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Signal reconstruction by random sampling in chirp space

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Abstract Reconstruction by irregular sampling and sampling at rates below the Nyquist rate are important issues for the experimental characterization of dynamical systems. For the space of functions that can be approximated by chirps, we prove a reconstruction theorem by random sampling at arbitrary rates.

Keywords Random sampling · Chirps

1 Introduction

Reconstruction of functions from observation of a discrete time series is the typical method for the experimental characterization of the dynamical systems occurring in Nature [1–3]. For functions that contain no frequencies higher than Ω, regularly spaced sampling at rates above $\frac{1}{2\Omega}$ (the Nyquist rate) provides an asymptotically rigorous reconstruction through Shannon's theorem or its extensions [4–6]. However, sampling at irregular intervals is sometimes imposed by the nature of the phenomena being observed, as in the case of geological measurements, anemometry or radar signals. Reconstruction of the signal from such measurements poses special problems, because the usual spectral estimation methods are designed to work with equidistantly spaced time series. Special methods have therefore been developed to deal with the problem of irregular sampling of band-limited functions [7–13].

Another problem, in the reconstruction or interpolation of signals, occurs when the sampling rate is below the Nyquist rate, especially in view of the Beurling-Landau density theorem [13, 14]. In this case irregular sampling of an appropriate type, instead of being a nuisance, may be of help for the asymptotically exact reconstruction of signals. This occurs in the space $\mathcal{A}$ of almost periodic functions.

Recall that f is almost periodic [15, 16] if it is uniformly continuous and if $\forall \varepsilon > 0$ there is $\Lambda(\varepsilon)$ such that any interval $[a, a + \Lambda(\varepsilon)]$, contains a number $\tau(a, \varepsilon)$ such that

$$\sup_x \left| f\left(x + \tau(a, \varepsilon)\right) - f(x) \right| \leq \varepsilon. \tag{1}$$

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Given $f(x)$ almost periodic, $\forall\varepsilon > 0$ there is a trigonometric polynomial g approximating f uniformly; that is, there are $B_1(\varepsilon), \dots, B_a(\varepsilon)$ such that

$$g(x) = \sum_{k=1}^{a} B_k e^{i2\pi\omega_k x} \tag{2}$$

and,

$$\sup_{x\in\mathbb{R}} \left| f(x) - g(x) \right| \le \varepsilon. \tag{3}$$

The reconstruction result in the space of almost periodic functions is:

Theorem 1 (*Collet* [17]) *Let $x_n = n\lambda + X_n$ with X_n a sequence of i.i.d. random variables uniformly distributed in $[0, \lambda]$. Then, almost every configuration $\{x_n\}$ of the point process has the property that if f is any complex almost periodic function satisfying $f(x_n) = 0$, $\forall n \in \mathbb{Z}$, then $f \equiv 0$.*

The approximation by trigonometric polynomials and Collet's theorem provides a basis for asymptotically exact reconstruction algorithms at rates much below Nyquist's rate.

The choice of the appropriate basis is an important issue when reconstructing signals from limited data, and specifically for the reconstruction from non-uniform sampling. In particular, if the local signal frequencies vary in time, as for example in density measurements by reflectometry [18], the approximation by trigonometric polynomials is not very convenient. A more general *chirp basis* would be more appropriate. The use of a chirp basis for the signals is closely related to the use of the fractional Fourier transform.

