotated from one real slice to another. Half-integer spins might still be associated to the $SU(3)$ group in a nonlinear sense through an induced representation on a homogeneous space. However, rather than rotating the states away from the real slice a multiplicity of identical representation spaces is obtained. This is discussed in App. D.

By contrast with half-integer states, integer spin states may be *bona fide* elementary states of the complex group. However, these states are of a special nature. In the complex Lorentz group parity and time-reversal are continuously connected to the identity. Therefore in faithful continuous norm-conserving representations of this group both parity and time reversal must be implemented by unitary operators. On the other hand because of energy positivity,[35] the time reversal operation in a state $\psi_R \in V_R$ of the real Poincaré group must be implemented by an antiunitary operator. Therefore between these "real slice states" ψ_R and those that are faithful representations of the complex group $\psi_C \in V_C$, there is a superselection rule. Consider a linear superposition of two of these states

$$\Phi = \alpha\psi_R + \beta\psi_C$$

with α, β reals numbers. Now Φ and $e^{i\theta}\Phi = \alpha e^{i\theta}\psi_R + \beta e^{i\theta}\psi_C$ belong to the same ray and therefore should represent the same state. Applying the time reversal operator to both Φ and $e^{i\theta}\Phi$

$$T\Phi = \alpha T\psi_R + \beta T\psi_C\,,$$

$$Te^{i\theta}\Phi = \alpha e^{-i\theta} T\psi_R + \beta e^{i\theta} T\psi_C$$

$T\Phi$ and $Te^{i\theta}\Phi$ belong to different rays, hence T does not establish a ray correspondence in $V_R \oplus V_C$ unless $\alpha = 0$ or $\beta = 0$, that is, V_R and V_C belong to different superselection sectors. The fact that the integer spin states in V_C are in a different superselection sector does not mean that they cannot interact with the states in V_R. Whereas the superselection rule result concerns the structure of the direct sum $V_R \oplus V_C$, the nature of the interactions depends on the way the group transformations operate in the tensor product $V_R \otimes V_C$. Consider now the T operation acting on $V_R \otimes V_C$ and compute its action on a matrix element

$$\begin{aligned}&\left(T\left(\psi_R^{(1)} \otimes \psi_C^{(1)}\right), T\left(\psi_R^{(2)} \otimes \psi_C^{(2)}\right)\right)\\&\quad = \left(\psi_R^{(2)} \otimes \psi_R^{(1)}\right)\left(\psi_C^{(1)} \otimes \psi_C^{(2)}\right) = \left(\psi_R^{(2)} \otimes \psi_C^{(1)}, \psi_R^{(1)} \otimes \psi_C^{(2)}\right).\end{aligned}$$

Therefore, there is no choice of phases that can make T a unitary or antiunitary operator in the tensor product space. Therefore by Wigner's theorem T cannot be a symmetry in $V_R \otimes V_C$.[c]

In conclusion: Half-integer spin states and integer spin states with antiunitary time reversal transformations are confined to one real slice. Integer spin states of the full complex Poincaré group, when interacting with the inner states of the real slices can only mediate T-violating interactions (CP violation or possibly gravitational interactions).[37–40]

2.3. $M_0^2 = 0$, $p^\mu \neq 0$ *representations*

For the $M_0^2 = 0$, $p^\mu \neq 0$ case the algebra of the little group $G_2^{\mathbb{C}}$ for the momentum $\left(p^0, 0, 0, p^0\right)$ is a semidirect sum

$$\mathcal{L}G_2^{\mathbb{C}} = N^c \Diamond H^c$$

($[N^c, N^c] \subset N^c$; $[H^c, N^c] \subset N^c$; $[H^c, H^c] \subset H^c$), N^c, the algebra of the normal subgroup $\mathcal{N}^c$, being a two-dimensional Heisenberg algebra

$$\begin{gathered}N^c = \{l_1 = L_1 + R_2;\ m_1 = M_1 + U_2;\ l_2 = L_2 - R_1;\\ m_2 = M_2 + U_1;\ m_3 = M_3 + C_3 - C_0\}\end{gathered}$$

and H^c is the algebra of $\mathcal{H}^c = U(2)$

$$H^c = \left\{R_3; U_3; \frac{1}{2}(C_1 - C_2); C_1 + C_2\right\}.$$

The representations of $G_2^{\mathbb{C}}$ are also obtained by the induced representation method. Given a representation α of $\mathcal{N}^c$ one finds the isotropy subgroup $\mathcal{H}^c(\alpha)$ of $\mathcal{H}^c$, that is,

$$\alpha(hnh^{-1}) = \alpha(n)\,, \quad \forall n \in \mathcal{N}^c\,, \quad \forall h \in \mathcal{H}^c\,.$$

[c] A similar situation has been discussed in the past[36] for states transforming under a different group, the analytical continuation complex Lorentz group $\left(\Lambda^T G \Lambda = G\right)$, which also connects the identity to time reversal although not to parity.

