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# Structure-generating mechanisms in agent-based models

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**Abstract**

The emergence of dynamical structures in multi-agent systems is analysed. Three different mechanisms are identified, namely: (1) sensitive-dependence (in the agent dynamics) and convex coupling, (2) sensitive-dependence and extremal dynamics and (3) interaction through a collectively generated field. The dynamical origin of the emergent structures is traced back either to a modification, by interaction, of the Lyapunov spectrum or to multistable dynamics. © 2001 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Organization and structure are ubiquitous in natural phenomena and much of the scientific endeavor is aimed at the discovery of patterns in the raw data supplied by Nature. A pattern, when discovered, is useful either to obtain a compressed description of the phenomenon or to predict its outcome. Prediction through compressed descriptions is the way many living beings deal with the external world, ants included [1]. Patterns in stationary processes are mostly an information processing feature of the observer. However, for evolving composite systems, the emergence of collective patterns is also a determining factor in the process of coevolution.

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A great deal of research has been done in the past to understand what a pattern (or structure) is, how it is represented and how it may be used for prediction. This ranges from time series prediction (for reviews see Refs. [2,3]), to stochastic model identification [4–6], to dynamical system reconstruction [7,8], to coding [9,10] and to the quantitative characterization of the complexity of patterns [11–13]. References here cannot be exhaustive nor make justice to a very large body of interesting work. They are only representative of the type of work developed in each approach.

Concerning on how a pattern may be used for prediction, a good part of the work done in the past fits in the general scheme of Crutchfield's *computational mechanics* [14]. A different question, however, concerns the dynamical mechanisms by which collective structures arise on composite systems. And why their dynamical behavior is so different from the dynamics of the components when in isolation.

Cross and Hohenberg [15] discussed pattern formation in systems modelled by partial differential equations, by analyzing the instabilities of the homogeneous states. Near the instabilities the dynamics is described by amplitude equations which characterize the collective variables. Here a different approach is followed. Systems composed of many agents in interaction, each one having *simple dynamics*, are considered. No collective variables are defined in the model to begin with, and any collective structures that may be observed appear as emergent properties of the dynamical process.

The dynamics of each agent must contain enough dynamical freedom for interesting behavior to be obtained when the agents are put in interaction. Therefore what is meant by simple dynamics of the agents is that the dynamics is simple to describe in law, but not that it has simple orbits. In short, a dynamical law with small *sophistication* [16], but capable of generating orbits of high Kolmogorov complexity. A paradigmatic example is multiplication by $p$ (mod. 1) ($p = 2, 3, \ldots$).

$$x_{n+1} = p x_n \,(\text{mod.}\, 1)\,, \tag{1}$$

where (mod.1) means the fractional part. It has an invariant measure absolutely continuous with respect to Lebesgue, positive Lyapunov exponents and Kolmogorov entropy, as well as orbits of all types.

In a multi-agent system, a temporal or spatial structure is defined as a phenomenon which has a time or space scale much larger than the corresponding scales of the individual agent dynamics. Structures and other collective properties of multi-agent systems may be rigorously characterized by ergodic invariants [17–19]. A short summary of the main ergodic invariants used to characterize multi-agent systems is included in Appendix A. When a unique invariant measure controls the dynamics, the ergodic invariants provide an adequate characterization of the system behavior. However in multistable systems, many different measures must be taken into account and a different type of parameters must be used. This is also sketched in Appendix A.

In the same way as there is no unique way to characterize the complexity of dynamical systems, each feature requiring a different complexity parameter [11,13], one should not expect to find a unique universal mechanism responsible for all structure-generating effects. In this paper three different mechanisms are identified, of which some examples are studied, namely

(1) Sensitive-dependence and convex coupling.
(2) Sensitive-dependence and extremal dynamics.
(3) Interaction through a collectively generated field (multistability and evolution).

In some of these mechanisms an important role is played by sensitive-dependence to initial conditions, that is, by the fact that the individual agent dynamics has positive Lyapunov exponents, as in Eq. (1). Sensitive-dependence to initial conditions, that is, the existence of at least one positive Lyapunov exponent is the accepted definition of *chaotic behavior*. Here however I prefer to use the term *sensitive-dependence* rather than chaos to emphasize the fact that, although the dynamics of each agent is chaotic if it evolves in isolation, when in interaction their behavior may be quite different.

Of course, without interaction the system would have a degenerate positive Lyapunov spectrum. The interaction lifts the degeneracy and it is when different directions in phase-space acquire different separation dynamics that collective dynamical structures are created. In particular, when a Lyapunov exponent approaches zero from above, it creates a feature with a very long time scale. The modification of the Lyapunov spectrum arises either from varying interaction strength or from extremal dynamics, that is, the mechanism by which only the agent under the largest stress is allowed to evolve.

A remarkable exception to the above paradigm of Lyapunov spectrum modification occurs when there are no direct interactions between the agents which, instead, react to a collective variable, that they themselves create. In this case, sensitive-dependence of the agent dynamics influences the fluctuations, but self-organization and collective variables are controlled by evolution processes and by multistability of the dynamics.

## 2. Sensitive-dependence and convex coupling

Here the individual agent dynamics is assumed to have positive Lyapunov exponents. Without interaction, the system would have a degenerate spectrum of positive Lyapunov exponents. Convex coupling, as in Eq. (2), has a contractive effect. Therefore, for sufficiently large interaction strength, some of the Lyapunov exponents approach zero from above. Physically, the mechanism of convex coupling relates, for example, to a situation where there is a limitation on the range of options and influences that determine the actions of the agents. Then if an agent receives an influence from someone else there is a correspondent decrease of the influence of its own state in the future evolution. On the other hand the intensity of the interactions between the agents may depend on the history of past interactions or on the number of agents that occupy the same volume of space.

This mechanism will now be illustrated by one example. It deals with a situation where the number of interacting agents varies, according to a reproduction and death scheme, and the strength of interaction depends on the number of agents at any given time. Of particular interest is the population control effect of the correlations.

The model is a system of Bernoulli agents on a circle with nearest-neighbor interactions by convex coupling, namely

$$x_i(t+1) = (1-c)f(x_i(t)) + \frac{c}{2}(f(x_{i+1}(t)) + f(x_{i-1}(t))) \tag{2}$$

with $f(x) = 2x \pmod{1}$ and periodic boundary conditions. The agents are assumed to live in a limited space, the intensity of the coupling being a function of the total number $N$ of agents, for example

$$c = c_0(1 - \mathrm{e}^{-\alpha N}) \,. \tag{3}$$

With fixed coupling, this model was used in the past to illustrate the behavior of the ergodic invariants for self-organization [19] and also in several other studies of the dynamics of coupled map lattices.

Here the coupling becomes a dynamical variable as well, by a *reproduction and death* mechanism defined as follows:

- After each $R$ time cycles, the system is examined, agents which at that moment have $x_i > 0.5$ are coded 1 and those for which $x_i \leqslant 0.5$ are coded 0.
- Then, configurations 0110 are candidates for reproduction with probability $p_r$ and configurations 0000 are candidates for death with probability $p_m$.
- Reproduction is the transition $0110 \to 0X110$ with the state of the new agent $X$ being chosen at random in the interval $(0, 1)$.
- Death is the transition $0000 \to 000$.

It is clear that without coupling ($c = 0$) the two configurations 0110 and 0000 appear, on average, the same number of times and the variation of the population density depends only on the relative values of $p_r$ and $p_m$. With coupling the situation is different and the model shows how correlations, generated by coupling, influence the inter-agent evolution mechanism.

