12 November 2001

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 290 (2001) 109–114

www.elsevier.com/locate/pla

Non-commutative space–time and the uncertainty principle

Eric Carlen [a], R. Vilela Mendes [b,*,1]

[a] *School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA*
[b] *Grupo de Física Matemática, Universidade de Lisboa, Av. Gama Pinto 2, 1699 Lisboa, Portugal*

Received 26 June 2001; accepted 8 October 2001
Communicated by J.P. Vigier

Abstract

The full algebra of relativistic quantum mechanics (Lorentz plus Heisenberg) is unstable. Stabilization by deformation leads to a new deformation parameter $\varepsilon\ell^2$, ℓ being a length and ε a $\pm$ sign. The implications of the deformed algebras for the uncertainty principle and the density of states are worked out and compared with the results of past analysis following from gravity and string theory. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 03.65.Bz

Physical theories are approximations to nature and physical constants may never be known with absolute precision. Therefore, a wider range of validity is expected to hold for theories that do not change in a qualitative manner under a small change of parameters. Such theories are called *stable* or *rigid*. The stable-model point of view originated in the field of non-linear dynamics, where it led to the notion of *structural stability* [1,2]. However, as emphasized by Flato [3] and Faddeev [4], the same pattern seems to occur in the fundamental theories of nature. Indeed, the most important physical revolutions of this century, the transition from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as the replacement of two unstable theories by two stable ones. Mathematically this corresponds to stabilizing deformations leading, in the first case, from the Galilean to the Lorentz algebra and, in the second, from the algebra of commutative phase space to the Moyal–Vey algebra (or equivalently to the Heisenberg algebra). The deformation parameters, which for non-zero values make the algebras stable, are $1/c$ (the inverse of the speed of light) and h (the Planck constant). Once deformed, the algebras are all equivalent for non-zero values of $1/c$ and h. Hence, relativistic mechanics and quantum mechanics may be derived from the conditions for stability of two mathematical structures, although the exact values of the deformation parameters cannot be fixed by purely algebraic considerations. Instead, the deformation parameters are fundamental constants to be obtained from experiment. In this sense not only is deformation theory the theory of stable theories, it is also the theory that identifies the fundamental constants.

Some time ago it was noticed [5] that stability of the subalgebras does not guarantee stability of the full algebra of relativistic quantum mechanics. The latter contains the Lorentz algebra $\{M_{\mu\nu}\}$ and the Heisenberg algebras $\{p_\mu, x_\nu\}$ plus the commutators that de-

* Corresponding author.
E-mail addresses: carlen@math.gatech.edu (E. Carlen), vilela@cii.fc.ul.pt (R. Vilela Mendes).

1 Work partially supported by NSF grant DMS 00-70589.

0375-9601/01/$ – see front matter
PII: S0375-9601(01)00673-9

fine the 4-vector nature of p_μ and x_μ. The full algebra turns out to be unstable and its stabilization by deformation leads to a new deformation parameter ℓ^2 with dimension (length)^2. The deformed commutators are

$$\begin{aligned}
&[x_\mu, x_\nu] = -i\varepsilon\ell^2 M_{\mu\nu},\\
&[p_\mu, x_\nu] = i\eta_{\mu\nu}\Im,\\
&[x_\mu, \Im] = i\varepsilon\ell^2 p_\mu,
\end{aligned} \tag{1}$$

where $\eta_{\mu\nu} = (1, -1, -1, -1)$, $c = \hbar = 1$, $\varepsilon = \pm 1$ and $\Im$ is the operator that replaces the trivial center of the Heisenberg algebra. The new relativistic quantum mechanics algebra implies that space–time is a noncommutative manifold and has other physical consequences, some of which are explored in Refs. [6,7]. Also, like $1/c$ and h, the deformation parameter ℓ is naturally identified as a new fundamental constant to be obtained from experiment. This constant sets the scale for the spectrum of the position operators. In addition to the magnitude ℓ, there is also the sign ε of the deformation parameter that is not fixed by stability considerations and must be determined experimentally.

Notice that because of non-commutativity of the space–time coordinates only one coordinate may be sharply specified. For the choice $\varepsilon = -1$ the space coordinates have discrete spectrum but the time coordinate a continuous spectrum and conversely for $\varepsilon = +1$ [7]. We focus here on the one-dimensional subalgebras (for one space coordinate and one momentum) that replace Heisenberg's algebra, which are

$$\begin{aligned}
&[x, p] = i\Im,\\
&[x, \Im] = i\varepsilon\ell^2 p,\\
&[p, \Im] = 0,
\end{aligned} \tag{2}$$

with either $\varepsilon = -1$ or $\varepsilon = +1$.