Then, for each representation β of $\mathcal{H}^c(\alpha)$, the product representation $\sigma = \alpha \times \beta$ defines a homogeneous vector bundle over $G_2^c/\mathcal{N}^c\mathcal{H}^c$. $\mathcal{N}^c\mathcal{H}^c$ acts on the fiber by the σ representation and the full G_2^c representation is obtained by composing σ with the translations in the base manifold $G_2^c/\mathcal{N}^c\mathcal{H}^c$. The states in the representation of G_2^c are labeled by a point in the base manifold $G_2^c/\mathcal{N}^c\mathcal{H}^c$ and by the fiber indices of the σ representation. To classify the representations of G_2^c all one has to do is to classify the σ-representations of $\mathcal{N}^c\mathcal{H}^c$ for each α.

The generator m_3 commutes with all generators in $\mathcal{L}G_2^c$. Therefore it is a constant in each irreducible representation. By the Stone–von Neumann theorem this constant uniquely characterizes an irreducible representation of $\mathcal{N}^c$, namely, for the algebra elements

$$\begin{aligned} l_1\psi^{(\mu)}(\eta,\xi) &= i\eta\psi^{(\mu)}(\eta,\xi)\,, \\ l_2\psi^{(\mu)}(\eta,\xi) &= i\xi\psi^{(\mu)}(\eta,\xi)\,, \\ m_1\psi^{(\mu)}(\eta,\xi) &= -\mu\frac{\partial}{\partial\eta}\psi^{(\mu)}(\eta,\xi)\,, \\ m_2\psi^{(\mu)}(\eta,\xi) &= -\mu\frac{\partial}{\partial\xi}\psi^{(\mu)}(\eta,\xi)\,, \\ m_3\psi^{(\mu)}(\eta,\xi) &= \frac{i}{2}\mu\psi^{(\mu)}(\eta,\xi) \end{aligned}$$

realized in the space of functions of two real variables η and ξ. Therefore there are two types of representations of G_2^c.

(1) For a nontrivial α-representation of $\mathcal{N}^c$ of the type above, the little group $\mathcal{H}^c(\alpha)$ is empty. Therefore β is trivial and the representations are simply labeled by the differentiable functions $\psi^{(\mu)}(\eta,\xi)$. This is analogous to the continuous spin group of the real Poincaré group.

(2) If α is trivial, that is, if the generators l_i, m_i are mapped on the zero operator, then $\mathcal{H}^c(\alpha) = U(2)$. The states are now labeled by the quantum numbers of $SU(2)$ and a $U(1)$ phase. Notice however, that each spin projection in a $SU(2)$ multiplet may correspond to a different particle in the "real slice" because only R_3 among the $SU(2)$ generators belongs to the real Lorentz group.

As seen from the commutation table in App. C the normalized generators of the $SU(2)$ subgroup, that label the states, are $\left\{\frac{R_3}{2}, \frac{U_3}{2}, \frac{1}{2}(C_1 - C_2)\right\}$. Therefore because in the representations of $SU(2)$, $\frac{R_3}{2}$ has an integer or half-integer spectrum, the spectrum of R_3 is integer and, once again, one finds that half-integer spins states cannot have the full complex Poincaré group as a symmetry group.

The fact that for the massless case half-integer spin states cannot be elementary states of the complex Poincaré group, only elementary states of the real group, implies, as in the massive case, that matter composed of half-integer elementary blocks cannot communicate between real slices. Integer spin states may be *bona fide* elementary states of the complex group. However, as seen before, their interactions with other states in each real slice still depend on the way the discrete transformations are implemented, leading, as discussed before to a superselection rule and T-violation.

The lowest massless spin multiplets that naturally appear on the complex Poincaré group are a zero spin singlet, a spin 1 with +1 and −1 projections and a multiplet containing +2 and −2 as well as a scalar.

