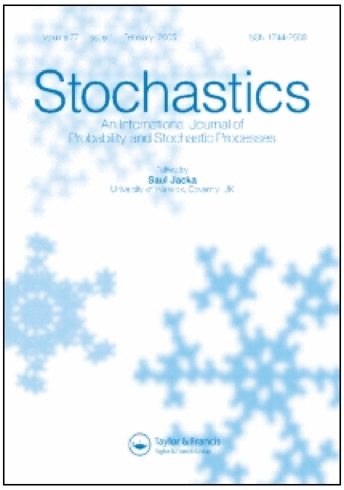

Stochastics An International Journal of Probability and Stochastic Processes
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713652481

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Online Publication Date: 01 June 2009

To cite this Article Vilela Mendes, R.(2009)'Stochastic solutions of some nonlinear partial differential equations',Stochastics An International Journal of Probability and Stochastic Processes,81:3,279 — 297

To link to this Article: DOI: 10.1080/17442500903080389

URL: http://dx.doi.org/10.1080/17442500903080389

Stochastics: An International Journal of Probability and Stochastics Processes
Vol. 81, Nos. 3–4, June–August 2009, 279–297

Stochastic solutions of some nonlinear partial differential equations

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(*Received 24 March 2008; final version received 6 February 2009*)

The stochastic construction of solutions for nonlinear partial differential equations is discussed. The technique provides new exact solutions and efficient numerical codes for localized solutions and parallel computing. A brief review of known solutions and some new results for the Poisson–Vlasov and Euler equations are included.

Keywords: stochastic solutions; KPP; superprocesses; Navier–Stokes; Poisson–Vlasov

AMS Subject Classification: 60H30; 35C15

1. Introduction: Stochastic solutions and their role

The solutions of linear elliptic and parabolic equations, both with Cauchy and Dirichlet boundary conditions, have a probabilistic interpretation. These are classical results which may be traced back to the work of Courant et al. [10] in the 1920s and became a standard tool in potential theory [3,4,6]. For example, for the heat equation

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) \quad \text{with} \;\; u(0,x) = f(x), \tag{1}$$

the solution may be written either as

$$u(t,x) = \frac{1}{2\sqrt{\pi}}\int \frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{2t}\right)f(y)\mathrm{d}y \tag{2}$$

or as

$$u(t,x) = \mathbb{E}_x f(X_t), \tag{3}$$

$\mathbb{E}_x$ meaning the expectation value, starting from x, of the process

$$\mathrm{d}X_t = \mathrm{d}W_t,$$

W_t being the Wiener process.

Equation (1) is a *specification* of the problem whereas (2) and (3) are *solutions* in the sense that they both provide algorithmic means of construction of a function satisfying the

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ISSN 1744-2508 print/ISSN 1744-2516 online

DOI: 10.1080/17442500903080389
http://www.informaworld.com

specification. An important condition for (2) and (3) to be considered as solutions is the fact that the algorithmic tools are independent of the particular solution, in the first case an integration procedure and in the second the simulation of a solution-independent process. This should be contrasted with stochastic processes constructed from a given particular solution, as has been done for example for the Boltzman equation [19].

In contrast with the linear problems, for nonlinear partial differential equations, explicit solutions in terms of elementary functions or integrals, are only known in very particular cases. However, if a solution-independent stochastic process is found that (for arbitrary initial conditions) generates the solution in the sense of Equation (3), a stochastic solution is obtained. In this way, the set of equations for which exact solutions are known would be considerably extended.

The exit measures provided by diffusion plus branching processes [12–16] as well as the stochastic representations recently constructed for the Navier–Stokes [5,21,28,29,34,35] and the Vlasov–Poisson equations [18,33] define solution-independent processes for which the mean values of some functionals are solutions to these equations. Therefore, they are exact stochastic solutions.

In the stochastic solutions, one deals with a process that starts from the point where the solution is to be found, a functional being then computed along the whole sample path or until it reaches a boundary. In addition to providing new exact results, the stochastic solutions are also a promising tool for numerical implementation. There are several reasons for why it is so. First, notice that whereas deterministic algorithms grow exponentially with the dimension d of the space, roughly N^d (L/N being the linear size of the grid), stochastic simulations only grow with the dimension of the process, typically of order d. In addition, because the stochastic representation processes always start from a definite point in the domain and paths starting from different points are independent from each other, these methods are quite efficient at computing localized solutions and are a natural choice for parallel and distributed implementation. Stochastic algorithms may also be used for domain decomposition purposes [1,2,31].

Because most stochastic solutions of nonlinear equations involve branching processes, in the numerical evaluation of a stochastic solution, care should be taken of large deviation phenomena. The fluctuations around the mean in a branching process are typically non-Gaussian. Therefore, a simple calculation of the standard deviation or other lower order momenta are not sufficient to check the reliability of the results. A large deviation analysis is recommended for numerical calculations using branching processes, which may be done by the empirical construction of the deviation function [30].

Stochastic solutions also provide an intuitive characterization of the physical phenomena, relating nonlinear interactions to cascading processes. By the study of exit times from a domain, they also provide access to quantities that cannot be obtained by perturbative methods [17,32].

One way to construct stochastic solutions is based on a probabilistic interpretation of the Picard series. The differential equation is written as an integral equation which is rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series.

Section 2 contains a brief review of some known stochastic solutions for nonlinear partial differential equations. This includes the McKean construction of a solution for the Kolmogorov–Petrovskii–Piskunov (KPP) equation, the superprocesses solution of the diffusion equation with nonlinearities of the type u^α ($\alpha \in (0, 2]$), the Navier–Stokes equation and the Fourier-transformed Poisson–Vlasov equation.

Then, in Sections 3 and 4 some new results are presented for the Poisson–Vlasov equation in configuration space and the Euler equation.

2. A review of stochastic solutions

In general, the stochastic solutions of nonlinear partial differential equations involve a mixture of two processes, either diffusion and branching or an exponential process and branching. The first contributions in this domain are very probably those of McKean [24–26].