In this example a rigorous characterization is possible of the structures that develop through interaction. As explained in Ref. [19] this characterization is obtained through the computation of the Lyapunov exponents from which a *structure index* is constructed (see Appendix A). Further insight is obtained from conditional exponents as well, but they will not be used here. The Lyapunov exponents for the dynamical system in (2) are

$$\lambda_k = \log\{2(1-c) + 2c\cos(\theta_k)\} \tag{4}$$

with $\theta_k = 2\pi k/N$, $k = 0, \ldots, N-1$. They are all positive for $c < 0.5$ and when the coupling varies above this value one observes the crossing through zero of each individual Lyapunov exponent and successive changes in the structure of the system. That means that, for $c \neq 0$, each collective mode has a different probability, a collective mode being frozen each time a Lyapunov exponent reaches the zero value.

The eigenvectors corresponding to each exponent are $\{\mathrm{e}^{\mathrm{i}n\theta_k}\}$, $k = 0, \ldots, N-1$. Therefore

$$y_k = \frac{1}{N}\sum_{n=1}^{N}\cos(n\theta_k) \tag{5}$$

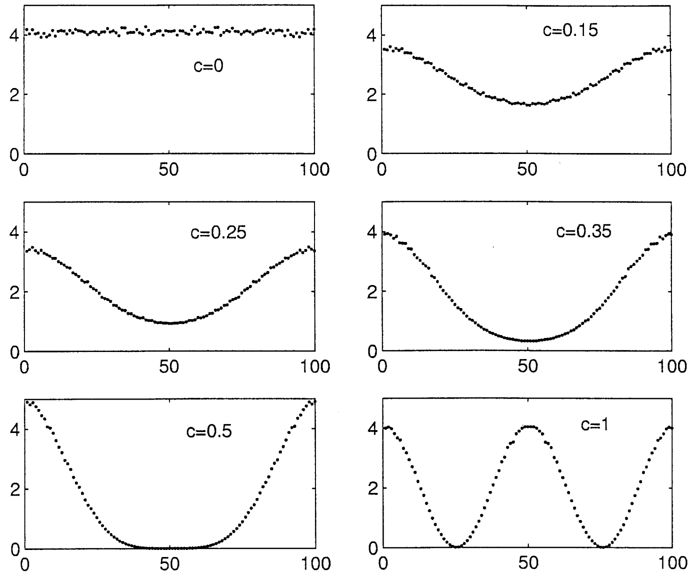


Fig. 1. Average energy of the collective modes for different coupling values.

are the coordinates of the collective eigenmodes. In Fig. 1 one shows the average energy $E_k = \langle y_k^2 \rangle$ of the collective modes for several values of the coupling (for a system with 100 agents). In all plots the mode $k = 0$, that is just the average over all agents, is not shown. The expected suppression and freezing of many modes is quite apparent. Notice that some modes that are suppressed for certain values of the coupling become restored for higher values. This is apparent, for example, for $c = 1$.

Fig. 2 shows the evolution of the population plotted against the reproduction–death cycle number. The probabilities are $p_r = 1$ and $p_m = 0.5$. This is a situation for which, without coupling, the population would grow indefinitely. However with the density-dependent coupling (3) the population becomes controlled with fluctuations around some average value. Three of the plots show this stabilization starting from different initial conditions. Even if the population stabilizing dynamical mechanism leads to a non-zero average value, a large fluctuation may lead to extinction, as shown in the last plot.

The population stabilizing mechanism is a consequence of the correlations that develop as a result of the unequal distribution of energy among the collective modes, caused by the coupling. Fig. 3 shows the relative probability of each one of the 16 different configurations of four neighbors $(x_1x_2x_3x_4)$, labelled by $x_1+2\times x_2+4\times x_2+8\times x_2$. As seen in the first plot, without coupling all configurations have the same probability.

Changes in the dynamical structure appear associated to the points where each Lyapunov exponent crosses zero. These are the points where the structure index diverges because the time scale associated to the vanishing Lyapunov exponent becomes infinite. In this sense these points are similar to statistical mechanics transition points. This

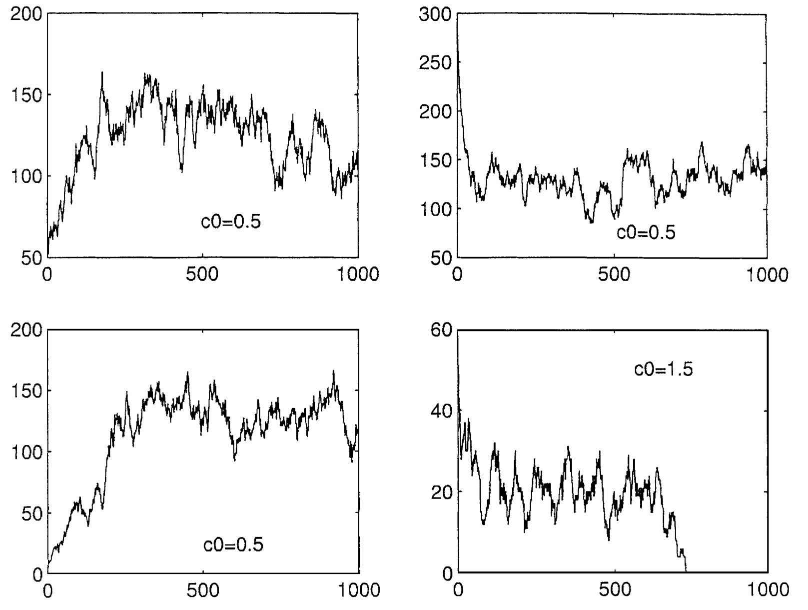


Fig. 2. Time evolution of the population.

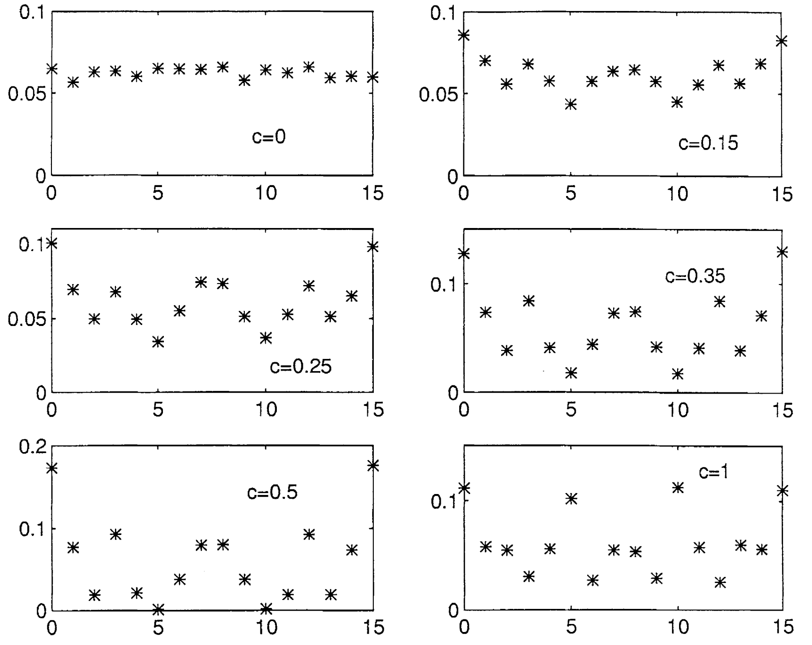


Fig. 3. Relative probability of four neighbor configurations.

transition is even more dramatic when a large number of Lyapunov exponents crosses zero simultaneously. This is, for example, the case for the globally coupled model studied in Ref. [17]. The transition points correspond to well defined (sharp) values of the parameters (coupling parameters, population density, etc.). A system may approach such points in the course of its life, by dynamical evolution of the population density, as seen above, or by environment changes. However, to generate a spontaneous approach to criticality, without fine tuning of the parameters, a different mechanism is required. As seen in the next section, extremal dynamics together with positive Lyapunov exponents of the individual dynamics is such a mechanism driving the system to the edge of criticality, that is, to the point where all Lyapunov exponents approach zero.