3. Quaternionic and Octonionic Space–Times

The generators of the quaternionic and octonionic Lorentz algebras ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{O}$) are

$$\left\{R_a; L_a; U_a^{(\alpha)}; M_a^{(\alpha)}; C_a^{(\alpha)}; C_0^{(\alpha)}\right\}$$

with $a = 1, 2, 3$ and $\alpha = i, j, k$, for the imaginary quaternionic units, $i^2 = j^2 = k^2 = -1$; $ij = -ji$; $ij = k$ and cyclic permutations and $\alpha = e_1$ to e_7 for the octonions. A matrix representation of these generators is contained in App. A.

The quaternionic Lorentz group has 36 generators and, with the 4 quaternionic translations, the corresponding Poincaré group has 40 generators. The octonionic Lorentz algebra has 76 generators that together with 8 translations adds up to 84 generators. It is an algebra closed under commutation, however, not a Lie algebra because of the nonassociativity of the octonions.

As for the complex Poincaré group one analyses the two cases: $M^2 > 0$ and $M^2 = 0$, $p^\mu \neq 0$.

3.1. $M_0^2 > 0$ *representations*

Bringing p to the form $\left(p^0, 0, 0, 0\right)$ the little algebra, G_2, is generated by 21 generators for the quaternionic case and 45 generators for the octonionic case

$$\left\{R_i; U_i^{(e)}; C_1^{(e)}; C_2^{(e)}; C_3^{(e)}\right\}$$

with the commutation table of App. B. These algebras are $u(3, \mathbb{Q})$ and $u(3, \mathbb{O})$. Notice however, that $u(3, \mathbb{O})$ is closed under commutation but not a Lie algebra. For example

$$\left[U_1^{(e_1)}, \left[U_2^{(e_2)}, U_3^{(e_6)}\right]\right] + \left[U_2^{(e_2)}, \left[U_3^{(e_6)}, U_1^{(e_1)}\right]\right] + \left[U_3^{(e_6)}, \left[U_1^{(e_1)}, U_2^{(e_2)}\right]\right]$$

$$= -\left[U_1^{(e_1)}, U_1^{(e_4)}\right] + \left[U_2^{(e_2)}, U_2^{(e_7)}\right] + \left[U_3^{(e_6)}, U_3^{(e_3)}\right] = -4\Big(C_1 + C_2 + C_3\Big)^{(e_5)} . \quad (19)$$

For $M_0^2 > 0$ one has in all cases (including the complex) the algebras $u(3, \mathbb{K})$ with $\mathbb{K} = \mathbb{C}, \mathbb{Q}, \mathbb{O}$ generated by the skew-Hermitian matrices

$$M = \left\{R_i; U_i^{(e)}; C_1^{(e)}; C_2^{(e)}; C_3^{(e)}\right\}$$

$M = -M^\dagger$, where here only the 3×3 block of the generators in App. A are considered

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_1^{(e)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e \\ 0 & e & 0 \end{pmatrix}, \quad U_2^{(e)} = \begin{pmatrix} 0 & 0 & e \\ 0 & 0 & 0 \\ e & 0 & 0 \end{pmatrix}, \quad U_3^{(e)} = \begin{pmatrix} 0 & e & 0 \\ e & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$C_1^{(e)} = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2^{(e)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_3^{(e)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e \end{pmatrix}$$

with $e \in \mathbb{K}'$ (an imaginary element of $\mathbb{K}$). For all these matrices $M = -M^\dagger$, with $e^\dagger = -e$, $e \in \mathbb{K}'$.

In these algebras there is a $u(1,\mathbb{K})$ subalgebra generated by $C_1^{(e)} + C_2^{(e)} + C_3^{(e)}$, whereas $\left\{R_i; U_i^{(e)}; C_1^{(e)} - C_2^{(e)}; C_1^{(e)} + C_2^{(e)} - 2C_3^{(e)}\right\}$ generates $su(3,\mathbb{K})$.

The root space decomposition may easily be generalized for these algebras. Let

$$\begin{aligned} h_1 &= -e\left(C_1^{(e)} - C_2^{(e)}\right), \\ h_2 &= \frac{-e}{\sqrt{3}}\left(C_1^{(e)} + C_2^{(e)} - 2C_3^{(e)}\right). \end{aligned} \tag{21}$$