Let $\mathcal{L}$ be the Fréchet space of infinitely differentiable functions, with a topology defined by a family of norms

$$\left\| f(t) \right\|_{n,k} = \sup_{t\in\mathbb{R}} \left| t^n f^{(k)}(t) \right|, \quad n, k = 0, 1, 2, \dots. \tag{4}$$

Then, the Fourier operator $\mathcal{F}_1$ is defined for $f \in \mathcal{L}$ as

$$(\mathcal{F}_1 f)(\omega) = F(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt, \tag{5}$$

the inverse transform being

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega)\, d\omega. \tag{6}$$

In the time–frequency plane with orthogonal axes t and ω, the Fourier transform has a geometrical interpretation as a rotation of the signal by $\alpha_1 = \frac{\pi}{2}$. The fractional Fourier transform (FrFT) [19–21] corresponds also to a rotation in t–ω plane, but now by a non-integer multiple of $\frac{\pi}{2}$, $\alpha_b = b\frac{\pi}{2}$; namely

$$\begin{aligned}(\mathcal{F}_b f)(\zeta) &= F_b(\zeta) \\ &= \frac{e^{-\frac{i}{2}(\operatorname{sgn}(\sin\alpha_b)\frac{\pi}{2} - \alpha_b)}}{(2\pi\,|\sin\alpha_b|)^{1/2}} \\ &\quad \times \int_{-\infty}^{\infty} e^{(-i\frac{t\zeta}{\sin\alpha_b} + \frac{i}{2}\cot\alpha_b(t^2+\zeta^2))} f(t)\, dt\end{aligned}$$

with inverse $\mathcal{F}_{-b}$.

Another way to look at the Fourier transform equation (4) is as a decomposition of f into a combination of harmonics (the basis functions $\{e^{-i\omega t}\}$). As a result, the Fourier transform is a convenient way to code the signal when it is a superposition of a (small) number of harmonics.

Instead of $e^{-i\omega t}$ the kernel of the FrFT is a linear chirp $e^{-i(\omega - ct)t}$ with $\omega = \frac{\zeta}{\sin\alpha_b}$ and $c = \frac{1}{2}\cot\alpha_b$. This indeed suggests that a more general basis to expand the signal, with arbitrary features in the ω–t plane, is a basis of chirps.

In this paper, we provide a generalization of Collet's theorem to a space of functions that may be approximated by chirps. It contains, as a particular case, the space of almost periodic functions. Notice, however, that whereas the FrFT is defined in the whole Fréchet space $\mathcal{L}$, both Collet's theorem and our result reconstruct functions in the more restricted setting of almost periodic and linear chirp functions.

2 A uniform approximation result for random sampling in chirp space

Instead of (1), the approximation by trigonometric polynomials ((2) and (3)) provides an alternative, and equivalent, characterization of almost periodic functions. Likewise, we define the space $\mathcal{LC}$ of *linear chirp functions* as the space of functions f such that $\forall\varepsilon > 0$ $\exists$ a finite number of real number sets $(\omega_1, c_1, \alpha_1, B_1), \dots, (\omega_k, c_k, \alpha_k, B_k)$ such that

$$g(x) = \sum_{j=1}^{k} B_j e^{i\{\omega_j + c_j(x-\alpha_j)\}x} \tag{7}$$

and

$$\sup_{x\in\mathbb{R}} \left| f(x) - g(x) \right| \leq \varepsilon. \tag{8}$$

The space of almost periodic functions corresponds to the $c = 0$ case.

The space $\mathcal{LC}$ of linear chirp functions is strictly larger than the space of almost periodic functions. It suffices to consider e^{ix^2}. If it were an almost periodic function, there would exist ξ such that

$$\left| e^{i(x+\xi)^2} - e^{ix^2} \right| < \varepsilon, \quad -\infty < x < \infty$$

with $\varepsilon < 1$. Choose $x^* = \frac{k\pi - \xi^2}{2\xi}$ and k odd. Then

$$\left| e^{i(x^*+\xi)^2} - e^{ix^{*2}} \right| = 2 > \varepsilon;$$

a contradiction.

For functions in $\mathcal{LC}$ we prove the following:

Theorem 2 *Let $x_n = n\lambda + X_n$ with X_n a sequence of i.i.d. random variables uniformly distributed in $[0, \lambda]$. Then, almost every configuration $\{x_n\}$ of the point process has the property that if f is a function in the linear chirp space satisfying*

$$f(x_n) = 0, \quad \forall n \in \mathbb{Z},$$

then $f \equiv 0$.