## 3. Criticality and extremal dynamics

Here one analyses the case where the individual dynamics is sensitive dependent, that is, it has at least one positive Lyapunov exponent, but the collective dynamics is of the extremal type. That means that, at each time step, only the agent under the largest stress is allowed to evolve. Depending on the specification of the dynamics, the largest stress may be the largest driving force or the largest or smallest value of a state variable. For example in the Bak–Sneppen [20] model the agent that evolves (together with its neighbors) is the one that has the smallest value of the state variable and in the train model [21,22] the agent that moves is the one that suffers the greatest driving force.

The prescription of extremal dynamics is a feature that simulates friction or resistance to change in the dynamical system. When parallel dynamics is replaced by this type of extremal dynamics, a dramatic effect takes place in the spectrum of Lyapunov exponents, in the limit of a large number $N$ of agents. For the computation of the Lyapunov exponents of the coupled system, instead of a tangent map matrix involving all the partial derivatives, one has now the product of matrices which have ones on the diagonal almost everywhere and only one non-trivial $r \times r$ block, $r$ being the number of neighbors that evolve at each time step. Therefore the Lyapunov exponents are obtained, on average, by the root $r/N$ of the $r \times r$ blocks. Therefore if the exponents of the $r \times r$ blocks are positive then, for large $N$, all the Lyapunov exponents approach zero from above, independently of any other characteristics of the dynamics. Hence sensitive-dependence of the individual dynamics plus extremal dynamics leads the system to the edge of criticality. Indeed, vanishing Lyapunov exponents means that there is no natural time scale for the separation dynamics. Recall that these are the points where the *structure index* diverges (see Appendix A).

Models of this type belong to the general class of *self-organized criticality* (SOC) [23] although not all SOC models that have been proposed display the above described mechanism in all its purity. At the end of the section a comment will be made about this.

A great deal of work on the SOC phenomenon has been done by many authors. Here I only want to emphasize those features that relate to the Lyapunov spectrum. For definiteness I will concentrate on a continuous version of the Bak–Sneppen model [19] which is a $C^\infty$-dynamical system defined as follows:

Let $\vec{x} \in [0,1)^N$ be the vector of coordinates of the agents and $\Gamma_i(\vec{x})$ the function

$$\Gamma_i \vec{x} = \prod_{j=i-n_V}^{j=i+n_V} \prod_{k \neq j} \left(1 + e^{\alpha(x_k - x_j)}\right)^{-1} . \quad (6)$$

For large $\alpha$ the function is nearly zero if $i$ is the index of one of the $(2n_V + 1)$ agents in the neighborhood of the minimum coordinate ($x_{\min}$) and is nearly one otherwise. The dynamics of the model is defined by

$$x_i(t+1) = \Gamma_i(\vec{x})x_i(t) + (1 - \Gamma_i(\vec{x}))f(x_i(t)) \quad (7)$$

with $f(x) = 2x$ (mod. 1). For $n_V = 1$ and in the limit of large $\alpha$ this is equivalent to the original Bak–Sneppen model [20]. Notice that the usual operation of finding the agent with the smallest barrier is replaced here by an infinitely differentiable operation and all the dynamical system techniques and results may be safely applied.

For large $\alpha$ the Lyapunov exponents are

$$\lambda \simeq \log(2)^{(2n_V+1)/N} \quad N \text{ times} . \quad (8)$$

Therefore, as stated before, as $N \to \infty$, $\lambda \to 0^+$. Hence, in this limit, there being no natural scale for the dynamics, exponential decay terms must disappear in all relaxation phenomena, leaving only the power law pre-factors.

This dynamical system displays some unusual features, related in particular to the nature of what has been improperly called its "attractor". In the $N \to \infty$ limit the one-agent probability density is uniform above 0.67 and zero below this threshold. The marginal density for any finite number $n$ of agents is the projection on a $n$-dimensional hyperplane of the $N$-dimensional hypercube of side $(1 - 0.67)$. This hypercube however is not an attractor because it carries zero or negligible (for finite $N$) measure. Also it is not a repeller because there are many neutral directions corresponding to the directions that are not neighbors to the minimum coordinate, not even a weak repeller because it is not an invariant set. I will call such a set a *ghost weak repeller*, sets of this type being characterized by the following conditions:

(i) existence of repelling and neutral directions,
(ii) zero or negligible measure,
(iii) full measure on the projection to hyperplanes up to dimension $N - k$, $N$ being the dimensionality of the system and $k$ a finite number.

Fig. 4 illustrates some of these properties for the model defined in Eq. (7) with $N = 100$, $n_V = 1$ and $\alpha = 1000$. The figure shows the one- and two-agent marginal distributions, the distribution of the distances to the hypercube of side $(1 - 0.67)$ and finally the scaling of the avalanches. The third plot emphasizes the negligible measure that is carried by the hypercube itself, to be compared with the structure of the marginal distributions.

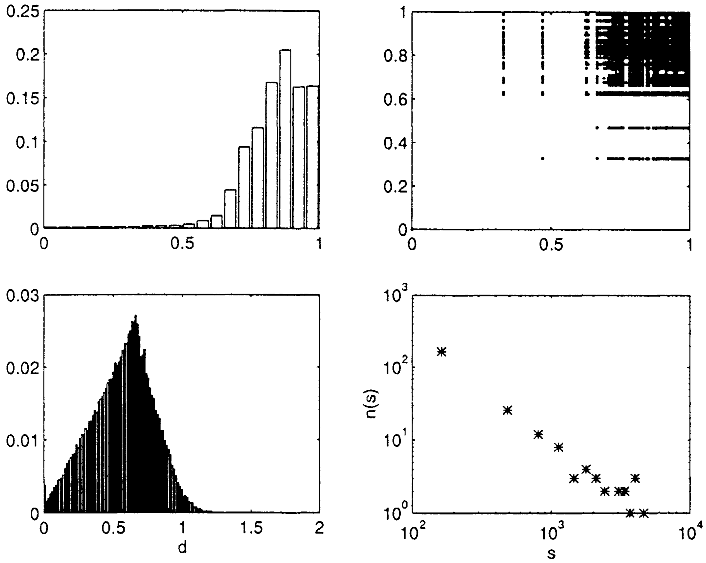


Fig. 4. One- and two-agent marginal distributions, distribution of distances to the hypercube and scaling of avalanches.

Whereas in the model discussed in the previous section, the creation of new structures, associated to the modification of the Lyapunov spectrum, occurred at particular values of the parameters, here no fine tuning of the parameters is needed, the system self-organizing itself spontaneously in a critical state. In the BS-model, as stated above, the establishment of the long-range temporal correlations is very naturally associated to the disappearance in the $N \to \infty$ limit of any characteristic time scale in the Lyapunov spectrum. Therefore the BS model is a very paradigmatic example of the SOC phenomenon in the sense that it becomes critical in the $N \to \infty$ limit, independently of any other details (value of $n_V$, function $f$ chosen for the dynamics of $x_{\min}$, etc.). This may not be the case for all SOC models that have been proposed.

For example, Zhang's model [24] was studied as a dynamical system by Blanchard et al. [25]. In their formalism, Zhang's model is a dynamical system of skew-product type with the first factor corresponding to the activation of one site and the second to the energy relaxation process. Their unit of time is not the natural unit of time, instead, each evolution step corresponds to an activation and a full avalanche. Therefore each step corresponds to different lengths of physical time, depending on the size of the avalanche. In their step units, there is one positive Lyapunov exponent ($\log N$) corresponding to the activation dynamics, the rest of the dynamics being contractive. Denote by $\bar{t}$ the average number of iteration steps (average duration of an avalanche), by $\bar{a}$ the average number of distinct relaxing sites in one avalanche and by $d$ the dimension of the lattice. Then, one estimates that the expanding Lyapunov exponent in physical time units is of order $(1/\bar{t})\log(N)$. If $\lambda_i$ is one of the contracting Lyapunov exponents, in physical units it is of order $\bar{a}(2d+1)\lambda_i/N$. Therefore all characteristic time scales will disappear only if $\bar{t}$ grows faster than $\log(N)$ when $N \to \infty$ and $\bar{a}$ grows slower than $N$ when $N \to \infty$. This means that in this model there may be ranges of parameters for which the SOC phenomenon is not observed. A similar conclusion is reached by the authors [25] using another approach.