Then

$$\begin{aligned}
\left[h_1, \left(-eU_3^{(e)} + R_3\right)\right] &= 2\left(-eU_3^{(e)} + R_3\right), \\
\left[h_2, \left(-eU_3^{(e)} + R_3\right)\right] &= 0, \\
\left[h_1, \left(-eU_3^{(e)} - R_3\right)\right] &= -2\left(-eU_3^{(e)} - R_3\right), \\
\left[h_2, \left(-eU_3^{(e)} - R_3\right)\right] &= 0, \\
\left[h_1, \left(-eU_1^{(e)} + R_1\right)\right] &= -\left(-eU_1^{(e)} + R_1\right), \\
\left[h_2, \left(-eU_1^{(e)} + R_1\right)\right] &= \sqrt{3}\left(-eU_1^{(e)} + R_1\right), \\
\left[h_1, \left(-eU_1^{(e)} - R_1\right)\right] &= \left(-eU_1^{(e)} - R_1\right), \\
\left[h_2, \left(-eU_1^{(e)} - R_1\right)\right] &= -\sqrt{3}\left(-eU_1^{(e)} - R_1\right), \\
\left[h_1, \left(-eU_2^{(e)} - R_2\right)\right] &= \left(-eU_2^{(e)} - R_2\right),
\end{aligned}$$

$$\left[h_2, \left(-eU_2^{(e)} - R_2\right)\right] = \sqrt{3}\left(-eU_2^{(e)} - R_2\right),$$
$$\left[h_1, \left(-eU_2^{(e)} + R_2\right)\right] = -\left(-eU_2^{(e)} + R_2\right),$$
$$\left[h_2, \left(-eU_2^{(e)} + R_2\right)\right] = -\sqrt{3}\left(-eU_2^{(e)} + R_2\right). \tag{22}$$

The roots are the same as in $su(3,\mathbb{C})$, the difference being that the nonzero root spaces now have dimension equal to $\dim \mathbb{K}'$, three for quaternions and seven for octonions. The root space of the zero root has dimension $2 \times \dim \mathbb{K}'$.

Representations of $su(3,\mathbb{Q})$ and $su(3,\mathbb{O})$ in quaternionic and octonionic Hilbert spaces are simply obtained from those of $su(3,\mathbb{C})$. Notice that $\left(-eU_3^{(e)} \pm R_3\right)$, $\left(-eU_1^{(e)} \pm R_1\right)$ and $\left(-eU_2^{(e)} \pm R_2\right)$ are the same for all e. Therefore given a complex representation of $su(3,\mathbb{C})$ with vectors $\left\{\psi_a^{(\lambda)}\right\}$ for the highest weight λ

$$\left(-eU_i^{(e)}\right)\psi_a^{(\lambda)} = \sum_{b=1}^{\dim\lambda} c_{ba}\psi_b^{(\lambda)} \tag{23}$$

with $c_{ba} \in \mathbb{C}$, one obtains

$$U_i^{(e)}\psi_a^{(\lambda)} = \sum_{b=1}^{\dim\lambda} ec_{ba}\psi_b^{(\lambda)} \tag{24}$$

with matrix elements $ec_{ba} \in \mathbb{Q}, \mathbb{O}$. The same procedure applies for the other generators and this makes sense in the framework of quaternionic or octonionic Hilbert spaces.

A different point of view would be to look for representations of these algebras with complex matrices. This may be achieved by defining for each state of the λ-representation $\dim \mathbb{K}' - 1$ new states

$$\psi_{a,e}^{(\lambda)} \doteq e\psi_a^{(\lambda)}$$

and Eq. (24) becomes

$$U_i^{(e)}\psi_a^{(\lambda)} = \sum_{b=1}^{\dim\lambda} c_{ba}\psi_{b,e}^{(\lambda)}$$

a representation with complex matrices in a space of dimension $\dim\lambda \times (\dim \mathbb{K}' - 1)$. Notice that one is identifying e_1 with the complex imaginary unit. The complex representations so obtained are not necessarily irreducible. The lowest nontrivial representations would be six-dimensional for quaternions and 18-dimensional for octonions, corresponding to two and six spin 1 states. Although octonions are a nonassociative algebra this procedure allows for a consistent construction of a representation space for $su(3,\mathbb{O})$. However, because of the nonassociativity of octonions the corresponding matrices are a representation of $su(3,\mathbb{O})$ only in the quasi-algebra sense.[41–44]

In the quaternionic case $su(3,\mathbb{Q})$ is a Lie algebra and its correspondence to an ordinary complex algebra may be explicitly exhibited. Representing the imaginary quaternion units by sigma matrices ($e_i \to -i\sigma_i$) the generators of $su(3,\mathbb{Q})$ become

$$R_{ii} = R_i \otimes \mathbf{1}_2\,, \quad U_{ij} = U_i \otimes (-i\sigma_j)\,, \quad C_{ij} = C_i \otimes (-i\sigma_j)\,.$$