For $\varepsilon = -1$ this is the algebra of the group of motions of the plane, ISO(2), and for $\varepsilon = +1$ the algebra of the group of motions of the hyperbolic plane, ISO(1,1).

For the ISO(2) case the irreducible representations T_r [8] can be realized as operators on $L^2(S^1)$ with respect to normalized Lebesgue measure on S^1, so that the scalar product is given by

$$(f_1, f_2) = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta) f_2^*(\theta)\,d\theta. \tag{3}$$

In this case p and $\Im$ are diagonal and the operators are

$$\begin{aligned}
&x = i\ell\frac{\partial}{\partial\theta},\\
&p = r\frac{1}{\ell}\sin\theta,\\
&\Im = r\cos\theta.
\end{aligned} \tag{4}$$

Fourier transforming, we have a representation on $\ell^2(Z)$ in which x is diagonal,

$$\begin{aligned}
&x = \ell n,\\
&p = \frac{1}{i\ell}\Delta_-,\\
&\Im = \Delta_+,
\end{aligned} \tag{5}$$

Δ_- and Δ_+ being the operators

$$\begin{aligned}
&\Delta_- f(x) = \frac{1}{2}\big(f(x+1) - f(x-1)\big),\\
&\Delta_+ f(x) = \frac{1}{2}\big(f(x+1) + f(x-1)\big).
\end{aligned} \tag{6}$$

The representations T_r are infinite-dimensional for all $r \neq 0$, a convenient basis being the set of exponentials $\exp(-in\theta)$,

$$\big\{e^{-in\theta};\ n \in Z\big\}, \tag{7}$$

or in the Fourier transformed representation,

$$\{\delta_n;\ n \in Z\}. \tag{8}$$

The states $e^{-in\theta}$ are eigenstates of the position operator x, which has a discrete spectrum $(= \ell Z)$. ℓ is the minimal length spacing and the maximum momentum p is r/ℓ.

For each localized state $e_n = (1/\sqrt{2\pi})e^{-in\theta}$, $P = p/\ell$ is a random variable with characteristic function

$$C(s) = \langle e_n, e^{isP} e_n\rangle = J_0(sr), \tag{9}$$

the corresponding probability density being

$$\nu(P) = \begin{cases} \frac{1}{\pi}\frac{1}{\sqrt{r^2 - P^2}}, & |P| < r,\\ 0, & |P| > r. \end{cases} \tag{10}$$

For the ISO(1,1) case ($\varepsilon = +1$) the irreducible representations T_r [8] are realized as operators on the space of smooth functions on the hyperbola ($\xi_1 = \cosh\mu$; $\xi_2 = \sinh\mu$; $\xi_1^2 - \xi_2^2 = 1$) with scalar product

$$(f_1, f_2) = \int_{-\infty}^{\infty} f_1(\mu) f_2^*(\mu)\,d\mu, \tag{11}$$

the operators being

$$x = i\ell \frac{\partial}{\partial \mu},$$
$$p = r\frac{1}{\ell}\sinh\mu,$$
$$\Im = r\cosh\mu. \tag{12}$$

More details on the physical consequences of this algebra will be given below.

The algebraic structure of non-commutative space–time and in particular the choice of the sign of ε has a strong effect on the spectrum. This is, for example, illustrated by the behavior of the energy levels in a strongly localizing potential. In the case $\ell = 0$, one knows that in a infinite square well of width Δ the energy levels E_n arising for

$$-\frac{1}{2m}\frac{d^2}{dx^2}\psi = E\psi, \tag{13}$$

with boundary conditions $\psi_n(-\Delta/2) = \psi_n(\Delta/2) = 0$, are

$$E_n = \frac{n^2\pi^2}{2m\Delta^2}, \tag{14}$$

with $n = 1, 2, \ldots$. In particular, the ground state energy E_0 diverges quadratically in the sharp localization limit $\Delta \to 0$.

In the case $\ell \neq 0$ and $\varepsilon = +1$, the equation that replaces (13) is

$$\frac{1}{2m}\frac{1}{4\ell^2}\left(e^{i\ell(d/dx)} - e^{-i\ell(d/dx)}\right)^2\psi = E\psi \tag{15}$$

leading, with the same boundary conditions, to the energy levels

$$E_n = \frac{1}{2m\ell^2}\sinh^2\left(\frac{n\pi\ell}{\Delta}\right), \tag{16}$$

with $n = 1, 2, \ldots$. Again, the ground state energy diverges in the sharp localization limit, but much more rapidly, and it coincides with (14) in the $\ell \to 0$ limit.