In the discrete version of the train model proposed by Vieira [22] the activation dynamics, being the addition of a fixed quantity $\delta f$ to the first block, is neutral as far as the Jacobian is concerned. The only non-zero contribution to the Lyapunov exponent comes from the derivative of the updating function $\phi'$. It has been noticed by the author [22] that SOC behavior is only obtained if $|\phi'| > 1$. With the estimate $(1/N)\log|\phi'|$ for the global Lyapunov exponent, in the $N \to \infty$ limit, this case is seen to correspond, once again, to an approach to zero from above.

## 4. Interaction through collective variables. Multistability and evolution

In all models studied before, there is some sort of direct interaction between the agents giving rise to the collective behavior. Another class of models is the one where the interaction is mediated by a collective variable. On the other hand, the collective variable is an aggregate result of the state variables and actions of the agents. Therefore the agents react to an aggregate variable that they themselves create.

In most models of this type, in addition to the interaction through the collective variable, there is also an evolution mechanism which plays an important role in organizing the system. Therefore the dynamics of the system is a composition of two dynamical laws. One is the (fast) dynamics of interaction through the collective variable, the other the (slow) evolution dynamics. Sometimes it turns out that the essential mechanism self-organizing the system is the evolution mechanism, a slow dynamics, whereas the fast dynamics only provides the multi-attractor background which is selected by the slow evolution. As an example of this mechanism two models will be studied.

### *4.1. Coupled map minority model*

Inspired on Brian Arthur's bar model [26], models have been proposed [27–29] where the agents choose the value of a variable ($\pm 1$, for example) and those that are on the minority group win a point. In a continuous version [19], which is qualitatively equivalent to the discrete one, a fixed number $c$ between zero and one is chosen, which one calls *the cut*. The cut divides the interval $[0,1]$ into two parts. Then each agent chooses a value $x_i$ between 0 and 1, and the average $x_m = (1/N)\sum_i x_i$ is computed. The winning agents are those for which $x_i$ lie on the side opposite to $x_m$. That is, the payoff of agent $i$ at time $t$ is

$$P_i(t) = \tfrac{1}{2}(1 - sign\{(x_m(t) - c)(x_i(t) - c)\}) . \tag{9}$$

At each time $t$ the dynamics of agent $i$ is a function of the average value $x_m$ at time $t$ and of a parameter $\alpha_i$ that characterizes his *strategy*

$$x_i(t+1) = f_i(x_m(t), \alpha_i) . \tag{10}$$

Here one considers for the function $f_i$ either a shifted tent map

$$f_i(x) = 2 + 2x\, sign\left(\frac{1}{2} - (x+\alpha_i)\right) \ (\mathrm{mod.}\ 1) \tag{11}$$

or a shifted $p$-ary multiplication

$$f_i(x) = p(x+\alpha_i) \quad (\mathrm{mod.}\ 1) \tag{12}$$

$\alpha_i$ being a number between zero and one, a different one for each agent.

At first the strategies, that is the $\alpha_i$'s, are randomly chosen. Then after each $r$ time steps, $k$ agents have their strategies modified. The $k'$ agents with less earnings in that period choose new $\alpha$'s at random and the remaining $k$–$k'$ copy the $\alpha$'s of the $k$–$k'$ best performers with a small error. This is the evolution dynamics of this model, the fast dynamics being the one in Eq. (10). The variables of the full dynamical systems are $(x_i, \alpha_k)$, these variables being coupled by the interplay of fast and evolution dynamics.

In the discrete minority models originally proposed, each agent has several strategies at his disposal and at each time step he chooses the strategy with the best virtual record. The periodic replacement of the worst strategies by the best ones, used here, is qualitatively equivalent and, in addition, provides a clear separation between the two types of dynamical laws that operate in the model. In particular by changing the ratio $k/r$ one may explore different time scales for the (*fast*) agent dynamics and the (*slow*) renewal and copy dynamics. This clear separation between the two dynamics is important because, as will be seen, they play very different roles in the self-organization of the system.

The most interesting feature of the system dynamics is the fact that after a certain time it approaches a regime where the average value $x_m$ oscillates around the value of the cut $c$, even when $c$ is very different from the random value 0.5. Fig. 5 shows the typical behavior of the system for $c = 0.7$, the map $f_i$ being the tent map. The two upper plots show the approach of $x_m$ to the self-organized steady-state through several steps corresponding to the evolution cycles and the fluctuations of $x_m$ around the cut. The number of agents is $N = 100$, $k = r = 10$ and $k' = 3$. The two lower plots in the figure show the distributions of $x_m$ and of the fraction of winning agents

$$P = \frac{1}{N}\sum_i P_i . \tag{13}$$

The average value and standard deviation of $x_m$ are $\overline{x_m} = 0.694$ and $\sigma(x_m) = 0.02$, to be compared with $\overline{x_m} = 0.5$ and $\sigma(x_m) = 0.288$ that would be obtained for a uniform random choice of values between zero and one for the agent variables. The fact that $x_m$ is close to the cut maximizes the percentage of winning agents, which for the data in Fig. 5 is $\bar{P} = 0.488$ with $\sigma(P) = 0.132$.

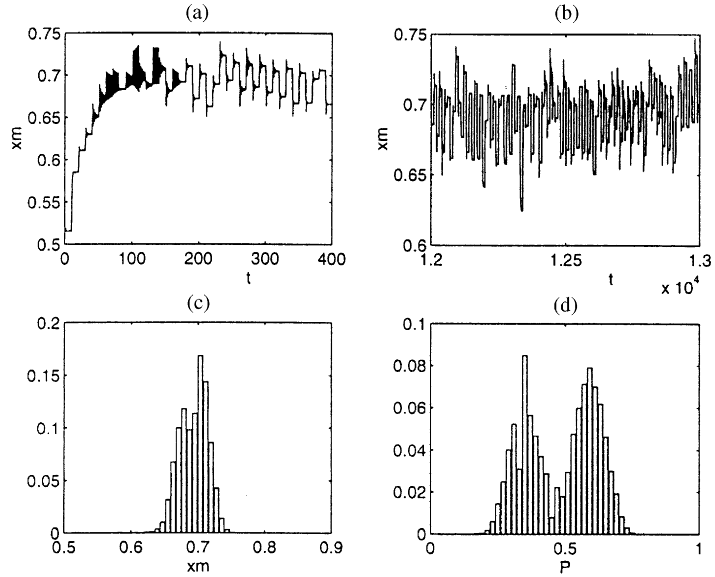


Fig. 5(a)–(d). The coupled map minority model (Shifted tent map).

The organization of the system's collective variable around the cut $c$ is easy to understand. Suppose that at a certain time $x_m < c$. Then, the evolution dynamics tends to copy the strategies of the agents that on average have $x_i > c$. This drives the average $x_m$ to higher values, closer to $c$. Conversely if $x_m > c$ the effect is just the opposite one. Hence $x_m$ tends to oscillate around $c$. For a minority model with agents having several strategies at their disposal, the choice of the strategies with the best virtual record has the same effect. Cavagna [30] pointed out the irrelevance of the memory size (the number of past time steps that the agents use in their strategies) and stated that the important issue is that all agents use the same collective information. A more accurate statement would be that the organization of the collective variable around $c$ depends only on the evolution dynamics, not on the details of the (fast) dynamics of the agents.