2.1 The KPP equation

The KPP equation [20] is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^2 - u. \tag{4}$$

One rewrites it as an integral equation,

$$u(t,x) = \mathrm{e}^{-t}\mathrm{e}^{(1/2)t(\partial^2/\partial x^2)}u(0,x) + \int_0^t \mathrm{d}\tau \mathrm{e}^{-(t-\tau)}\mathrm{e}^{1/2(t-\tau)(\partial^2/\partial x^2)}u^2(\tau,x) \tag{5}$$

and iteration of this equation corresponds to the following process: At time $t = 0$, a single particle starts a Brownian motion from the origin continuing for an exponential holding time τ with $P(\tau > t) = \mathrm{e}^{-t}$. Then the particle splits into two new particles which continue along independent Brownian paths starting from the branching point. These new particles are subjected to the same branching rule, so that at time t one has k particles with probability $P(n = k) = \mathrm{e}^{-t}(1 - \mathrm{e}^{-t})^{k-1}$. Then the solution of (4) is

$$u(t,x) = \mathbb{E}\left[f(x+\chi_1)f(x+\chi_2)\cdots f(x+\chi_k)\right], \tag{6}$$

$x+\chi_1, x+\chi_2, \ldots, x+\chi_k$ being the coordinates of surviving particles at time t and

$$f(x) = u(0,x), \tag{7}$$

the initial condition. Some recent papers provide extensions to the probabilistic study of equations of this type [7,22,23,27] and to a generalization involving fractional derivatives [8].

2.2 Superprocesses and nonlinear pde's

A (X_t, ψ)-superprocess is a measure-valued process which is a model for the random evolution of a cloud of particles. Each particle follows a trajectory ruled by the X_t-process and dies at a random time leaving a random offspring with size regulated by the function ψ (Figure 1). At time t one has N_t particles located at the points $\{x_i(t)\}$. To study the process one looks at the measure

$$\mu_t = \frac{1}{N_t}\sum_i \delta_{x_i(t)}. \tag{8}$$

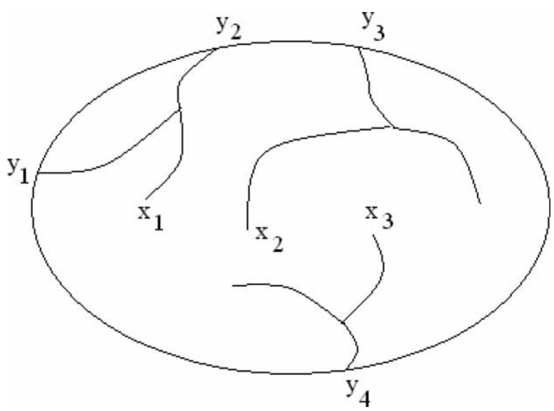

Figure 1. A sample path of the measure-valued (X, ψ)-superprocess.

To obtain the (X, ψ)-superprocess, one passes to the limit as the mass of each particle and its lifetime tends to zero and the number of particles tends to ∞ in such a way that the measure μ_t converges weakly to a measure in the domain D. The superprocess constructed in this way is the limit of a branching particle system [11,13].

An important concept is the *exit measure* (from the domain D). For the example in Figure 1, the initial measure μ is

$$\mu_0 = \frac{1}{3} \sum_{i=1}^{3} \delta_{x_i} \tag{9}$$

and the *exit measure* ν_D, given by the particles that reach the boundary, is

$$\nu_D = \frac{1}{4} \sum_{i=1}^{4} \delta_{y_i}.$$

In the literature one finds two notions of superprocesses, either as a system of mass distributions (measures) at each time t or as a system of exit measures from a domain. Notice that a mass distribution at fixed time t can also be interpreted as an exit measure from the space-time domain $(-\infty, t) \times D$.

For the particular case of *superdiffusions* X_t is a diffusion associated to an elliptic operator L. Superdiffusions provide representations for the positive solutions of the nonlinear equations

$$\begin{aligned} &Lu = \psi(u) \quad \text{in } D, \\ &u_{\partial D} = \varphi \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = Lu - \psi(u) \quad \text{in } [0, \infty) \times D, \\ &u(0, \cdot) = f(\cdot), \end{aligned} \tag{11}$$

φ being a bounded nonnegative function defined on ∂D. The solutions of these equations can be related to the Laplace transform of the measure-valued superprocesses.

Let $(\xi_t, t \geq 0)$ be a diffusion, $(P_t, t \geq 0)$ its transition semigroup and P_x the law of ξ starting at x. ψ is a C^2 function in $(0, \infty)$ which defines the branching mechanism. Results have been proven for ψ's of the form

$$\psi(u) = au + bu^2 + \int_0^\infty (\mathrm{e}^{-ru} - 1 + ru) n(\mathrm{d}r),$$

n being a Radon measure on $(0, \infty)$ satisfying $\int_0^\infty (r \wedge r^2) n(\mathrm{d}r) < \infty$.

In the linear problem with $Lu = 0$ in D and $u = \varphi$ in ∂D, the solution is obtained by starting the process at x and sampling the boundary condition at the point where the path hits the boundary. That is one starts with a point measure localized at x and integrates over the boundary with another point measure localized at the hitting point. Likewise for the nonlinear problems (10) and (11) one starts the superdiffusion with an initial measure $\mu_0 = \delta_x$ and then integrates the boundary condition with the exit measure $\mu_{\partial D}$ generated by the superprocess on the boundary. The solution of (10) is then given by

$$u(x) = -\ln \mathbb{E}^x \mathrm{e}^{-\langle \mu_{\partial D}, \varphi \rangle},$$

where $\mathbb{E}^x$ means $\mathbb{E}^{\delta_x}$.

Likewise for (11) one obtains

$$u(t, x) = -\ln \mathbb{E}^x \mathrm{e}^{-\langle \mu_t, f \rangle},$$

μ_t being now the mass distribution of the superprocess at time t. As stated before, the measure μ_t may also be interpreted as an exit measure of a space-time domain $(-\infty, t) \times D$.

For further results and a detailed study of superprocesses, see the monographs [15,16].