For $\ell \neq 0$ and $\varepsilon = -1$, the situation is rather different. First of all, we must require that Δ be an integral multiple of ℓ in this case, say $\Delta = k\ell$. The equation is

$$-\frac{1}{2m}\frac{1}{4\ell^2}\left(e^{\ell(d/dx)} - e^{-\ell(d/dx)}\right)^2\psi = E\psi, \tag{17}$$

with energy spectrum

$$E_n = \frac{1}{2m\ell^2}\sin^2\left(\frac{n\pi\ell}{\Delta}\right). \tag{18}$$

This time the energies are all finite, bounded above by $1/(2m\ell^2)$, independent of $\Delta = k\ell$. However, in this case n runs only from $n = 1$ to $n = k/2$ for a total of k states.

We begin by analyzing the consequences of the deformation on phase-space volume counting rules. That is, confine n fermions in a box of size Δ so that each one must occupy a different state. Let p_n be the magnitude of the momentum of the nth particle. Now add an $(n+1)$st particle, whose momentum is p_{n+1} in magnitude. The additional phase-space volume required to accommodate the new particle is

$$\Delta p \Delta x = (p_{n+1} - p_n)\Delta, \tag{19}$$

since in the box $\Delta x = \Delta$.

In the case $\ell = 0$, we have $p_n = \sqrt{2mE_n} = n\pi/\Delta$, and so

$$\Delta p = p_{n+1} - p_n = \frac{\pi}{\Delta}. \tag{20}$$

Therefore, in the case $\ell = 0$,

$$\Delta p \Delta x = \pi \tag{21}$$

independent of n, which is the usual phase-space volume counting rule.

For $\ell \neq 0$ and $\varepsilon = +1$, one easily computes from $p_n = \sqrt{2mE_n}$ that

$$\begin{aligned}\Delta p &= p_{n+1} - p_n \\ &= \frac{2}{\ell}\sinh\left(\frac{\pi\ell}{2\Delta}\right)\cosh\left(\frac{\pi\ell}{\Delta}\left(n + \frac{1}{2}\right)\right).\end{aligned} \tag{22}$$

Therefore, in this case,

$$\Delta p \Delta x = \frac{2\Delta}{\ell}\sinh\left(\frac{\pi\ell}{2\Delta}\right)\cosh\left(\frac{\pi\ell}{\Delta}\left(n + \frac{1}{2}\right)\right). \tag{23}$$

This reduces to the previous result in the limit $\ell \to 0$, but for $\ell \neq 0$, the required increase in phase-space volume, to add another particle, increases rapidly with n. When n is large, this effect can be significant even for very small values of ℓ.

Finally, for $\ell \neq 0$ and $\varepsilon = -1$, we again fix $\Delta = k\ell$, and in the same way we compute that at the nth energy level,

$$\Delta p \Delta x = 2k\sin\left(\frac{\pi}{2k}\right)\cos\left(\frac{\pi}{k}\left(n + \frac{1}{2}\right)\right). \tag{24}$$

Again, the phase-space volume increment depends on n, but in such a way that adding a particle at higher energy requires less and less phase-space volume.

Phase-space volume counting plays an important role in statistical mechanics, and it is conceivable that statistical mechanical considerations could lead to bounds on the possible values of ℓ and the sign of ε.

We now turn to the implications of the deformed non-commutative space–time algebra to the uncertainty principle and compare it with previous analysis and conjectures concerning modifications of this principle following from gravity and string theory.

Phase-space volume counting is directly connected with the uncertainty principle, and recently a number of authors have considered the introduction of a fundamental length through a modified uncertainty principle. Indeed, hints for the existence of a fundamental length had already appeared in string theory [9–12], as well as through considerations of the effect of gravitation on the measurement process [13,14]. This has led to the proposal of a generalized uncertainty principle

$$\Delta x \geqslant \frac{\hbar}{2\Delta p} + \frac{C}{4}\frac{\Delta p}{\hbar}, \tag{25}$$

C being a quantity proportional to the string tension or to the square of Planck's length. If Δp is finite, inequality (25) implies

$$\Delta x \geqslant \sqrt{\frac{C}{2}}, \tag{26}$$

that is, there would be a non-zero minimal length that can be probed with finite energy states. On the other hand, the statistical mechanics consequences of (25) have also been explored [15].