However, the dynamics of the agents, that is, the nature of their strategies, has an effect on the type of fluctuations around $c$. The Lyapunov exponents for the dynamics of $x_m$ and for the dynamics of the agents characterize these fluctuations. The dynamics of $x_m$ is

$$x_m(t+1) = \frac{1}{N}\sum_i f_i(x_m(t) + \alpha_i) \tag{14}$$

the Lyapunov exponent being

$$\lambda = \lim_{k\to\infty}\frac{1}{k}\log\left(\frac{1}{N}\left|\sum_i f_i'(x_m(t)+\alpha_i)\right|\cdots\frac{1}{N}\left|\sum_i f_i'(x_m(t+k)+\alpha_i)\right|\right) . \tag{15}$$

For the tent map, assuming, for a large number agents, a uniform distribution of the $\alpha$'s over the interval, $(1/N)|\sum_i f_i'|$ is of order $1/\sqrt{N}$, hence $\lambda$ is negative of order $-\frac{1}{2}\log N$. For the $p$-ary map $\lambda = p$.

For the dynamics of the agents the Jacobian matrix is

$$DT = \begin{pmatrix} \frac{1}{N}f_1' & \frac{1}{N}f_1' & \cdots & \frac{1}{N}f_1' \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{1}{N}f_N' & \frac{1}{N}f_N' & \cdots & \frac{1}{N}f_N' \end{pmatrix} . \tag{16}$$

The eigenvalues of $(DT^k)^T(DT^k)$ are $N-1$ zeros and one equal to

$$N\left(\frac{1}{N^2}\sum_i f_i'^2\right)\left(\frac{1}{N}\sum_i f_i'\right)^2\cdots\left(\frac{1}{N}\sum_i f_i'\right)^2 . \tag{17}$$

Therefore there is only one non-trivial Lyapunov exponent identical to the Lyapunov exponent of the $x_m$ dynamics.

It is the evolution dynamics that organizes the system, driving $x_m$ towards the cut. The fast dynamics controls the nature of the fluctuations around this value. For the tent map, all the Lyapunov exponents being negative, in the time intervals of duration $r$ between the evolution steps the dynamics settles down, at a fast rate, to a fixed point or periodic orbit. Nevertheless the behavior of the collective variable around its average value, as shown in Fig. 5 is quite irregular. The reason why this is compatible with the fast contraction, associated to negative Lyapunov exponents, is the sensitivity of the attractor to small changes on the agents strategies, that is, to the variables where the evolution dynamics acts. Therefore the $(x_i, \alpha_k)$-dynamical system is a multi-attractor system. For each fixed set of strategies and initial conditions the system converges rapidly to a period orbit. However small changes of the $\alpha$-variables, induced by the evolution process, change the attractor to which the system converges.

For the $p$-ary maps the existence of one positive Lyapunov exponent changes the nature of the fluctuations around the mean collective value, meaning that in this case the system is still a multi-attractor one, but the attractors are not necessarily periodic. This is shown in Fig. 6 where the same quantities as in Fig. 5 are plotted. The fluctuations now partly spoil the self-organization induced by evolution. The average values and standard deviations for the data in Fig. 6 are $\overline{x_m} = 0.554$, $\sigma(x_m) = 0.145$, $\bar{P} = 0.378$ and $\sigma(P) = 0.223$. As before, the evolution dynamics controls the collective behavior and acts as a selector of the attractors of the fast agent dynamics. The difference to the previous case is that here, rather than periodic orbits, one has non-periodic attractors.

In contrast to the mechanisms discussed before, where the approach towards $0^+$ of the positive Lyapunov exponents is the source of the self-organized collective variables, here it is a multi-attractor evolution mechanism that determines the emergence of such

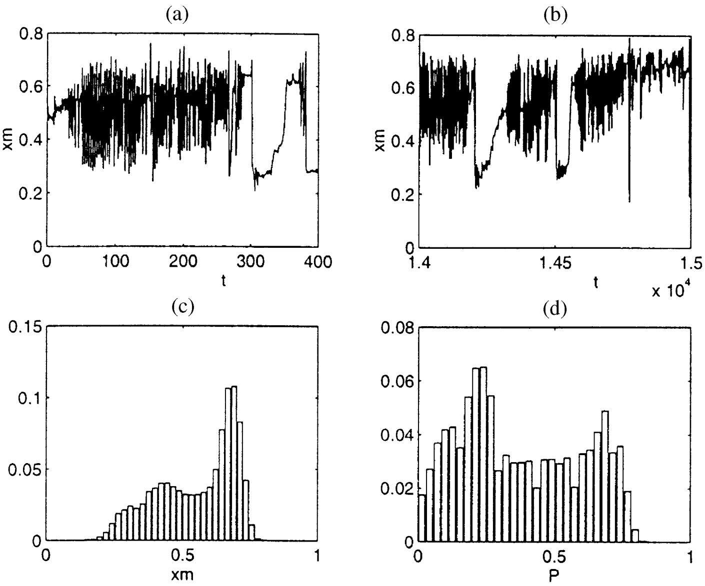


Fig. 6(a)–(d). The coupled map minority model (2-ary multiplication).

variables and, on the contrary, positive Lyapunov exponents may, to some extent, spoil the self-organization of the system.

### 4.2. *A market-like game*

In the minority model, as we have seen, the primary interaction between the agents takes place through an external collective variable, which they themselves create. In addition there is another dynamical (slower) mechanism namely the copy of the best strategies by the worst players or, alternatively the choice by each player of the best virtual strategy among a number of strategies put at their disposal. This paradigm is very much alike what happens in a market, where, among other things, investors react to the stock prices, which they themselves influence through their investments. At the same time they evolve in time trying to adjust their strategies in order to maximize their profits. Therefore, once again, we have an interaction through a collective variable and an evolution mechanism driven by the desire to maximize a cost function (the profit). From the lessons learned in the minority model, one is led to expect the collective variables to be controlled by the evolution mechanism with the agent dynamics providing the attractor background.

Many factors play a role in a real market. Here no attempt is made to take into account all the relevant factors, nor to build an accurate model of the market place. The objective is to isolate some of the mechanisms that presumably play a role in the market and, by stripping the model from other (inessential?) complications, to exhibit and understand the *purified* effect of these factors. In a real market, many factors, endogenous and exogenous, play a role and one should not expect to find such a clear cause–effect relationship between dynamical laws and actual behavior. Nevertheless, as in other branches of science, the splitting apart of the dynamical components of a phenomena, may improve its understanding [31].

We consider a set of investors playing *against* the market, that is they have some effect on an existing market that is influenced by other factors (other investors and general economic effects). This assumption implies that in addition to the impact function of this group of investors on the market, the rest of the impact is represented by a stochastic process. Therefore

$$z_{t+1} = f(z_t, \omega_t) + \eta_t \tag{18}$$

represents the change in the log price ($z_t = \log p_t$) with $\omega_t$ being the total investment made by the group of traders and $\eta_t$ the stochastic process that represents all the other effects.

In addition, no conservation law is assumed for the total amount of *stock* $s$ and *cash* $m$ detained by the group of traders. If $p_t$ is the price of the traded asset at time $t$, the purpose of the group of investors is to have an increase, as large as possible, of the total wealth $m_t + p_t \times s_t$ at the expense of the rest of the market.

For purposes of comparison with the minority model, here the collective variable $z$ plays the role of the average value $x_m$ and the difference between the initial wealth and the wealth at time $t$

$$\Delta_t = \sum_i \left( m_t^{(i)} + p_t \times s_t^{(i)} \right) - \sum_i \left( m_0^{(i)} + p_0 \times s_0^{(i)} \right) \tag{19}$$

plays the role of the total payoff $P$.