2.3 The Navier–Stokes equation

In the Navier–Stokes equation for incompressible fluids

$$\frac{\partial u_k}{\partial t} + \frac{\partial}{\partial x_j} u_k u_j = \nu \frac{\partial^2 u_k}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_k} + f_k, \tag{12}$$

$(\partial u_k / \partial x_k) = 0$, the convective nonlinearity $u \cdot \nabla u$ is of a type different from the ones studied in the superprocesses. Variational formulations of this equation in a stochastic context were developed. Nevertheless, a general stochastic representation for the solution of equations with nonlinearities of this type remained, for a long time, an open problem. The breakthrough was finally made on a paper by LeJan and Sznitman [21] further developed by other authors [5,28,29,34,35]. Transforming Equation (12) to the Fourier space

$$\hat{u}(\xi, t) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-ix \cdot \xi} u(x, t) \mathrm{d}x,$$

one obtains

$$\frac{\partial \hat{u}_k}{\partial t} + i(2\pi)^{-n/2}\xi_j(\hat{u}_k * \hat{u}_j) = -\nu|\xi|^2\,\hat{u}_k - i\xi_k\hat{p} + \hat{f}_k, \tag{13}$$

which by Leray projection, multiplication and division by $|\xi|$, $s \to t - s$ and assuming the initial data and the forcing to be divergence-free, may be rewritten in integral form

$$\begin{aligned}\hat{u}(\xi,t) = {} & \mathrm{e}^{-\nu|\xi|^2 t}\,\hat{u}_0(\xi) + \int_0^t \mathrm{d}s\mathrm{e}^{-\nu|\xi|^2 s}\Bigg\{\frac{|\xi|}{\left(\sqrt{2\pi}\right)^n} \\ & \int_{\mathbb{R}^n}\hat{u}(\eta, t-s)\otimes_\xi \hat{u}(\xi-\eta, t-s)\mathrm{d}\eta + \hat{f}(\xi, t-s)\Bigg\}\end{aligned} \tag{14}$$

with

$$a\otimes_\xi b = -i\left(\frac{\xi}{|\xi|}\cdot a\right)\left(b - \left(\frac{\xi}{|\xi|}\cdot b\right)\frac{\xi}{|\xi|}\right).$$

Now each one of coefficients in the equation is normalized to obtain probability measures. Define

$$\chi(\xi,t) = \frac{\hat{u}(\xi,t)}{h(\xi)};\quad \chi_0(\xi) = \frac{\hat{u}_0(\xi)}{h(\xi)};\quad m(\xi) = \frac{2|\xi|h*h(\xi)}{\nu|\xi|^2\left(\sqrt{2\pi}\right)^n h(\xi)}, \tag{15}$$

$$\phi(\xi,t) = \frac{2\hat{f}(\xi,t)}{\nu|\xi|^2 h(\xi)};\quad K_\xi(\eta, \xi-\eta) = \frac{h(\eta)h(\xi-\eta)}{h*h(\xi)}, \tag{16}$$

h being called a *majorizing kernel*. Then

$$\begin{aligned}\chi(\xi,t) = {} & \mathrm{e}^{-\nu|\xi|^2 t}\chi_0(\xi) + \int_0^t \mathrm{d}s\,\nu|\xi|^2\mathrm{e}^{-\nu|\xi|^2 s}\Bigg\{\frac{m(\xi)}{2} \\ & \int_{\mathbb{R}^n}\chi(\eta, t-s)\otimes_\xi\chi(\xi-\eta, t-s)K_\xi(\eta, \xi-\eta)\mathrm{d}\eta \quad + \frac{\phi(\xi, t-s)}{2}\Bigg\}.\end{aligned} \tag{17}$$

Two examples of majorizing kernels for three dimensions satisfying $h*h(\xi) \leq |\xi|h(\xi)$ are $h(\xi) = 1/\pi^3|\xi|^2$; $h(\xi) = (\beta/2\pi)(\mathrm{e}^{-\beta|\xi|}/|\xi|)$.

2.3.1 *Probabilistic interpretation of Equation (17)*

Let S be an exponentially distributed random variable with parameter $\nu|\xi|^2$, k a Bernoulli random variable with parameter 1/2, η_1 and η_2 a pair of correlated random variables distributed as $K_\xi(\mathrm{d}\xi_1, \mathrm{d}\xi_2)$. S, k and η_1, η_2 are assumed to be independent processes. Then

$$\begin{aligned}\chi(\xi,t) = {} & \mathbb{E}\big\{\chi_0(\xi)\mathbf{1}_{[S\geq t]} + \phi(\xi, t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[k=0]} \\ & + m(\xi)\chi(\eta_1, t-S)\otimes_\xi\chi(\eta_2, t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[k=1]}\big\} \\ = {} & \mathbb{E}\{V^{(1)}(\xi,t)\}.\end{aligned}$$

Iterating the random variable $V(\xi, t)$ one obtains in the limit (if it exists)

$$V^{(\infty)}(\xi,t) = \chi_0(\xi)\mathbf{1}_{[S\geq t]} + \phi(\xi, t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[k=0]} + m(\xi)V^{(\infty)}(\eta_1, t-S)\otimes_\xi V^{(\infty)}(\eta_2, t-S)\mathbf{1}_{[S<t]}\mathbf{1}_{[k=1]}$$

and

$$\chi(\xi,t) = \mathbb{E}\{V^{(\infty)}(\xi,t)\}.$$

2.3.2 *The stochastic process*

It is an exponential process plus a branching one, leading to a tree backward in time. Starting at t, the mode ξ propagates backwards in time with holding parameter $\nu|\xi|^2$. At the death of ξ, a Bernoulli coin is tossed. If $k = 0$, no new modes are born. If $k = 1$, two new modes η_1 and $\eta_2 = \xi - \eta_1$ are born with probability density $K_\xi(\eta_1, \eta_2)$. The process is repeated for the new modes with exponential lifetimes with parameters $\nu|\eta_1|^2$ and $\nu|\eta_2|^2$.

In the Figure 2, a sample path of the process is displayed. Input nodes are marked (O) and operational nodes ($\bullet$). At an input node at time $t^* > 0$, (ξ^*, t^*), the tree samples the forcing at that point $\phi(\xi^*, t^*)$. At an input node below $t = 0$, the tree samples the initial data $\chi_0(\xi)$. All the sample values are combined at the operational nodes by the $m(\xi)a\otimes_\xi b$ functional.

For the example shown in the figure, the contribution is

$$m(\xi)\{m(\eta_1)\chi_0(\eta_{11})\otimes_{\eta_1}\chi_0(\eta_{12})\}\otimes_\xi\phi(\eta_2, t - S_\theta - S_2),$$

the solution being an average over all trees.

Renormalizing the times, the branching process is identical to a Galton–Watson process which is known to terminate in finite time with probability one. The number of factors in the multiplicative functional is finite and with the bounds $|\chi_0(\xi)| \leq 1$ and

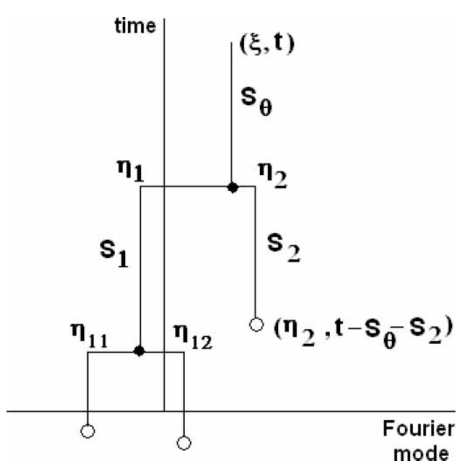

Figure 2. A sample path of the backwards in time branching process leading to a stochastic solution of Navier–Stokes.