For the proof one needs the following theorem.

Theorem 3 *For almost every configuration $\{x_n\}$ of the random process, one has*

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} e^{i(\omega x_n + c x_n^2)} = 0$$

for real ω and c with $\omega \neq 0$.

The proof may follow in a similar vein as the proof of Proposition 5 in [17]. A simpler argument uses the invariant measure properties of the random dynamical system in the circle

$$y_n = \frac{\omega}{2\pi}(n\lambda + X_n) + \frac{c}{2\pi}(n\lambda + X_n)^2 \pmod 1. \tag{9}$$

Let first $c = 0$, $\omega \neq 0$ and (without loss of generality) $\lambda = 1$. What the dynamical system (9) does is to cover the circle with intervals of length $\frac{\omega}{2\pi}$ and, in each one, to choose a point at random. Because X_n has uniform distribution in each interval, the distribution of y_n is also uniform. Therefore, by the ergodic theorem,

$$\lim_{L\to\infty} \frac{1}{2L+1} \sum_{-L\leq n\leq L} e^{i\omega x_n} = \left\langle e^{i2\pi y_n} \right\rangle_{S^1} = 0 \tag{10}$$

for generic sequences $\{X_n\}$.

For c and $\omega \neq 0$, the proof is more delicate. At each step n of the random dynamical system (9), the interval where the random point is chosen is

$$\left[\frac{\omega}{2\pi} n + \frac{c}{2\pi} n^2, \frac{\omega}{2\pi}(n+1) + \frac{c}{2\pi}(n+1)^2 \right]. \tag{11}$$

For all

$$n > n^* = \frac{2\pi - \omega}{2c} - \frac{1}{2},$$

the interval in (11) wraps one or more times over the circle. In the interval (11) the distribution of the random points would be

$$\rho_n(y) = \frac{2\pi}{\sqrt{\omega^2 + 8\pi c y}}.$$

When the interval wraps around the circle, the maximum possible deviation from uniformity of the density is

$$\Delta\rho_n = \frac{2\pi}{\sqrt{\omega^2 + 4c(\omega n + c n^2)}}. \tag{12}$$

Therefore, we have a sequence $\{y_n\}$ of independent random variables with values in $[0, 1]$ and each one has a density $\rho_n(y)$ with

$$\rho_n(y) = 1 + r_n(y)$$

where

$$\sup_{x\in[0,1]} \left| r_n(x) \right| = a_n.$$

We are interested in the case in which

$$\lim_{n\to\infty} a_n = 0 \quad \text{as } (1/n) \tag{13}$$

and we want to check whether

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sin(2\pi j X_j) = 0, \tag{14}$$

and also to relate the speed of convergence in the second limit to the speed of convergence in the first limit.

For each n, define Y_n by

$$Y_n = \sin(2\pi X_n) \tag{15}$$

and let μ_n be its mean. Define S_N by

$$S_N = \sum_{n=1}^{N} Y_n. \tag{16}$$

Also, define $M_n(\xi)$ by

$$M_n(\xi) = E\big(e^{\xi Y_n}\big). \tag{17}$$

By Jensen's inequality,

$$M_n(\xi) \geq e^{\xi \mu_n}.$$

The following lemma is a simple variant of a well-known estimate of Cramer. The variation, probably not new, is that our sequence of random variables, $\{Y_n\}$, is not identically distributed.