We now wish to relate the conjectured generalized uncertainty principle (25) with the results following from non-commutative space–time algebraic structure obtained by deformation theory. The algebraic structure of relation (25) has been studied by a number of authors [16–18]. Here we will refer in particular to the results of Kempf et al. [18]. Relation (25) is shown to follow from a commutation relation

$$[x,p] = i\left(1 + \frac{C}{2}p^2\right). \tag{27}$$

In fact, from the Schwartz inequality one has ($\hbar = 1$)

$$\Delta x\,\Delta p \geqslant \frac{1}{2}\left|i(x\psi, p\psi) - i(p\psi, x\psi)\right| \tag{28}$$

which, if the domain $D([x,p])$ of $[x,p]$ coincides with $D(x)\cap D(p)$, is equivalent to

$$\Delta x\,\Delta p \geqslant \frac{1}{2}\left|\langle [x,p]\rangle\right|. \tag{29}$$

Using (27), inequality (25) follows.

For purposes of comparison with (2), we note from (4) and (12) that, with $r = 1$,

$$[x,p] = i\left(1 + \varepsilon(\ell p)^2\right)^{1/2}. \tag{30}$$

Expanding in ℓp to leading order, we obtain

$$[x,p] \approx i\left(1 + \frac{\varepsilon}{2}(\ell p)^2\right) \tag{31}$$

which, in the case $\varepsilon = +1$, agrees with (27) if we identify $C = \ell^2$. Therefore the deformation parameter ℓ^2 is seen to play the same role as the squared Planck length or the string tension that appear in the generalized uncertainty relation (25). However, there are some fundamental differences. The first is that the commutation relations (27) do not correspond to a Lie algebra deformation. The spectral structure is also different. To understand this, consider an explicit symmetric operator realization of (27) by operators in $\mathbb{R}$, namely

$$p = p,$$
$$x = i\left(1 + \frac{C}{2}p^2\right)\frac{d}{dp} + i\frac{C}{2}p. \tag{32}$$

This x operator has normalizable eigenvectors

$$\psi(p) = \left(1 + \frac{C}{2}p^2\right)^{-1/2} \times \exp\left(\sqrt{\frac{2}{C}}a\tan^{-1}\left(\sqrt{\frac{C}{2}}p\right)\right), \tag{33}$$

with $x\psi = a\psi$. However, these states have infinite energy. The same happens of course for the generalized position eigenstates in the usual Heisenberg algebra. The important difference is that, contrary to the Heisenberg case, here these ψ states cannot be approximated, arbitrarily close, by finite energy states [18]. This is the reason for the upper bound (26) on Δx.

The situation in the deformed algebra (2), with $\varepsilon = +1$, is different. From (12) it follows that in any nontrivial representation (of the subalgebra) one has the relation

$$\Im = \left(1 + \ell^2 p^2\right)^{1/2}. \tag{34}$$

From this and inequality (29) one obtains the following uncertainty principle:

$$\Delta x\,\Delta p \geqslant \frac{1}{2}\left|\left\langle\left(1+\ell^2 p^2\right)^{1/2}\right\rangle\right|. \tag{35}$$

In leading ℓ^2 order it looks like (25), however the physical content is somewhat different. In particular, if the uncertainty principle (35) is used to compute partition functions as in [15], the phase-space measure will be $dx\,dp/\sqrt{1+\ell^2p^2}$ rather than $dx\,dp/(1+\beta p^2)$.

Eq. (35) surely implies that the overall position-momentum uncertainty grows when one probes a system with large momentum particles, as in (25). However, there is no lower bound on Δx if one is ready to accept a sufficiently large, but finite, uncertainty in Δp. This is already seen in Eq. (23). As another example consider a normalized Gaussian

$$\psi(\mu) = (2\pi\alpha)^{-1/4}\exp\left(-\frac{\mu^2}{4\alpha}\right). \tag{36}$$

Using representation (12) one computes the expectation values $(\psi, x\psi) = (\psi, x\psi) = 0$ and

$$\left(\psi, x^2\psi\right) = \frac{\ell^2}{4\alpha},$$
$$\left(\psi, p^2\psi\right) = \frac{1}{4\ell^2}\left(2e^{2\alpha}-1\right). \tag{37}$$

Therefore one sees that, by increasing α, Δx may be made arbitrarily small. However, Δp grows much faster than for the Heisenberg algebra, namely

$$\Delta x\,\Delta p = \frac{1}{4}\left(\frac{2e^{\alpha}-1}{\alpha}\right)^{1/2}, \tag{38}$$

to be compared with $\Delta x\,\Delta p = 1/2$ for the Heisenberg algebra.

In conclusion: From the deformed algebra in the $\varepsilon = +1$ case, one obtains a modified uncertainty relation (35) which contains the expected higher uncertainty associated to large momentum probes, but no non-zero lower bound on Δx.