### 4.2.1. The market impact function

Let $p$ be the price of some asset, $z = \log(p)$ and $\omega_t$ the total sum of the buying and selling orders (in money units) for the asset. Buying orders are positive and selling ones negative. An important factor in the models is the effect of the magnitude of these orders on the price change of the asset, the so called *market impact function*. Let small orders have an impact according to the loglinear law [32,33]

$$z_{t+1} - z_t = \frac{\omega_t}{\lambda} + \eta_t \,. \tag{20}$$

The constant $\lambda$, called the *liquidity*, controls the volatility of the market. Eq. (20) corresponds naturally to a first order expansion and satisfies the condition

$$p(p(p_0, \omega^{(1)}), \omega^{(2)}) = p(p_0, \omega^{(1)} + \omega^{(2)}) \tag{21}$$

which one expects to be valid for small orders. However, as pointed out by Zhang [34] there is experimental evidence that this is not an accurate representation for large orders. Therefore a slightly different market impact function will be used. The reasoning used to motivate it, has some relation to Zhang's although the result is somewhat different.

When using Eq. (20) in a discrete dynamics model we are somehow neglecting the fact that the market takes different times to fulfill (and to react to) small and large orders. Therefore this should be taken into account when reducing the dynamics to a sequence of equal time steps. In particular the reaction of the market may be parametrized by a change in the $\lambda$ coefficient, which being related in first approximation to a random walk may vary by a factor proportional to $\sqrt{t}$. Taking the time $t$ to fill an order to be proportional to its size, one obtains

$$z_{t+1} - z_t = \frac{\omega_t}{\lambda_0 + \lambda_1 |\omega_t|^{1/2}} + \eta_t \,. \tag{22}$$

For small orders one recovers the loglinear approximation and for very large orders Zhang's square root law.

#### 4.2.2. The agent strategies

In first-order, two main types of informations are taken into account by the investors, namely the difference between price and perceived actual value (the misprice)

$$zv_t - z_t = \log(v_t) - \log(p_t) \tag{23}$$

and the variation in time of the price (the price trend)

$$z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}) \,. \tag{24}$$

One may also consider differences of prices going further back in time. However, the qualitative effect on the dynamics is basically the same and, in line with the main aim of isolating the fundamental constituents of the process, only these two pieces of informations will be considered. Consider now a non-decreasing function $f(x)$ such that $f(-\infty) = 0$ and $f(\infty) = 1$. Two useful examples are

$$f_1(x) = \theta(x) \,,$$
$$f_2(x) = \frac{1}{1 + \exp(-\beta x)} \,. \tag{25}$$

The information about misprice and price trend is coded on a four-component vector $\gamma$

$$\gamma_t = \begin{pmatrix} f(zv_t - z_t) f(z_t - z_{t-1}) \\ f(zv_t - z_t)(1 - f(z_t - z_{t-1})) \\ (1 - f(zv_t - z_t)) f(z_t - z_{t-1}) \\ (1 - f(zv_t - z_t))(1 - f(z_t - z_{t-1})) \end{pmatrix} \,. \tag{26}$$

The strategy of each investor is also a four-component vector $\alpha^{(i)}$ with entries $-1$, 0, or 1. $-1$ means to sell, 1 means to buy and 0 means to do nothing. Hence, at each time, the investment of agent $i$ is $\alpha^{(i)} \cdot \gamma$. A fundamental (value-investing strategy) that buys when the price is smaller than the value and sells otherwise would be $\alpha^{(i)} = (1, 1, -1, -1)$ and a pure trend-following (technical trading) strategy would be

$\alpha^{(i)} = (1, -1, 1, -1)$. In this setting the total number of possible strategies is $3^4 = 81$. For future reference the strategies will be labelled by a number

$$n^{(i)} = \sum_{k=0}^{3} 3^k \left( \alpha_k^{(i)} + 1 \right) . \tag{27}$$

Therefore the fundamental strategy is strategy no. 72 and the pure trend-following one is no. 60.

A factor that is sometimes considered in market models is the dependence of the strategies on the values of the prices not only at $t$ and $t-1$ but also on a larger set of past times. However, if the lesson that is learned from minority models also applies here, the size of the memory and the details of the agents strategies are not very important as far as the collective properties of the model are concerned. What seemed to be important was the copy mechanism in the evolution dynamics and the value of the Lyapunov exponents to control the fluctuations around (and away) from the value of the collective variable, the latter being mainly controlled by the evolution dynamics.

The evolution dynamics that is considered here for the market model is similar to the one in the minority model. After a number $r$ of time steps, $s$ agents copy the strategy of the $s$ best performers and, at the same time, have some probability to mutate that strategy. This evolution aims at attaining the goal of improving gains, while at the same time allowing for some renewal of the strategies. The percentage of each strategy changes in time and one may find whether some of them become dominating or stable and when this may occur.

Figs. 7–10 show the results of some simulations of the model with 100 agents. The parameters that were kept fixed are $r = 50$, $s = 10$, $\lambda_0 = 10{,}000$. The simulations differ by the choice of the initial conditions and the existence or non-existence of evolution. For Fig. 7 an initial condition is chosen with all traders in the fundamental strategy and evolution is activated. It is seen that on average the price follows value, although its fluctuations are amplified. In Figs. 7–10 the left upper plot shows the evolution of price minus value, the right upper plot the statistics of one-time step price increments and the left lower plot the evolution of $\sum_i (m_t^{(i)} + p_t \times s_t^{(i)})$. The right lower plot in Figs. 7–9 shows the time evolution of the strategies coded according to (27). The fundamental strategy is seen to be stable, in the sense that it becomes dominant, not being invaded by any other of the strategies that are created by the mutation process. There are however a few other strategies that, after being created, survive the selection process. This is true for example for the strategies $45 = (0, 1, -1, -1)$, $18 = (-1, 1, -1, -1)$, $73 = (1, 1, -1, 0)$ and $75 = (1, 1, 0, -1)$. These surviving strategies are however similar to the fundamental one. When there is dominance of the fundamental strategies, the price increments $dp$ have a Gaussian distribution. On the other hand the collective objective of increasing gains $\Delta_t$ (Eq. (19)) is achieved, as shown in the third plot of Fig. 7.

For the simulation of Fig. 8 the initial condition contains 50% of fundamental strategies (no. 72) and 50% of trend-following ones (no. 60). It is seen that the trend

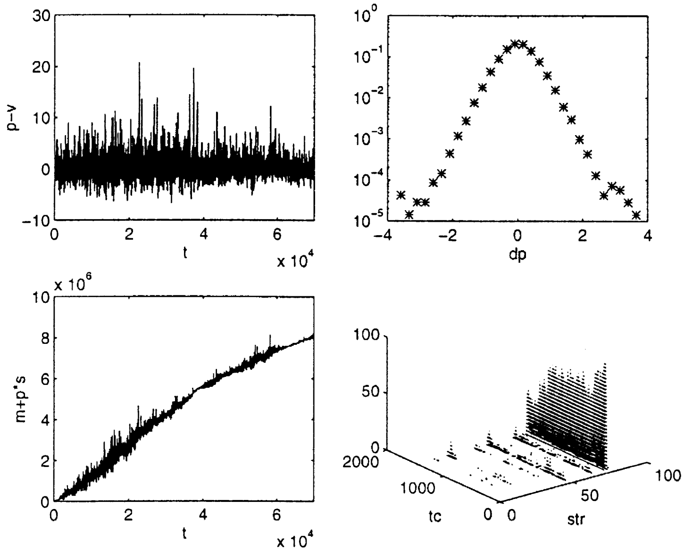


Fig. 7. Market game simulation with evolution. Initial condition: all traders in the fundamental strategy.