$|\phi(\xi, t)| \le 1$ on the initial condition and forcing, together with a choice of majorizing kernel satisfying $h*h(\xi) \le |\xi|h(\xi)$, the multiplicative functional is bounded by one in absolute value almost surely.

Other stochastic approaches have been developed for the Navier–Stokes equation. In some cases, they provide elegant existence proofs. However, they follow a different point of view and the stochastic representations that are obtained cannot be classified as stochastic solutions in the sense described in the introduction. This is because the stochastic process that is used is not independent of the solution, instead it is obtained implicitly. For example, in the work of Constantin and Iyer [9], the drift term in the stochastic differential equation is computed implicitly as the expected value of an expression involving the flow it drives.

2.4 The Fourier-transformed Poisson–Vlasov equation

The (multi-species) Poisson–Vlasov equation in 3 + 1 space-time dimensions is

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \nabla_x f_i - \frac{e_i}{m_i} \nabla_x \Phi \cdot \nabla_v f_i = 0 \tag{18}$$

$(i = 1, 2)$, with

$$\Delta_x \Phi = -4\pi \left\{ \sum_i e_i \int f_i(\vec{x}, \vec{v}, t) d^3 v \right\}. \tag{19}$$

In the Navier–Stokes equation, the dissipation provides, through $\nu|\xi|^2$, a natural clock for the exponential process. In conservative equations like Euler or Vlasov, there is no natural such clock. We may however multiply the solution f by a invertible function of time $g(t)$ and write a stochastic representation for the product $g(t)f$. Another issue is the fact that in the Vlasov equation one needs to control the growth associated both to ∇_x and ∇_v. In Refs. [18,33], these issues were dealt with by the choice $g(t) = e^{-\lambda t}$ and a renormalization of time. Here, this construction will be reviewed as well as an alternative choice which avoids the renormalization of time at the cost of a more complex branching process. Notice that also for Navier–Stokes in two dimensions [28], multiplication by the factor $\exp(-\lambda t)$ is used to obtain a convergent stochastic representation of the solution.

Because of the localized nature of the stochastic solutions, as discussed in the introduction, solutions of both the Fourier-transformed and the configuration space equations are useful for the applications. If, in a plasma confinement device, one is interested in the behaviour of the solution at a particular point (for example at a point in the scrape-off layer) then it is the solution in configuration space that should be used. If however one is interested in the overall nature of the turbulent fluctuations it is probably the study of high Fourier modes in the Fourier-transformed equation that will be useful.

Fourier transforming Equations (18) and (19), with

$$F_i(\xi, t) = \frac{1}{(2\pi)^3} \int d^6\eta f_i(\eta, t) e^{i\xi \cdot \eta}, \tag{20}$$

$\eta = (\vec{x}, \vec{v})$ and $\xi = (\vec{\xi}_1, \vec{\xi}_2) \stackrel{\circ}{=} (\xi_1, \xi_2)$, one obtains

$$0 = \frac{\partial F_i(\xi, t)}{\partial t} - \vec{\xi}_1 \cdot \nabla_{\xi_2} F_i(\xi, t) + \frac{4\pi e_i}{m_i} \int d^3\xi_1' F_i(\xi_1 - \xi_1', \xi_2, t) \frac{\vec{\xi}_2 \cdot \vec{\xi}_1'}{|\xi_1'|^2} \sum_j e_j F_j(\xi_1', 0, t). \tag{21}$$

Changing variables to

$$\tau = \gamma(|\xi_2|)t, \tag{22}$$

where $\gamma(|\xi_2|)$ is a positive continuous function satisfying

$$\gamma(|\xi_2|) = 1 \qquad \text{if } |\xi_2| < 1$$
$$\gamma(|\xi_2|) \geq |\xi_2| \quad \text{if } |\xi_2| \geq 1$$

leads to

$$\frac{\partial F_i(\xi, \tau)}{\partial \tau} = \frac{\vec{\xi}_1}{\gamma(|\xi_2|)} \cdot \nabla_{\xi_2} F_i(\xi, \tau) - \frac{4\pi e_i}{m_i} \int d^3\xi_1' F_i(\xi_1 - \xi_1', \xi_2, \tau) \times \frac{\vec{\xi}_2 \cdot \hat{\xi}_1'}{\gamma(|\xi_2|)|\xi_1'|} \sum_j e_j F_j(\xi_1', 0, \tau) \tag{23}$$

with $\hat{\xi}_1 = (\vec{\xi}_1 / |\xi_1|)$. Equation (23) written in integral form, is

$$F_i(\xi, \tau) = e^{\tau(\vec{\xi}_1/\gamma(|\xi_2|)) \cdot \nabla_{\xi_2}} F_i(\xi_1, \xi_2, 0) - \frac{4\pi e_i}{m_i} \int_0^\tau ds e^{(\tau - s)(\vec{\xi}_1/(\gamma(|\xi_2|))) \cdot \nabla_{\xi_2}} \times \int d^3\xi_1' F_i(\xi_1 - \xi_1', \xi_2, s) \frac{\vec{\xi}_2 \cdot \hat{\xi}'_1}{\gamma(|\xi_2|)|\xi_1'|} \sum_j e_j F_j(\xi_1', 0, s). \tag{24}$$