The matrices of the algebra $\mathcal{A} = \{R_{ii}, U_{ij}, C_{ij}\}$ satisfy

$$A^T\Omega + \Omega A = 0\,, \quad A \in \mathcal{A}\,,$$

where Ω is the symplectic form

$$\Omega = \mathbf{1}_3 \otimes (i\sigma_2)\,.$$

Therefore, the algebra is the algebra of the symplectic group in 6 dimensions. With

$$\Omega_{12} = -\Omega_{21} = \Omega_{34} = -\Omega_{43} = \Omega_{56} = -\Omega_{65} = 1$$

all other elements Ω_{ij} being zero, the irreducible representations are obtained from (symplectic) traceless tensors

$$\Omega_{ij} F_{\cdots ij \cdots} = 0$$

of definite permutation symmetry corresponding to Young diagrams $(\sigma_1, \sigma_2, \sigma_3)$ with at most three lines.

The lowest-dimensional representations have dimensions $0\,(000)$, $6\,(100)$, $14\,(110)$ and (111), $21\ (200), \ldots$. Homogeneous polynomial basis for the defining (6) and the adjoint (21) representations are $(x_i; i = 1\cdots 6)$ and $\left(x_i^2,\ x_i x_j,\ i < j;\ i,j = 1\cdots 6\right)$. As in the complex Poincaré case only integer spins exist in the irreducible representations. In the 6-representation there are two independent spin one states associated to (x_1, x_3, x_5) and (x_2, x_4, x_6).

3.2. $M_0^2 = 0$, $p^\mu \neq 0$ *representations*

Here for definiteness the quaternionic case will be analyzed. As before, bringing p to the form $\left(p^0, 0, 0, p^0\right)$, the little group G_2^q is generated by the following 21 generators

$$\left\{R_3; U_3^\alpha; C_1^\alpha; C_2^\alpha; L_1 + R_2; L_2 - R_1; M_1^\alpha + U_2^\alpha; M_2^\alpha + U_1^\alpha; M_3^\alpha + C_3^\alpha - C_0^\alpha\right\}$$

the commutation table for these generators being listed in App. C.

From the commutation table one sees that the algebra of the little group G_2^q is the semidirect sum

$$\mathcal{L}G_2^q = N^q \Diamond H^q$$

$N^q = \left\{l_1, l_2, m_1^\alpha, m_2^\alpha, m_3^\alpha\right\}$ and $H^q = \left\{R_3, U_3^\alpha, C_1^\alpha - C_2^\alpha, C_1^\alpha + C_2^\alpha\right\}$.

The generators m_3^α commute with all the other generators, constants in an irreducible representation, are denoted $\frac{i}{2}\mu_i$, $\frac{i}{2}\mu_j$, $\frac{i}{2}\mu_k$. Then, from the commutation table in App. C one concludes that the invariant algebra N^q consists of a set of overlapping Heisenberg algebras which, on the space of differentiable functions of 6 variables $\vec{\eta} = \eta_i,\ \eta_j,\ \eta_k,\ \xi_i,\ \xi_j,\ \xi_k$ have the representation

$$\begin{aligned}
l_1\psi(\vec{\eta},\vec{\xi}) &= i(\eta_i + \eta_j + \eta_k)\psi(\vec{\eta},\vec{\xi})\,,\\
m_1^i\psi(\vec{\eta},\vec{\xi}) &= i\left(-\mu_i\frac{\partial}{\partial\eta_i} + i\frac{\mu_k}{\mu_j}\eta_j - i\frac{\mu_j}{\mu_k}\eta_k\right)\psi(\vec{\eta},\vec{\xi})\,,\\
m_1^j\psi(\vec{\eta},\vec{\xi}) &= i\left(-\mu_j\frac{\partial}{\partial\eta_j} + i\frac{\mu_i}{\mu_k}\eta_k - i\frac{\mu_k}{\mu_i}\eta_i\right)\psi(\vec{\eta},\vec{\xi})\,,\\
m_1^k\psi(\vec{\eta},\vec{\xi}) &= i\left(-\mu_k\frac{\partial}{\partial\eta_k} + i\frac{\mu_j}{\mu_i}\eta_i - i\frac{\mu_i}{\mu_j}\eta_j\right)\psi(\vec{\eta},\vec{\xi})\,,\\
l_2, m_2^\alpha &\to (\eta \to \xi)\,,\\
m_3^\alpha\psi(\vec{\eta},\vec{\xi}) &= \frac{i}{2}\mu_\alpha\psi(\vec{\eta},\vec{\xi})\,.
\end{aligned}$$

As in the complex case there are two classes of representations

(1) For a nontrivial representation of N^q, as above, the little group $\mathcal{H}^q(\alpha)$ is empty and the states are labeled by the functions $\psi(\vec{\eta},\vec{\xi})$. It is the continuous spin case.