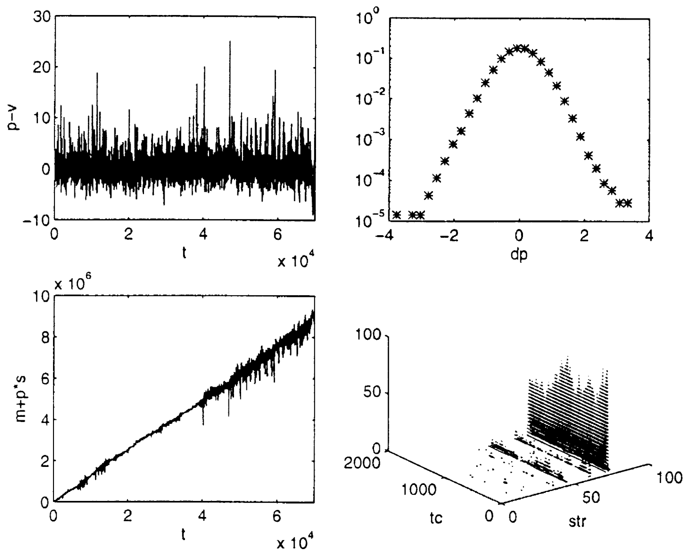


Fig. 8. Market game simulation with evolution. Initial condition: 1/2 fundamental and 1/2 trend-following.

following strategies do not survive the selection process and are eliminated, after a transient period. The statistically stable situation that is obtained is similar to the one shown in Fig. 7. However the fundamental strategy ceases to be stable if it occurs in the initial condition in smaller amounts ($\leqslant 40\%$). The dependence on the initial

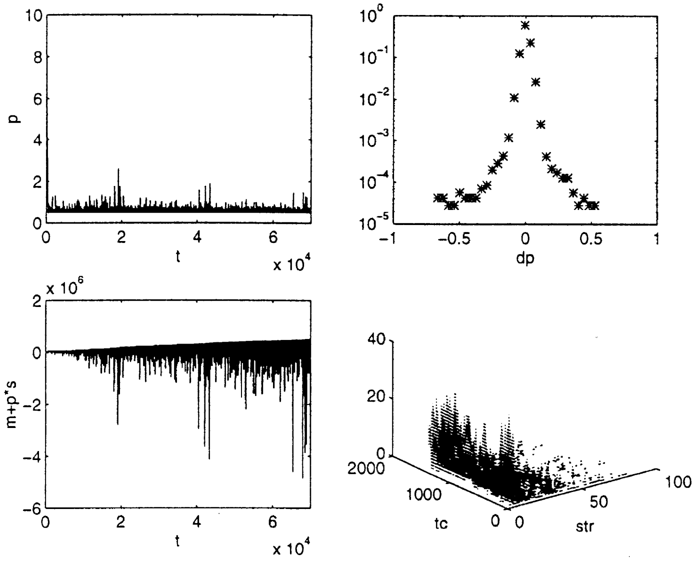


Fig. 9. Market game simulation with evolution. Initial condition: random strategies.

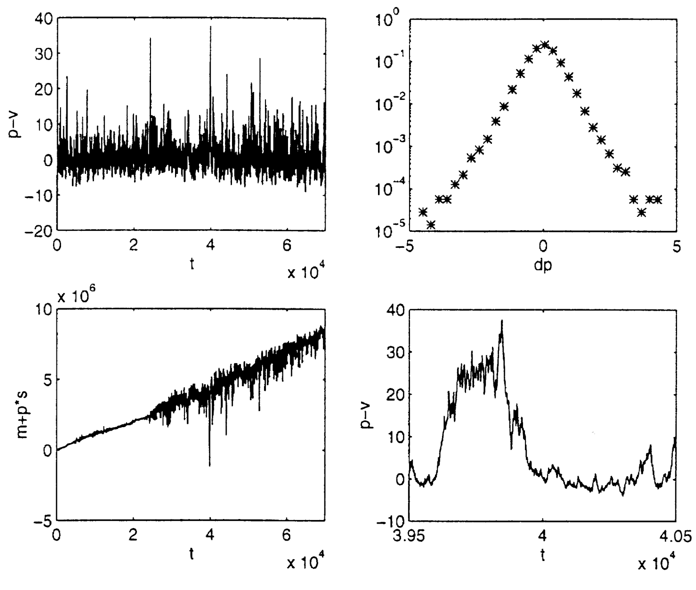


Fig. 10. Market game simulation without evolution. 1/2 fundamental of strategies and 1/2 trend-following.

condition is manifest in Fig. 9, where one starts for a completely random mixture of strategies in the initial condition. In this case, although the selection mechanism is still favoring at each evaluation cycle the best performers, the system never organizes itself to make $\Delta_t$ grow.

Finally the simulation in Fig. 10 is performed without evolution, with a fixed 50% of fundamental strategies (no. 72) and 50% of trend-following ones (no. 60). One sees in this case a large number of bubbles and crashes in the price evolution and the price increments distribution has fat tails. The right lower plot in Fig. 10 is an expanded view of the bubble around time step 39,800.

To understand the nature of the dynamics that leads to the results of the simulations is useful to compute the Lyapunov exponents for the log-price ($z_t$) dynamics. The Jacobian for the dynamics

$$\begin{pmatrix} z_t \\ z_{t-1} \end{pmatrix} \to \begin{pmatrix} z_{t+1} \\ z_t \end{pmatrix} \tag{28}$$

is

$$M_t = \begin{pmatrix} 1 + \dfrac{\partial}{\partial z_t} \dfrac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1 \left|\sum_i \omega^{(i)}\right|} & \dfrac{\partial}{\partial z_{t-1}} \dfrac{\sum_i \omega^{(i)}}{\lambda_0 + \lambda_1 \left|\sum_i \omega^{(i)}\right|} \\ 1 & 0 \end{pmatrix} \tag{29}$$

the Lyapunov spectrum being obtained from

$$\lim_{N \to \infty} \left| M_{t+N-1}^{\mathrm{T}} \cdots M_t^{\mathrm{T}} M_t \cdots M_{t+N-1} \right|^{1/2N} . \tag{30}$$

Lyapunov exponents were computed for $f = f_2$ (Eq. (25)) for several values of $\beta$ and a 50–50 admixture of fundamental and trend-following strategies. Typically, that is for a very large range of $\beta$, one obtains one Lyapunov number equal to zero and the other is $1 - \varepsilon$, $\varepsilon > 0$ being a small quantity. Therefore, typically, one has two negative Lyapunov exponents, one of them close to $0^-$.

Although more complex than the minority model, the similarities are very clear. In the minority model there is an *objective variable* (the payoff $P$) which drives the evolution and a *collective variable* $x_m$ to which the agents react in the short term. Here the objective variable that controls the evolution is $p \times s + m$ and the price is the collective variable. In both cases the average behavior of the collective variable is controlled by evolution and the fluctuations by the (fast) agent dynamics. Whether the objective variable reaches a stable behavior depends of course on the interaction between the two dynamical laws. Multi-stability of the coupled dynamics, rather than the Lyapunov spectrum, seems in both cases to be the main structure-generating mechanism.

## Appendix A. Some parameters characterizing the dynamics of multi-agent systems

In this appendix one collects the definitions and some properties of a few parameters which may be used to characterize in a quantitative manner the self-organization of multi-agent systems. Two different cases are considered. The first concerns systems where only one invariant measure controls the dynamics and the second is the case where many different measures come into play.

### A.1. Ergodic invariants

Let a dynamical system evolve on the support of some measure $\mu$ which is left invariant by the dynamics. An *ergodic invariant* is a dynamical characterization of this measure

$$I(\mu) = \lim_{T \to \infty} \frac{1}{T} \sum_{n=1}^{T} (f^n x_0) \tag{A.1}$$

for $x_0$ $\mu$-almost everywhere.