A stochastic representation is obtained for the following function

$$\chi_i(\xi_1, \xi_2, \tau) = e^{-\lambda\tau} \frac{F_i(\xi_1, \xi_2, \tau)}{h(\xi_1)} \tag{25}$$

with λ a constant and $h(\xi_1)$ a positive function to be specified later on. The integral equation for $\chi(\xi_1, \xi_2, \tau)$ is

$$\chi_i(\xi_1, \xi_2, \tau) = e^{-\lambda\tau} \chi_i\left(\xi_1, \xi_2 + \tau \frac{\xi_1}{\gamma(|\xi_2|)}, 0\right) - \frac{8\pi e_i}{m_i \lambda} \frac{(|\xi_1|^{-1} h * h)(\xi_1)}{h(\xi_1)} \times \int_0^\tau ds \lambda e^{-\lambda s} \int d^3\xi'_1 p(\xi_1, \xi'_1) \chi_i\left(\xi_1 - \xi'_1, \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) \times \frac{(\xi_2 + s(\xi_1/\gamma(|\xi_2|))) \cdot \hat{\xi}'_1}{\gamma(|\xi_2 + s(\xi_1/\gamma(|\xi_2|))|)} \sum_j \frac{1}{2} e_j e^{\lambda(\tau - s)} \chi_j(\xi_1', 0, \tau - s) \tag{26}$$

with

$$(|\xi_1'|^{-1}h*h) = \int d^3\xi_1' |\xi_1'|^{-1} h(\xi_1 - \xi_1') h(\xi_1') \tag{27}$$

and

$$p(\xi_1, \xi_1') = \frac{|\xi_1'|^{-1} h(\xi_1 - \xi_1') h(\xi_1')}{(|\xi_1'|^{-1} h*h)}. \tag{28}$$

Equation (26) has a stochastic interpretation as an exponential process (with a time shift in the second variable) plus a branching process. $p(\xi_1, \xi_1')d^3\xi_1'$ is the probability that, given a ξ_1 mode, one obtains a $(\xi_1 - \xi_1', \xi_1')$ branching with ξ_1' in the volume $(\xi_1', \xi_1' + d^3\xi_1')$. $\chi(\xi_1, \xi_2, \tau)$ is computed from the expectation value of a multiplicative functional associated to the processes. Convergence of the multiplicative functional hinges on the fulfilling of the following conditions:

$$\text{(A)} \quad |F_i(\xi_1, \xi_2, 0)/h(\xi_1)| \le 1$$

$$\text{(B)} \quad (|\xi_1'|^{-1}h*h)(\xi_1) \le h(\xi_1)$$

. Condition (B) is satisfied, for example, for

$$h(\xi_1) = \frac{c}{(1 + |\xi_1|^2)^2} \quad \text{and} \quad c \le \frac{1}{3\pi}. \tag{29}$$

Indeed, computing $|\xi_1'|^{-1}h*h$ one obtains

$$\begin{aligned} c^2\Gamma(\xi_1) = & \ (|\xi_1'|^{-1}h*h)(\xi_1) \\ = & \ 2\pi c^2 \left\{ \frac{2\ln(1 + |\xi_1|^2)}{|\xi_1|^2(|\xi_1|^2 + 4)^2} + \frac{1}{|\xi_1|^2(|\xi_1|^2 + 4)} \right. \\ & \left. + \frac{|\xi_1|^2 - 4}{2|\xi_1|^3(|\xi_1|^2 + 4)^2} \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{2 - 2|\xi_1|^2}{4|\xi_1|} \right) \right) \right\}. \end{aligned} \tag{30}$$

Then $(1/h(\xi_1))(|\xi_1'|^{-1}h*h)$ is bounded by a constant for all $|\xi_1|$, and choosing c sufficiently small, condition (B) is satisfied. Once $h(\xi_1)$, consistent with (B) is found, condition (A) only puts restrictions on the initial conditions.

Because $e^{-\lambda\tau}$ is the survival probability during time τ of an exponential process with parameter λ and $\lambda e^{-\lambda s}ds$ the decay probability in the interval $(s, s + ds)$, $\chi_i(\xi_1, \xi_2, \tau)$ in Equation (26) is obtained as the expectation value of a multiplicative functional for the following backward-in-time process:

Starting at (ξ_1, ξ_2, τ), a particle of species i lives for an exponentially distributed time s up to time $\tau - s$. At its death a coin l_s (probabilities $1/2, 1/2$) is tossed. If $l_s = 0$, two new particles of the same species as the original one are born at time $\tau - s$ with Fourier modes $(\xi_1 - \xi_1', \xi_2 + s(\xi_1/\gamma(|\xi_2|)))$ and $(\xi_1', 0)$ with probability density $p(\xi_1, \xi_1')$. If $l_s = 1$, the two new particles are of different species. Each one of the newborn particles continues its backward-in-time evolution, following the same death and birth laws. When one of the particles of this tree reaches time zero it samples the initial condition. The multiplicative functional of the process is the product of the following contributions:

- At each branching point where two particles are born, the coupling constant is

$$g_{ij}(\xi_1, \xi_1', s) = -\mathrm{e}^{\lambda(\tau-s)} \frac{8\pi e_i e_j}{m_i \lambda} \frac{(|\xi_1'|^{-1} h * h)(\xi_1)}{h(\xi_1)} \frac{(\xi_2 + s(\xi_1/\gamma(|\xi_2|)))\cdot\hat{\xi}_1'}{\gamma(|\xi_2 + s(\xi_1/\gamma(|\xi_2|))|)} \tag{31}$$

- When one particle reaches time zero and samples, the initial condition the coupling is

$$g_{0i}(\xi_1, \xi_2) = \frac{F_i(\xi_1, \xi_2, 0)}{h(\xi_1)}. \tag{32}$$

The multiplicative functional is the product of all these couplings for each realization of the process $X(\xi_1, \xi_2, \tau)$, this process being obtained as the limit of the following iterative process:

$$\begin{aligned} X_i^{(k+1)}(\xi_1, \xi_2, \tau) = {} & \chi_i\left(\xi_1, \xi_2 + \tau\frac{\xi_1}{\gamma(|\xi_2|)}, 0\right)\mathbf{1}_{[s>\tau]} + g_{ii}(\xi_1, \xi_1', s) \\ & \times X_i^{(k)}\left(\xi_1 - \xi_1', \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) X_i^{(k)}(\xi_1', 0, \tau - s)\mathbf{1}_{[s<\tau]}\mathbf{1}_{[l_s=0]} \\ & + g_{ij}(\xi_1, \xi_1')X_i^{(k)}\left(\xi_1 - \xi_1', \xi_2 + s\frac{\xi_1}{\gamma(|\xi_2|)}, \tau - s\right) \\ & X_j^{(k)}(\xi_1', 0, \tau - s)\mathbf{1}_{[s<\tau]}\mathbf{1}_{[l_s=1]}. \end{aligned} \tag{33}$$

Then $\chi_i(\xi_1, \xi_2, \tau)$ is the expectation value of the functional.

$$\chi_i(\xi_1, \xi_2, \tau) = \mathbb{E}\{\Pi(g_0 g_0' \cdots)(g_{ii} g_{ii}' \cdots)(g_{ij} g_{ij}' \cdots)\} \tag{34}$$