Lemma 1 *Suppose that for each N there is a convex function $\Phi_N(\xi)$ such that*

$$\sum_{n=1}^{N} \ln M_n(\xi/N) \leq \Phi_N(\xi). \tag{18}$$

Let $I_N(y)$ denote the Legendre transform of Φ_N; i.e.,

$$I_N(y) = \sup_{\xi}\big(y\xi - \Phi_N(\xi)\big). \tag{19}$$

Also, let

$$m_N = \frac{1}{N}\sum_{n=1}^{N} \mu_n. \tag{20}$$

Then, for any number $y > m_N$,

$$\ln\big(\Pr(S_N/N > y)\big) \leq -I_N(y). \tag{21}$$

Proof By the generalized Bienaymé–Chebychev inequality, for all $\xi \geq 0$,

$$\Pr(S_N/N > y) \leq e^{-y\xi} E\big(e^{\xi S_N/N}\big).$$

But by the independence,

$$E\big(e^{\xi S_N/N}\big) = \prod_{n=1}^{N} M_n(\xi/N),$$

and so

$$\ln\big(\Pr(S_N/N > y)\big) \leq -\xi y + \sum_{n=1}^{N} \ln M_n(\xi/N). \tag{22}$$

Since $\ln M_n(\xi) \geq \xi\mu_n$ for each n and ξ, it follows that $m_N\xi \leq \Phi_N(\xi)$. Then

$$y\xi - \Phi_N(\xi) \leq y\xi - m_N\xi \leq 0$$

for each $\xi < 0$ and $y > m_N$. Then

$$I_N(y) = \sup_{\xi\geq 0}\big(y\xi - \Phi_N(\xi)\big).$$

Therefore, (21) follows from (22).

The same way of reasoning shows that for any number $y < m_N$,

$$\ln\big(\Pr(S_N/N < y)\big) \leq -I_N(y).$$

We now apply this in the following setting. From (13) we may deduce estimates on each $\ln M_n(\xi)$ of the form

$$\ln M_n(\xi) \leq \mu_n\xi + C\xi^2.$$

Then,

$$\sum_{n=1}^{N} \ln M_n(\xi) \leq N\big(m_N\xi + C\xi^2\big).$$

In this case, we can apply this very lemma with

$$\Phi_N(\xi) = m_N\xi + \frac{C}{N}\xi^2.$$

Computing the Legendre transform, we then have

$$I_N(y) = \frac{N}{4C}(m_N - y)^2.$$

Now, define b_N by

$$b_N^2 = \frac{8C\ln N}{N}.$$

Then

$$\Pr(S_N/N > m_N + b_N) \leq e^{-Nb_N^2/4C} = N^{-2}.$$

Since this is summable, the probability that $S_N/N > m_N + b_N$ infinitely often is zero. This large deviation estimate implies (14). A similar reasoning applies for the cosine series. This completes the proof of Theorem 3. □

Proof of Theorem 2

Let $f(x_n)=0$, $\forall n\in\mathbb{Z}$, and $g(x)$ be its ε-approximation by linear chirp polynomials. Then

$$\left|\lim_{L\to\infty}\frac{1}{2L+1}\sum_{-L\le n\le L} e^{-i\{\omega x_n+cx_n^2\}}g(x_n)\right|$$
$$=\left|\lim_{L\to\infty}\frac{1}{2L+1}\sum_{-L\le n\le L} e^{-i\{\omega x_n+cx_n^2\}}\big(g(x_n)-f(x_n)\big)\right|$$
$$\le\varepsilon$$

for all ω and c. Inserting now (7) in the left-hand side of the above equation, one obtains

$$\left|\lim_{L\to\infty}\frac{1}{2L+1}\sum_{-L\le n\le L}\sum_{j=1}^{k} B_j e^{i\{(\omega_j-c_j\alpha_j-\omega)x_n+(c_j-c)x_n^2\}}\right|$$
$$\le\varepsilon.$$

Choosing $\omega=\omega_j-c_j\alpha_j$, $c=c_j$ and using Lemma 1, one concludes that for almost every configuration $\{x_n\}$,

$$|B_j|\le\varepsilon$$

for all j in the linear chirp approximation.