#### A.1.1. Lyapunov and conditional exponents

Let $f : M \to M$, with $M \subset R^m$, $\mu$ a measure invariant under $f$ and a splitting of $M$ induced by $\Sigma = R^k \times R^{m-k}$. The *conditional exponents* [35,36,17] are the eigenvalues $\xi_i^{(k)}$ and $\xi_i^{(m-k)}$ of the limits

$$\lim_{n \to \infty} (D_k f^{n*}(x) D_k f^n(x))^{1/2n} ,$$

$$\lim_{n \to \infty} (D_{m-k} f^{n*}(x) D_k f^n(x))^{1/2n} , \tag{A.2}$$

where $D_k f^n$ and $D_{m-k} f^n$ are the $k \times k$ and $m - k \times m - k$ diagonal blocks of the full Jacobian. For $k = m$, $\xi_i^{(m)} = \lambda_i$ are the *Lyapunov exponents*. Both the Lyapunov and the conditional exponents are ergodic invariants, with existence $\mu$-almost everywhere guaranteed by the conditions of Oseledec's multiplicative ergodic theorem, in particular the integrability condition

$$\int \mu(\mathrm{d}x) \log^+ \|T(x)\| < \infty . \tag{A.3}$$

$T$ being either the Jacobian or its $k \times k$ and $m - k \times m - k$ diagonal blocks. The set of regular points is Borel of full measure and

$$\lim_{n \to \infty} \frac{1}{n} \log \|D_k f^n(x) u\| = \xi_i^{(k)} \tag{A.4}$$

with $0 \neq u \in E_x^i / E_x^{i+1}$, $E_x^i$ being the subspace of $R^k$ spanned by eigenstates corresponding to eigenvalues $\leqslant \exp(\xi_i^{(k)})$.

#### A.1.2. Dynamical selforganization

Self-organization in a system concerns the dynamical relation of the whole to its parts. The conditional Lyapunov exponents, being quantities that separate the intrinsic dynamics of each component from the influence of the other parts in the system, provide a *measure of dynamical selforganization* $I(S, \Sigma, \mu)$

$$I(S, \Sigma, \mu) = \sum_{k=1}^{N} \{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\} \tag{A.5}$$

the sum being over all relevant partitions $R^k \times R^{m-k}$ and

$$h_k(\mu) = \sum_{\xi_i^{(k)} > 0} \xi_i^{(k)} ; \; h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)} > 0} \xi_i^{(m-k)} \; ; \; h(\mu) = \sum_{\lambda_i > 0} \lambda_i .$$

*A.1.3. Conditional entropies*

Consider cylindrical partitions adapted to the splitting $R^k \times R^{m-k}$,

$$\eta^{(k)} = \left\{ C_1^{(k)}, C_2^{(k)}, \dots \right\} ,$$

$$\eta^{(m-k)} = \left\{ C_1^{(m-k)}, C_2^{(m-k)}, \dots \right\} , \tag{A.6}$$

where $C_i^{(k)}$ and $C_i^{(m-k)}$ are $k$ and $m-k$-dimensional cylinder sets in $R^m$.

Let now $\zeta$ be a generator partition for the dynamics $\{f, \mu\}$. The *conditional entropies* associated to the splitting $R^k \times R^{m-k}$ are

$$\hbar^{(k)} = \sup_{\eta^{(k)}} \lim_{n \to \infty} \frac{1}{n+1} H\left(\zeta \vee f^{-1}\zeta \vee \cdots \vee f^{-n}\zeta | \eta^{(k)}\right) ,$$

$$\hbar^{(m-k)} = \sup_{\eta^{(m-k)}} \lim_{n \to \infty} \frac{1}{n+1} H\left(\zeta \vee f^{-1}\zeta \vee \cdots \vee f^{-n}\zeta | \eta^{(m-k)}\right) \tag{A.7}$$

$H(\chi|\eta)$ being

$$H(\chi|\eta) = -\int_{M/\eta} \sum_i \mu\left(C_i^{(\chi)}|\eta\right) \ln \mu\left(C_i^{(\chi)}|\eta\right) \mathrm{d}\mu . \tag{A.8}$$

That is, the conditional entropies are the supremum over all cylinder partitions of the sum of the conditional Kolmogorov–Sinai entropies.

*A.1.4. The structure index*

A structure (in a collective system) is a phenomenon with a characteristic scale very different from the scale of the component units in the system. A structure in space is a feature at a length scale larger than the characteristic size of the components and a structure in time is a phenomenon with a time scale larger than the cycle time of the individual components. A (temporal) *structure index* may then be defined by

$$S = \frac{1}{N} \sum_{i=1}^{N_s} \frac{T_i - T}{T} , \tag{A.9}$$

where $N$ is the total number of components (agents) in the coupled system, $N_s$ is the number of structures, $T_i$ is the characteristic time of structure $i$ and $T$ is the cycle time of the isolated components (or, alternatively the characteristic time of the fastest structure). A similar definition applies for a *spatial structure index*, by replacing characteristic times by characteristic lengths.

Structures are collective motions of the system. Therefore their characteristic times are the characteristic times of the separation dynamics, that is, the inverse of the positive Lyapunov exponents. Hence, for the temporal structure index, one may write

$$S = \frac{1}{N} \sum_{i=1}^{N_+} \left( \frac{\lambda_0}{\lambda_i} - 1 \right) \tag{A.10}$$

the sum being over the positive Lyapunov exponents $\lambda_i$. $\lambda_0$ is the largest Lyapunov exponent of an isolated component or some other reference value.

The temporal structure index diverges whenever a Lyapunov exponent approaches zero from below. Therefore the structure index diverges at the points where long time correlations develop.

More details on the construction of the ergodic invariants and their interpretation as relevant properties of multi-agent systems may be found in Ref. [19].

### A.2. Multi-stability parameters

If a dynamical system has multiple attractors for the same set of parameters, then each attractor has its own invariant measure. However in this case these measures are not of great practical interest. Instead, a global (Lebesgue) measure $\mu$ is defined in phase space, with respect to which the probability to be in the basin of attraction of each one of the attractors is computed. Several parameters may be used to characterize the multistable system.

One is the *number $n_A(N)$ of distinct attractors* as a function of the number $N$ of degrees of freedom (number of agents) of the system. Alternatively one may define the *scaling function for the number of attractors $g_n(N)$* such that

$$\lim_{N\to\infty} \frac{n_A(N)}{g_n(N)} = constant\ . \tag{A.11}$$

The diversity of possible dynamical behaviors when the initial conditions are chosen at random is characterized by the *attractor entropy*

$$S(N) = \sum_i \mu(b_i) \log \mu(b_i) \tag{A.12}$$

$b_i$ being the *basin of attraction* corresponding to the $i$ attractor. As in (A.11) a scaling function $g_A(N)$ may be defined for the entropy.

When a multistable system is perturbed, by noise or by fluctuations in the parameters, migration between attractors takes place which, in addition to the intensity of the perturbation, is strongly influenced by the stability of the attractors and by the nature of the boundaries of the basins of attraction. Given a metric in phase space, the stability of the attractors may be characterized by its *average strength $\bar{s}$* defined as the average of the minimum distances $d_{\min}(i)$ between the attractors and the boundary of their basins of attraction, scaled by average size of a basin of attraction

$$\bar{s} = \frac{1}{n_A(N)^{1-1/d}} \sum_i d_{\min}(i) \tag{A.13}$$

$d$ being the geometrical dimension of the phase space.

Another important factor controlling attractor migration is the *Hausdorff dimension of the basin boundaries*. If this dimension is high (in some cases it may approach $d$) the noise-perturbed system may spend most of the time in such a riddled boundary, without ever settling in any particular attractor. This situation leads to a high degree of unpredicability, even higher than the usual chaotic (positive Lyapunov exponent) regime.

A easier to measure characterization of the effect of attractor strength and basin boundary structure on the migration dynamics is the *mean first-passage time* $\bar{\tau}(\varepsilon)$ between attractors as a function of the noise intensity $\varepsilon$.

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