For example, for the realization in Figure 3 the contribution to the multiplicative functional is

$$\begin{aligned} & g_{ji}(\xi_1, \xi_1', \tau - \tau_1) g_{ij}(\xi_1 - \xi_1', \xi_1'', \tau_1 - \tau_2) g_{ii}(\xi_1', \xi_1''', \tau_1 - \tau_3) \\ & \times g_{0i}(\xi_1' - \xi_1''', k_3) g_{0i}(\xi_1''', 0) g_{0j}(\xi_1'', 0) g_{0i}(\xi_1 - \xi_1' - \xi_1'', k_2) \end{aligned} \tag{35}$$

and

$$\begin{aligned} k &= \xi_2, \\ k_1 &= k + (\tau - \tau_1)\frac{\xi_1}{\gamma(|\xi_2|)}, \\ k_2 &= k_1 + (\tau_2 - \tau_1)\frac{(\xi_1 - \xi_1')}{\gamma(|k_1|)}, \\ k_3 &= (\tau_3 - \tau_1)\xi_1'. \end{aligned} \tag{36}$$

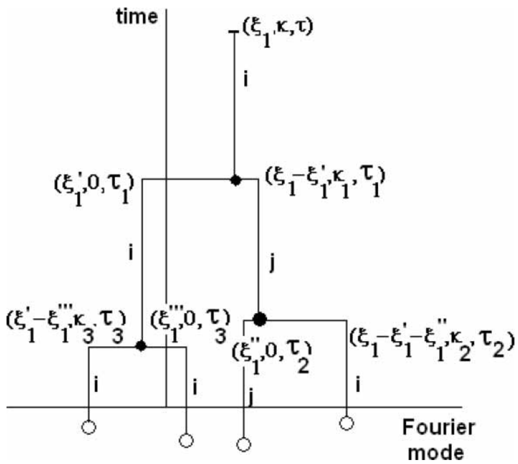

Figure 3. A sample path of the stochastic process $X(\xi_1, \xi_2, \tau)$.

With the conditions (A) and (B), and choosing $\lambda \geq |8\pi e_i e_j/\min_i\{m_i\}|$ and $c \leq \mathrm{e}^{-\lambda\tau}(1/3\pi)$, the absolute value of all coupling constants is bounded by one. The branching process, being identical to a Galton–Watson process, terminates with probability one and the number of inputs to the functional is finite (with probability one). With the bounds on the coupling constants, the multiplicative functional is bounded by one in absolute value almost surely. Once a stochastic representation is obtained for $\chi(\xi_1, \xi_2, \tau)$, one also has, by (25), a stochastic solution of the Fourier-transformed Poisson–Vlasov equation and one obtains:

Theorem 1. The stochastic process $X(\xi_1, \xi_2, \tau)$, above described, with kernel $h(\xi_1)$ satisfying condition (B), provides a stochastic solution for the Fourier-transformed Poisson–Vlasov equation $F_i(\xi_1, \xi_2, t)$ for any arbitrary finite value of the arguments, provided the initial conditions at time zero satisfy the boundedness conditions (A).

Instead of renormalizing the time (Equation (22)) and defining $\chi_i(\xi_1, \xi_2, \tau) = \mathrm{e}^{-\lambda\tau}(F_i(\xi_1, \xi_2, \tau)/h(\xi_1))$, one might write a stochastic representation for

$$\Theta_i(\xi_1, \xi_2, t) = \mathrm{e}^{-t|\xi_2|}\frac{F_i(\xi_1, \xi_2, t)}{h(\xi_1)}, \tag{37}$$

which leads to the integral equation

$$\begin{aligned}\Theta_i(\xi_1, \xi_2, t) = {} & \mathrm{e}^{-t|\xi_2|}\Theta_i(\xi_1, \xi_2 + t\xi_1, 0) - \frac{8\pi e_i N(\xi_1, \xi_2, t)}{m_i}\frac{(|\xi_1|^{-1}h * h)(\xi_1)}{h(\xi_1)} \\ & \times \int_0^t \mathrm{d}s \Pi(\xi_1, \xi_2, s) \int d^3\xi'_1 p(\xi_1, \xi'_1)\Theta_i(\xi_1 - \xi'_1, \xi_2 + s\xi_1, t - s) \\ & \times \frac{(\xi_2 + s\xi_1)\cdot\hat{\xi}'_1}{|\xi_2 + s\xi_1|}\sum_j \frac{1}{2} e_j \Theta_j(\xi'_1, 0, t - s).\end{aligned} \tag{38}$$

The probability density $p(\xi_1, \xi_1')$ and the conditions on $h(\xi_1)$ are the same as before. The main different is the survival probability of the propagating particles, namely $e^{-t|\xi_2|}$ is the survival probability up to time t and $ds\Pi(\xi_1, \xi_2, s)$ the dying probability in time ds, with

$$\Pi(\xi_1, \xi_2, s) = \frac{|\xi_2 + s\xi_1| e^{(t-s)|\xi_2+s\xi_1|-t|\xi_2|}}{N(\xi_1, \xi_2, t)}, \tag{39}$$

the normalizing function being obtained from

$$N(\xi_1, \xi_2, t) = \frac{1}{1 - e^{-t|\xi_2|}} \int_0^t ds |\xi_2 + s\xi_1| e^{(t-s)|\xi_2+s\xi_1|-t|\xi_2|}. \tag{40}$$

The stochastic process is qualitatively the same as before and for the convergence of the solution functional one requires $|\Theta_i(\xi_1, \xi_2, 0)| \le 1$ and the constant c in the definition of the majorizing kernel $h(\xi_1)$ to satisfy condition (29) and $|8\pi c e_i N(\xi_1, \xi_2, t)/m_i| \le 1$.