Because this result holds for all ε and the linear chirp basis functions are kernels to the fractional Fourier transform, one concludes that the function f has zero fractional Fourier spectrum. Therefore, it is the zero function.

As in the case of functions in the almost periodic space, the above result may be used to estimate functions in the linear chirp space by random sampling. If from a time series $h(x_n)$ one obtains, by the appropriate algorithm, a linear chirp approximation $g(x)$ coinciding with the sampled function on a typical sequence $\{x_n\}$, that is

$$f(x_n)=g(x_n)-h(x_n)=0,$$

then, in the above defined space, one knows that $g(x)=h(x)$ for all x. □

3 Nonlinear chirps

By a nonlinear chirp, we mean a linear combination of functions of the form $e^{ip(x)}$ where p is a real polynomial of some degree $m>2$. As before, let c denote the leading coefficient. The analysis leading to Theorem 3 easily extends to include nonlinear as well as linear chirps. Although these nonlinear chirp functions have no connection with the fractional Fourier transform, it is of interest to see how far one may extend the analysis, especially as the extension requires very little modification.

To see how this goes, observe that for large $|x|$, the leading term in $p(x)$ dominates, and so there is an $N<\infty$ such that for all $|x|>N$,

$$\left|p'(x)\right|\ge\big(|c|m/2\big)|x|^{m-1}>2\pi. \tag{23}$$

In particular, for $|n|>N$, $p(x)$ is monotone on $[n,n+1]$.

Let $\sigma_n(y)$ be the density for the random variable $y=p(x)$ with x chosen uniformly in $[n,n+1]$ for $|n|>N$. We may assume without loss of generality that $p(x)$ is increasing in this interval, and in this case

$$\sigma_n(y)=\frac{1}{p'(x(y))},$$

where $p(n)\le y\le p(n+1)$, and $x(y)$ is the unique solution of $p(x)=y$.

As before, the interval $[p(n),p(n+1)]$ "wraps" one or more times around the circle, inducing the density $\rho_n(y)$ on the circle. The variation between the maximum and the minimum of ρ_n is no more than the maximum value of $\sigma_n(y)$. By our assumption (23), this is no more than $2m|c|/n^{m-1}$.

Therefore, we have that

$$\rho_n(y)=1+r_n(y)$$

with

$$\sup_{x\in[0,1]}\left|r_n(x)\right|=a_n=\mathcal{O}\big(n^{m-1}\big).$$

This is an improvement over (13), and from here, the proof of the analog of Theorem 3 proceeds as before.

4 Remarks and conclusions

Although referring to a different space, that is almost-periodic instead of band-limited, Collet's result [17] considerably extends the practical range of reconstruction by sampling, in the sense that it shows how asymptotic reconstruction of stationary signals is possible at any rate, provided the sampling is random. As emphasized by several authors, variable frequency signals are ubiquitous in Nature, from Doppler-shifted signals to bat echolocation, from bird song to human language, from critical phenomena to EEG signals, etc. Chirps are everywhere [22]. Therefore, extension of the random reconstruction result to chirp spaces seems to be of practical interest.

Reconstruction by random sampling may be carried out by the following algorithm:

(1) Compute

$$F(f,c)=\frac{1}{N}\sum_{n=1}^{N}s(t_n)\exp\left\{-i\left(2\pi f t_n+ct_n^2\right)\right\}$$

for the random sampled signal $s(t)$.

Find the dominant maximum of $|F(f,c)|$ in the (f,c) plane. Let (f_1,c_1) be the location of this maximum and

$$A_1=F(f_1,c_1).$$

(2) Subtract $A_1\exp\{i(2\pi f_1 t+ct^2)\}$ from the signal

$$s_1(t_n)=s(t_n)-A_1\exp\left\{i\left(2\pi f_1 t+c_1 t^2\right)\right\}.$$

(3) Repeat step (1) for $s_1(t)$ looking for another dominant maximum away from (f_1,c_1). Let (f_2,c_2) be the location of this maximum and $A_2=F(f_2,c_2)$.