(2) For a trivial (identically zero) representation of N^q the little group is $\mathcal{H}^q$ itself. H^q is the algebra of $Sp(2) \sim SO(5)$. A Cartan subalgebra is $\{iR_3, iC_+^i = i(C_1^i + C_2^i)\}$ and the root vectors are

$$\begin{gathered}
iU_3^j - \varepsilon_2 U_3^k - \varepsilon_1(C_-^j + \varepsilon_2 iC_-^k)\,,\\
iC_+^j - \varepsilon C_+^k\,, \quad iU_3^i - \varepsilon C_-^i
\end{gathered}$$

ε_1 and ε_2 are independent $\pm$ signs. The first 4 root vectors have weights $2\binom{\varepsilon_1}{\varepsilon_2}$ and the other 4 have weights $2\binom{0}{\varepsilon}$ and $2\binom{\varepsilon}{0}$.

In conclusion: Both in the quaternionic and octonionic cases, the same restrictions as in the complex case, are obtained concerning the separation of the states of the full group from those of the real Poincaré group in the real slices. These are, as before, the nonexistence of half-integer spins as elementary states and the superselection rule for integer spins. The main difference from the complex case is the multiplicity of states, which, of course, has implications on the nature of the hypothetical conversion of kinematical into internal symmetries (see App. D).

4. Conclusions

A study of matter representations in space–times over the complex, quaternions and octonions has been performed. The main conclusions are:

(1) No elementary half-integer spin states exist consistent with the full operations of this higher-dimensional division algebras. Hence, no "rotation" between real slices for half-integer elementary states. It would explain why half-integer spin matter stays confined to a real slice, essentially unaware of the larger space–time where it might be embedded.
(2) Integer spin states exist as elementary states of the larger groups. However there is a superselection rule operating between these states and those of the real slices, potentially leading to T-violating interactions.
(3) When, in the nonlinear sense of App. D, half-integer spin states are associated to the higher groups, rather than a transformation between real slices, a multiplicity of states is generated. In this sense the transformations of the higher-dimensional kinematical group play the role of internal symmetries.
(4) These general conclusions apply to complex, quaternionic and octonionic spaces, the differences being mostly on the multiplicity of states.
(5) If indeed the actual structure of space–time is associated to the higher-dimensional division algebras with a foliation in real slices, an obvious question would be how many of these real slices are populated by ordinary matter? Is there a quantum like restriction on the density of real slices? Is it related to quantization of noncommutative coordinates?

Appendix A.

A matrix representation of the generators of the complex, quaternionic and octonionic Lorentz transformations from

$$\left(1+\omega^{\dagger}\right)G(1+\omega)=G \tag{A.1}$$

it follows:

$$\omega^{*}_{\sigma\mu}g_{\sigma\nu}=-g_{\mu\sigma}\omega_{\sigma\nu} \tag{A.2}$$

and a set of independent generators is obtained from the conditions $\omega^{*}_{00}=-\omega_{00}$, $\omega^{*}_{k0}=\omega_{0k}$ and $\omega^{*}_{ik}=-\omega_{ki}$

$$R_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\quad R_2=\begin{pmatrix}0&0&0&0\\0&0&0&-1\\0&0&0&0\\0&1&0&0\end{pmatrix},\quad R_3=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{pmatrix},$$

$$L_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad L_2=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},\quad L_3=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix},$$

$$U_1^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix}, \quad U_2^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix}, \quad U_3^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_1^\alpha = \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2^\alpha = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_3^\alpha = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix},$$

$$C_1^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3^\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \quad C_0^\alpha = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

R_i, L_i $(i = 1, 2, 3)$ are generators of real rotations and real boosts. For the other generators:

(1) $\alpha = i$ in the complex case;
(2) $\alpha = i,\ j,\ k$ with $i^2 = j^2 = k^2 = -1$, $ij = -ji$, $ik = -ki$, $jk = -kj$, $ij = k$ and cyclic permutations in the quaternionic case;
(3) $\alpha = e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_7$ with $e_i^2 = -1$ and multiplication table for the octonionic case.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