3. The Poisson–Vlasov equation in configuration space

Let $G_i(\vec{x}, \vec{v}, t)$ be

$$G_i(\vec{x}, \vec{v}, t) = e^{-\lambda t} \frac{f_i(\vec{x}, \vec{v}, t)}{\varphi_i(\vec{x} - t\vec{v}, \vec{v})}, \tag{41}$$

$\varphi(\vec{x}, \vec{v})$ being a function to be specified later. From (18) and (19), one obtains the following integral equation

$$\begin{aligned} G_i(\vec{x}, \vec{v}, t) = {} & e^{-\lambda t} G_i(\vec{x} - t\vec{v}, \vec{v}, 0) - 2\sum_j \frac{1}{2} \frac{e_i e_j}{m_i \lambda} \int_0^t ds \lambda e^{-\lambda s} A^{(j)}_{x,v,t,s} e^{\lambda(t-s)} \\ & \times \int d^3x' d^3u p^{(j)}_{x,v,t,s}(\vec{x}', \vec{u}) G_j(\vec{x}', \vec{u}, t-s) \left(\widehat{\vec{x} - s\vec{v} - \vec{x}'} \right) \\ & \times \frac{1}{\varphi_i(\vec{x} - t\vec{v}, \vec{v})} (\nabla_v + s\nabla_x) \varphi_i(\vec{x} - t\vec{v}, \vec{v}) G_i(\vec{x} - s\vec{v}, \vec{v}, t-s) \end{aligned} \tag{42}$$

with

$$p^{(j)}_{x,v,t,s}(\vec{x}', \vec{u}) = \frac{1}{A^{(j)}_{x,v,t,s}} \frac{\varphi_j(\vec{x}' - u(t-s), \vec{u})}{|\vec{x} - s\vec{v} - \vec{x}'|^2}, \tag{43}$$

$A_{x,v,t,s}$ being the normalization constant

$$A^{(j)}_{x,v,t,s} = \int d^3x' d^3u \frac{\varphi_j(\vec{x}' - u(t-s), \vec{u})}{|\vec{x} - s\vec{v} - \vec{x}'|^2}. \tag{44}$$

The simplest choice for the functions $\varphi_i(\vec{x},\vec{v})$ is

$$\varphi_i(\vec{x},\vec{v}) = f_i(\vec{x},\vec{v},0) \tag{45}$$

the initial conditions of the Cauchy problem. Then, the probabilistic interpretation requires finiteness of

$$A^{(j)}_{x,v,t,s} = \int d^3x' d^3u \frac{f_j(\vec{x}' - u(t-s),\vec{u},0)}{|\vec{x} - s\vec{v} - \vec{x}'|^2}, \tag{46}$$

a quantity that has the nature of a retarded field intensity generated by the initial condition.

From Equation (42), one concludes that a stochastic solution of the configuration space Poisson–Vlasov equation is obtained by the following $Y(\vec{x},\vec{u},t)$ process:

Starting at $(\vec{x},\vec{u},t)$, a particle of species i propagates backwards in time for an exponentially distributed time s_1 up to time $t - s_1$. At this time, the particle is at the position $\vec{x} - s_1\vec{v}$ and a new particle of the same or of the opposite type (probabilities $1/2, 1/2$) is born at position $\vec{x}'$ and velocity $\vec{u}$ with probability distribution $p^{(j)}_{x,v,t,s}(\vec{x}',\vec{u})$. The original particle receives a label (s_1) and continue its propagation for another exponentially distributed time s_2 up to time $t - s_1 - s_2$. There the process is repeated, that is, it receives a label (s_2) and a new particle is born at $\vec{x}'', \vec{u}'$. Each one of the newborn particles follows a similar rule, until all particles reach time zero (Figure 4).

The solution is obtained by a multiplicative functional

$$G_i(\vec{x},\vec{v},t) = \mathbb{E}\{\Pi(g_1 g_2 \cdots)(t_1 t_2 \cdots)\}, \tag{47}$$

the $g's$ being the coupling constants at the creation of new particles and the $t's$ the terminal contribution of the particles that reach time zero.

- The coupling constants at the creation of each new particle is

$$g_{ij}(x,v,t,s) = \frac{2e_i e_j}{m_i \lambda} A^{(j)}_{x,v,t,s} e^{\lambda(t-s)}. \tag{48}$$

- The terminal contribution of a particle that in the course of its evolution has received

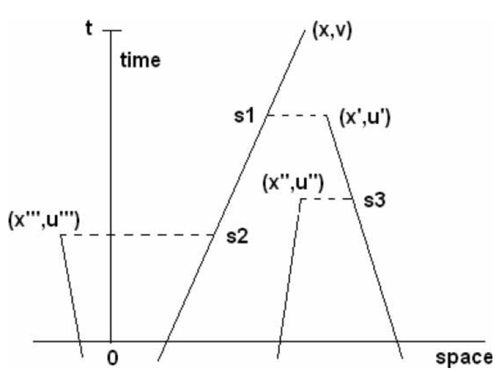

Figure 4. A sample path of the process $Y(\vec{x},\vec{u},t)$.

the labels $s_1, s_2, \ldots, s_n$ is

$$\frac{1}{f_i(\vec{x}-t\vec{v},\vec{v},0)}(\nabla_v+s_1\nabla_x)(\nabla_v+s_2\nabla_x)\cdots(\nabla_v+s_n\nabla_x)f_i(\vec{x}-t\vec{v},\vec{v},0). \tag{49}$$

Convergence of the multiplicative functional in the stochastic solution (47) requires

$$\left|\frac{2e_ie_j}{\min(m_i)\lambda}\max_s\left(A^{(j)}_{x,v,t,s}\right)\mathrm{e}^{\lambda(t-s)}\right|\leq 1 \tag{50}$$

and

$$\left|\frac{1}{f_i(\vec{x}-t\vec{v},\vec{v},0)}(\nabla_v+s_1\nabla_x)(\nabla_v+s_2\nabla_x)\cdots(\nabla_v+s_n\nabla_x)f_i(\vec{x}-t\vec{v},\vec{v},0)\right|\leq 1 \tag{51}$$

that is satisfied for sufficiently small and smooth initial conditions. Notice however that these are somewhat more restrictive conditions than for the stochastic solution of the Fourier-transformed equation. The result is summarized in the following:

THEOREM 2. The stochastic process $Y(\vec{x},\vec{u},t)$, above described, provides a stochastic solution for the configuration space Poisson–Vlasov equation provided the initial conditions satisfy the constraints (50) and (51).

Provided conditions similar to (50) and (51) are satisfied, other choices for the regularizing functions $\varphi_i(\vec{x},\vec{v})$ are possible. The identification of $\varphi_i(\vec{x},\vec{v})$ with the initial conditions is the choice that provides the simplest form for the multiplicative functional (47). Notice however that to obtain the solution $f_i(\vec{x},\vec{v},t)$ of the Poisson–Vlasov equation from $G_i(\vec{x},\vec{v},t)$ in all points of the domain, one should restrict to nowhere vanishing initial conditions or to approximate $f_i(\vec{x},\vec{v},0)$ by a sequence of nowhere vanishing functions.