(4) Compute

$$s_2(t_n)=s_1(t_n)-A_2\exp\left\{i\left(2\pi f_2 t+c_2 t^2\right)\right\}.$$

Repeat the process until no new maxima appear. Then repeat the whole process starting from (1) until a stable estimate of $s(t)$ is obtained.

This, of course, is the kind of procedure that one naively expects to lead to an estimate of the signal in chirp space. What our result improves upon is not on this or similar algorithms but on the guarantee of the asymptotic convergence of the random sampling approximation.

Actually, the power of random sampling may be illustrated by a simple example. Let

$$s(t)=\sum_{i=1}^{3}A_i\exp\left\{i\left(2\pi f_i t+c_i t^2\right)\right\}$$

be a 3-chirp signal. When the function $|F(f,c)|$ is computed either by random or regular sampling above Nyquist's rate, the identification of the maxima in the (f,c)-plane is quite similar. However, for (average) sampling rates below Nyquist's the difference is quite remarkable. Figure 1 compares the behavior of $|F(f,c)|$ for random and regular samplings at the same average rate, equal to $\frac{1}{4}$ of the Nyquist rate. The

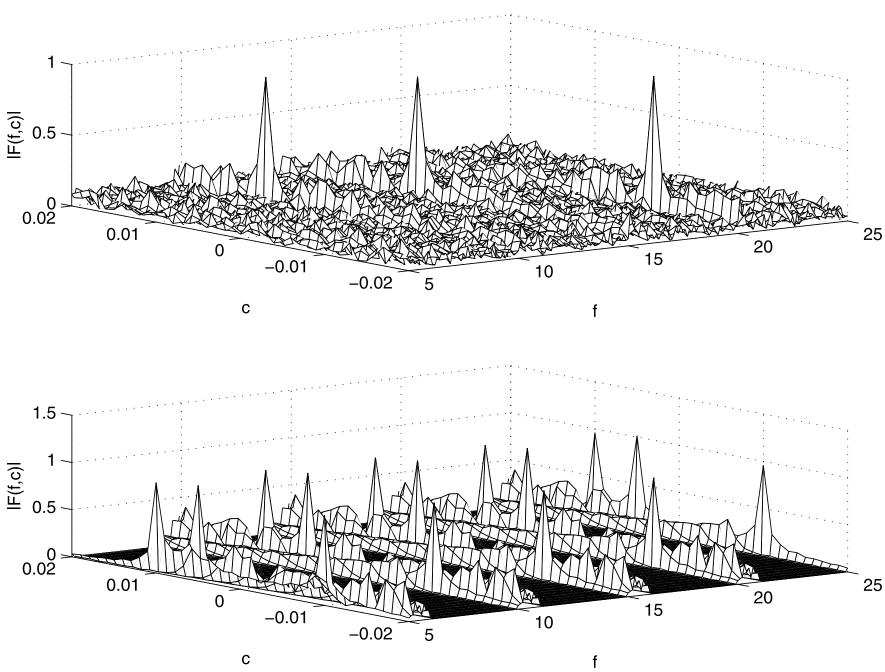

Fig. 1 The power of random sampling: Comparison of the overlap function $|F(f,c)|$ for a 3-chirp signal. Random (*upper plot*) versus regular (*lower plot*) sampling at the same average rate (1/4 Nyquist's of the smallest frequency): $f_1=10$, $f_2=15$, $f_3=20$, $c_1=0.01$, $c_2=0.005$, $c_3=-0.01$. Signal duration $=100$ s

accurate identification of the chirp parameters by random sampling is quite impressive, whereas for regular sampling the result is pure nonsense. What one sees in the regular sampling case is the beatings between the signal frequencies and the sampling frequency. Notice that to make the regular sampling as unbiased as possible we have randomized the initial time of the sequence.

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