For the $\varepsilon = -1$ case, as seen above, each space coordinate has a discrete spectrum, in units of ℓ. Here also the uncertainty principle suffers some modification, but it has more to do with the discrete nature of the spectrum than with this particular algebra. A similar situation already arises for the uncertainty relation between angle and angular momentum with eigenstates of angular momentum satisfying

$$\Delta L_z \Delta\phi = 0, \tag{39}$$

in apparent contradiction with Eq. (29). However, it does not contradict the (domain-correct) Eq. (28) which in this case is not equivalent to (29). In fact, by integration by parts, what is obtained, instead of (29), is [19,20]

$$\Delta L_z \Delta\phi \geqslant \frac{1}{2}\left|1-2\pi\left|\psi(2\pi)^2\right|\right|. \tag{40}$$

For the case $\ell \neq 0$, $\varepsilon = -1$, computing $\Delta x\,\Delta p$ for a localized state $e_n = (1/\sqrt{2\pi})e^{-in\theta}$ one obtains $\Delta x\,\Delta p = 0$, which does not contradict Eqs. (28) or (29), because $(e_n, \cos\theta\, e_n) = 0$.

This calculation, however, fail to convey the true physical meaning of the uncertainty relations which should be a statement about the minimal size of the phase-space cell that must be assigned to a quantum state. Therefore, a formulation of the uncertainty principle that applies both to continuous and to discrete spectrum may use, instead of the product $\Delta x\,\Delta p$, the product of the inverses of the density of states $\mu(x)^{-1}\mu(p)^{-1}$. For example, for a free particle quantized in a box of size L, the density of momentum eigenstates is $\mu(p) = L/2\pi$ and for each one of these states the density of particles in a unit length is $\mu(x) = 1/L$. Then

$$\mu(x)^{-1}\mu(p)^{-1} = 2\pi,$$

a reasonable statement about the average size of the phase-space cell.

On the other hand, for continuous spectrum, $\mu(x)^{-1} \sim \Delta x$, $\mu(p)^{-1} \sim \Delta p$, and the uncertainty principle would have its usual meaning.

For algebra (2) with $\varepsilon = -1$ the density of eigenstates $e^{-in\theta}$ of the position operator is $\mu(x) = \ell^{-1}$. On the other hand, the density of states in momentum space is obtained by integrating Eq. (10) over a unit interval around $p = \langle p\rangle = 0$ leading to $\mu(p) = \ell/\pi$. Then

$$\mu(x)^{-1}\mu(p)^{-1} = \pi.$$

References

[1] A. Andronov, L. Pontryagin, Dokl. Akad. Nauk USSR 14 (1937) 247.

[2] S. Smale, Amer. J. Math. 88 (1966) 491;
S. Smale, Bull. Amer. Math. Soc. 73 (1967) 747.
[3] M. Flato, Czech J. Phys. B 32 (1982) 472.
[4] L.D. Faddeev, Asia-Pacific Phys. News 3 (1988) 21;
L.D. Faddeev, in: Cerdeira, Lundqvist (Eds.), Frontiers in Physics, High Technology and Mathematics, World Scientific, 1989, pp. 238–246.
[5] R. Vilela Mendes, J. Phys. A: Math. Gen. 27 (1994) 8091.
[6] R. Vilela Mendes, Phys. Lett. A 210 (1996) 232.
[7] R. Vilela Mendes, J. Math. Phys. 41 (2000) 156.
[8] N.Ya. Vilenkin, A.U. Klimyk, Representation of Lie Groups and Special Functions, Vols. 1, 2, Kluwer, Dordrecht, 1991–1993.
[9] D.J. Gross, P.F. Mende, Nucl. Phys. B 303 (1988) 407.
[10] D. Amati, M. Cialfaloni, G. Veneziano, Phys. Lett. B 216 (1989) 41.
[11] K. Konishi, G. Paffuti, P. Provero, Phys. Lett. B 234 (1990) 276.
[12] T. Yoneya, hep-th/0004074.
[13] L.J. Garay, Int. J. Mod. Phys. A 10 (1995) 145.
[14] R.J. Adler, D.I. Santiago, gr-qc/9904026.
[15] M. Lubo, hep-th/0009162.
[16] M. Maggiore, Phys. Lett. B 319 (1993) 83.
[17] A. Kempf, J. Math. Phys. 35 (1994) 4483;
A. Kempf, J. Math. Phys. 37 (1996) 2121;
A. Kempf, J. Math. Phys. 38 (1997) 1347.
[18] A. Kempf, G. Mangano, R. Mann, Phys. Rev. D 52 (1995) 1108.
[19] D. Judge, Phys. Lett. 5 (1963) 189.
[20] K. Kraus, Z. Phys. 188 (1965) 374.