4. The Euler equation

The construction of the stochastic solution in this section is similar to the one for the Fourier transformed Poisson–Vlasov equation. Consider a Euler equation in 3 + 1 space-time dimensions

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u=\nabla p \tag{52}$$

for an incompressible fluid, $\nabla\cdot u=0$. Passing to the Fourier transform

$$\hat{u}(\xi,t)=\frac{1}{(2\pi)^{3/2}}\int u(x,t)\mathrm{e}^{i\xi\cdot x}\mathrm{d}x, \tag{53}$$

one obtains

$$\frac{\partial\hat{u}_k(\xi,t)}{\partial t}-\frac{i}{(2\pi)^{3/2}}\sum_{j=1}^{3}\xi_j(\hat{u}_j*\hat{u}_k)(\xi,t)=-i\xi_k\hat{p}(\xi),\quad k=1,2,3, \tag{54}$$

$\hat{p}$ being the Fourier transform of the pressure p.

Change the time variable to

$$\tau = \gamma(|\xi|)t, \tag{55}$$

$\gamma(|\xi|)$ being a function with the same properties as the one defined in (22). Then

$$\frac{\partial \hat{u}_k(\xi,\tau)}{\partial \tau} - \frac{i}{(2\pi)^{3/2}} \sum_{j=1}^{3} \frac{\xi_j}{\gamma(|\xi|)} (\hat{u}_j * \hat{u}_k)(\xi,\tau) = -i \frac{\xi_k}{\gamma(|\xi|)} \hat{p}(\xi), \quad k = 1,2,3 \tag{56}$$

.

A stochastic representation is written for

$$\chi(\xi,\tau) = \mathrm{e}^{-\lambda\tau} \frac{\hat{u}(\xi,\tau)}{h(\xi)} \tag{57}$$

with λ a constant and $h(\xi)$ a positive majorizing kernel. Let us denote by P_ξ the orthogonal projection on the space $< \xi >^{\perp}$, more precisely

$$P_\xi v = v - < v, \hat{\xi} > \hat{\xi}$$

$\hat{\xi} = \xi/|\xi|$. Since $\nabla \cdot u = 0$, $\hat{u} \in < \xi >^{\perp}$. Applying the operator to the Equation (56), one obtains the following vectorial integral equation

$$\begin{aligned} \chi(\xi,\tau) =& \mathrm{e}^{-\lambda\tau}\chi(\xi,0) - \frac{i(h*h)(\xi)}{\lambda(2\pi)^{3/2}\, h(\xi)} \int_0^\tau \mathrm{d}s \lambda \mathrm{e}^{-\lambda s} \\ & \int_{\mathbb{R}^3} \chi(\eta,\tau-s) \otimes_\xi \chi(\xi-\eta,\tau-s) e^{\lambda(\tau-s)} q(\xi,\eta) d\eta, \end{aligned} \tag{58}$$

where

$$q(\xi,\eta) = \frac{h(\xi-\eta)h(\eta)}{(h*h)(\xi)} \tag{59}$$

and

$$a \otimes_\xi b = P_\xi \left(\frac{\xi}{\gamma(|\xi|)} \cdot ab \right). \tag{60}$$

Equation (58) has a stochastic interpretation as an λ-exponential process plus a branching process. $q(\xi,\eta)d\eta$ is the probability that, given a ξ mode, one obtains a $(\xi-\eta,\eta)$ branching with η in the volume $(\eta, \eta + d\eta)$. $\chi(\xi,\tau)$ is computed from the expectation value of a multiplicative functional associated to the processes. Convergence of the multiplicative functional is obtained by imposing the following conditions:

$$(C) \quad \left| \frac{\hat{u}(\xi,0)}{h(\xi)} \right| \leq 1,$$

$$(D) \quad (h*h)(\xi_1) \leq h(\xi_1)(2\pi)^{3/2} \lambda \mathrm{e}^{-\lambda\tau}.$$

In three dimensions a function $h(\xi_1)$ satisfying (D) is, for example

$$h(\xi_1) = \sqrt{\frac{2}{\pi}} \frac{\lambda \mathrm{e}^{-\lambda\tau}}{(1 + |\xi_1|^2)^2}.$$

As before the solution is obtained by the following process, denoted $Z(\xi, \tau)$: Starting at (ξ, τ), a particle lives for an exponentially distributed time s up to time $\tau - s$. At its death two new particles are born at time $\tau - s$ with Fourier modes $\xi - \eta$ and η with probability density $q(\xi, \eta)$. Each one of the new born particles continues its backward-in-time evolution, following the same death and branching laws. When one of the particles of this tree reaches time zero it samples the initial condition. The multiplicative functional of the process has the following contributions:

- At each branching point

$$m(\xi, s) = \frac{ih{*}h(\xi)}{(2\pi)^{3/2} h(\xi)} \frac{\mathrm{e}^{\lambda(\tau - s)}}{\lambda}. \tag{61}$$

- When one particle reaches time zero and samples the initial condition

$$\chi_0(\xi) = \chi_0(\xi, 0) = \frac{\hat{u}(\xi, 0)}{h(\xi)}. \tag{62}$$

The multiplicative functional is the composition of all these factors for each realization of the process $Z(\xi, \tau)$, obtained as the limit of the iterative process

$$Z^{(k+1)}(\xi, \tau) = \chi(\xi, 0)\mathbf{1}_{[s>\tau]} + m(\xi, s) Z^{(k)}(\xi - \eta, \tau - s) Z^{(k)}(\eta, \tau - s)\mathbf{1}_{[s<\tau]}.$$

Then, $\chi(\xi, \tau)$ is the expectation value of the functional obtained by composing the contributions (61) and (62) with the $\otimes_\xi$ product defined in (60)

$$\begin{aligned}\chi(\xi, \tau) = \ & \mathbb{E}\Big\{ m(\xi, s)\{m(\eta_1, s_1)\cdots\chi_0(\eta_{k+1})\otimes_{\eta_k}\chi_0(\eta_{k+2})\}\} \\ & \otimes_\xi \{m(\eta'_1, s'_1)\cdots\chi_0(\eta'_{k+1})\otimes_{\eta'_k}\chi_0(\eta'_{k+2})\}\Big\}.\end{aligned}$$

THEOREM 3. The stochastic process $Z(\xi, \tau)$ provides a stochastic solution (up to a finite time t) for the Fourier-transformed Euler equation provided conditions (C) and (D) are satisfied.

Note

1. http://www.label2.ist.utl.pt/vilela/.

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