The correspondence of these generators to the $(M_{\mu\nu}, N_{\mu\nu})$ ones are:

$$R_i = \frac{1}{2}\epsilon_{ijk}M_{jk}\,, \quad L_i = M_{0i}\,, \quad M_i = N_{0i}\,, \quad U_i = N_{jk}\,, \quad C_\mu = -\frac{1}{2}N_{\mu\mu}g_{\mu\mu}\,.$$

Other conventions are found in the literature for the labeling of the octonionic imaginary units (see for example Ref. 45). The above one is used here because it is probably the one that better allows for a unified treatment of the four normed

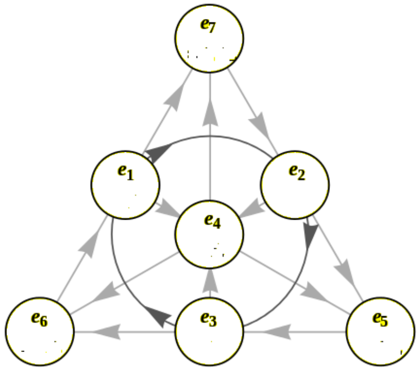

Fig. A.1. A Fano multiplication diagram for the octonions.

division algebras. Notice also that, because of consistency with the cyclic permutation order, the numeric labeling of the generators of the complex Lorentz group differs from the one in Ref. 30.

Appendix B.

Commutation table for the generators of the little (quasi-)group in the massive case $\left(M_0^2 > 0\right)$. Here and in App. C, $X^{[\alpha,\beta]}$ means $2X^{|\alpha\beta|}\,\mathrm{sign}(\alpha\beta)$ and $X^{(\alpha\beta)}$ means $X^{|\alpha\beta|}\,\mathrm{sign}(\alpha\beta)$.

	R_1	R_2	R_3	U_1^β	U_2^β	U_3^β	C_1^β	C_2^β	C_3^β
R_1	0	$-R_3$	R_2	$2\left(C_2^\beta - C_3^\beta\right)$	U_3^β	$-U_2^\beta$	0	$-U_1^\beta$	U_1^β
R_2		0	$-R_1$	$-U_3^\beta$	$2\left(-C_1^\beta + C_3^\beta\right)$	U_1^β	U_2^β	0	$-U_2^\beta$
R_3			0	U_2^β	$-U_1^\beta$	$2\left(C_1^\beta - C_2^\beta\right)$	$-U_3^\beta$	U_3^β	0
U_1^α				$0(\alpha=\beta)$ $(C_2 + C_3)^{[\alpha,\beta]}$ $(\alpha\neq\beta)$	$R_3(\alpha=\beta)$ $U_3^{(\alpha\beta)}(\alpha\neq\beta)$	$-R_2(\alpha=\beta)$ $U_2^{(\alpha\beta)}(\alpha\neq\beta)$	0	$R_1(\alpha=\beta)$ $U_1^{(\alpha\beta)}(\alpha\neq\beta)$	$-R_1(\alpha=\beta)$ $U_1^{(\alpha\beta)}(\alpha\neq\beta)$
U_2^α					$0(\alpha=\beta)$ $(C_1 + C_3)^{[\alpha,\beta]}$ $(\alpha\neq\beta)$	$R_1(\alpha=\beta)$ $U_1^{(\alpha\beta)}(\alpha\neq\beta)$	$-R_2(\alpha=\beta)$ $U_2^{(\alpha\beta)}(\alpha\neq\beta)$	0	$R_2(\alpha=\beta)$ $U_2^{(\alpha\beta)}(\alpha\neq\beta)$
U_3^α						$0(\alpha=\beta)$ $(C_1 + C_2)^{[\alpha,\beta]}$ $(\alpha\neq\beta)$	$R_3(\alpha=\beta)$ $U_3^{(\alpha\beta)}(\alpha\neq\beta)$	$-R_3(\alpha=\beta)$ $U_3^{(\alpha\beta)}(\alpha\neq\beta)$	0
C_1^α							$C_1^{[\alpha,\beta]}$	0	0
C_2^α								$C_2^{[\alpha,\beta]}$	0
C_3^α									$C_3^{[\alpha,\beta]}$

Appendix C.

Commutation table for the generators of the little (quasi-)group in the massless case $\left(M_0^2 = 0, p_\mu \neq 0\right)$

$$l_1 = L_1 + R_2 \,, \quad l_2 = L_2 - R_1 \,,$$

$$m_1^\alpha = M_1^\alpha + U_2^\alpha \,, \quad m_2^\alpha = M_2^\alpha + U_1^\alpha \,, \quad m_3^\alpha = M_3^\alpha + C_3^\alpha - C_0^\alpha \,.$$

	l_1	l_2	m_1^β	m_2^β	m_3^β	R_3	U_3^β	C_1^β	C_2^β
l_1	0	0	$2m_3^\beta$	0	0	l_2	m_2^β	m_1^β	0
l_2	0	0	0	$2m_3^\beta$	0	$-l_1$	m_1^β	0	m_2^β
m_1^α			$m_3^{[\alpha,\beta]}$	0	0	m_2^α	$-l_2(\alpha=\beta)$ $m_2^{(\alpha\beta)}(\alpha\neq\beta)$	$-l_1(\alpha=\beta)$ $m_1^{(\alpha\beta)}(\alpha\neq\beta)$	0
m_2^α				$m_3^{[\alpha,\beta]}$	0	$-m_1^\alpha$	$-l_1(\alpha=\beta)$ $m_1^{(\alpha\beta)}(\alpha\neq\beta)$	0	$-l_2(\alpha=\beta)$ $m_2^{(\alpha\beta)}(\alpha\neq\beta)$
m_3^α					0	0	0	0	0
R_3						0	$2\left(C_1^\beta - C_2^\beta\right)$	$-u_3^\beta$	u_3^β
U_3^α							$\left(C_1 + C_2\right)^{[\alpha,\beta]}$	$r_3(\alpha=\beta)$ $u_3^{(\alpha\beta)}(\alpha\neq\beta)$	$-r_3(\alpha=\beta)$ $u_3^{(\alpha\beta)}(\alpha\neq\beta)$
C_1^α								$C_1^{[\alpha,\beta]}$	0
C_2^α									$C_2^{[\alpha,\beta]}$

Appendix D. Half-Integer Spin States in Homogeneous Spaces

Half-integer spin states (of the subgroup generated by $\{R_i\}$) are not contained in the irreducible representations of $U(3)$ (the little group of massive states of $U(3,1)$), nevertheless half-spin state representations may be associated to $U(3)$ in a nonlinear way. Here one considers the complex Poincaré group case.

The set $\{R_i\}$ generates a $SU(2)$ subgroup of $U(3)$. With a coset decomposition of $U(3)$ one obtains a six-dimensional homogeneous space $M = U(3)/SU(2)$. Let us label the cosets by the letter p and for each coset chose an element $\sigma(p) \in U(3)$ such that a generic element of that coset is $\sigma(p)h$ with $h \in SU(2)$. To the homogeneous space M associate a vector bundle Γ with base M and fibers V carrying an half-integer representation of $SU(2)$. Locally the bundle is $N(p) \times V$, $N(p)$ being a neighborhood of $p \in M$. An arbitrary element of Γ is $\Phi(p, \alpha)$ where α carries the quantum numbers of the $SU(2)$ representation.

The action of an arbitrary element of $g \in U(3)$ on $\Phi(p, \alpha)$ is obtained as follows. Notice that

$$g\sigma(p) = \sigma(p')h(g, p, p')$$

with $h(g, p, p') \in SU(2)$. Then

$$g\Phi(p, \alpha) = D(h(g, p, p')) \circ \Phi(p', \alpha)$$

$D(\,\cdot\,)$ being a representation matrix of $SU(2)$. If instead of the states Φ one considers sections $\psi(p) \in V$ of the bundle

$$g \circ \psi(p) = D\big(h^{-1}(g, p, p')\big) \circ \psi\big(\tau_g^{-1} p'\big)$$

$\tau_g : p \to p'$ being the action of g on M.

This construction implements a representation of $U(3)$ carrying half-integer spin states of the $SU(2)$ generated by $\{R_i\}$. However, this representation cannot be reduced into irreducible representations of $U(3)$.

The physical interpretation is that, when operating on the half-integer spin states of the real slice Poincaré group, the additional generators of the complex Lorentz group instead of carrying the states to a different real slice, move them in an internal space of dimension six. In this sense the kinematical transformations of the complex Lorentz group become internal symmetries. Notice however, that this situation is mathematically different from, for example, assuming an internal $SU(3)$ color symmetry and generating a color triplet and antitriplet space. Here the label space (the base of the bundle) is six-dimensional but the representation is in fact infinite-dimensional.

For the quaternionic and octonionic space–times the situation is similar in the sense that also half-integer spin states do not appear as elementary states, but the dimension of the internal spaces (the base of the bundles) is larger.

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