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# Lyapunov exponent in quantum mechanics. A phase-space approach

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**Abstract**

Using the symplectic tomography map, both for the probability distributions in classical phase-space and for the Wigner functions of its quantum counterpart, we discuss a notion of Lyapunov exponent for quantum dynamics. Because the marginal distributions, obtained by the tomography map, are always well-defined probabilities, the correspondence between classical and quantum notions is very clear. Then we also obtain the corresponding expressions in Hilbert space. Some examples are worked out. Classical and quantum exponents are seen to coincide for local and non-local time-dependent quadratic potentials. For non-quadratic potentials classical and quantum exponents are different and some insight is obtained on the taming effect of quantum mechanics on classical chaos. A detailed analysis is made for the standard map. Providing an unambiguous extension of the notion of Lyapunov exponent to quantum mechanics, the method that is developed is also computationally efficient in obtaining analytical results for the Lyapunov exponent, both classical and quantum. 

*Keywords:* Lyapunov exponents; Phase-space; Quantum tomography

## 1. Introduction

Classical chaotic motion is characterized by the existence of positive Lyapunov exponents or positive Kolmogoroff–Sinai entropy. Because the definition of these quantities is based on the properties of classical trajectories in phase-space, it is not obvious what the corresponding quantities in quantum mechanics should be. This situation led to the proposal of many different quantities to characterize the quantum behavior of classically chaotic systems [1–26]. To be sure that one constructs quantum mechanical functionals, with exactly the same physical meaning as the classical quantities which characterize classical chaos, the natural suggestion would be to use also a phase-space formulation for quantum mechanics, rather than the usual Hilbert space formulation. The difficulty here lies in the fact that quantum phase-space is a non-commutative manifold with the usual pointwise product of functions being replaced by the $*$-product [27]. One possible solution is to use the tools of non-commutative geometry for this formulation. Another approach, however, is to look for (commutative) quantities which have the same formal

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structure both in classical and in quantum mechanics. The Wigner function [28], which some authors have attempted to use for this purpose, is not the appropriate choice because, unlike the classical probability distributions, it is not positive definite. In fact, the Wigner function is only correctly interpreted in a non-commutative geometry setting or, alternatively, as an operator symbol which by the Weyl map corresponds to a well-defined operator in Hilbert space [27].

There is however a set of phase-space quantities that have the same mathematical nature in both classical and quantum mechanics. This is the set of marginal distributions of the symplectic tomography formulation [29–36], which are in both cases well-defined probability distributions. In Section 2, we review the symplectic tomography formulation of classical and quantum mechanics. In both cases the dynamics is defined by a set of marginal probability distributions, Eqs. (6) and (17). The difference between classical and quantum mechanics comes only on the modification of the equations of motion. In this formulation this is the analog of the Moyal deformation [27,37] of the phase-space algebra. The equation of motion for the quantum marginal distributions is derived both in a compact closed form, Eq. (21), and as a series expansion, Eq. (22), which emphasizes the deformation nature of the transition from classical to quantum mechanics.

Once the appropriate phase-space quantities are identified and classical Lyapunov exponents (and local entropies) are formulated in terms of probability distributions, the transition to quantum mechanics is rather straightforward. This is discussed in Section 3. Having obtained the Lyapunov exponent for quantum mechanics from the marginal probability distributions, Eq. (36), one is then also able to obtain in Section 4 the corresponding Hilbert space expression.

The remainder of the paper is dedicated to the computation of some examples, namely kicked systems on the line, on the 2-torus and on the circle. The results that are obtained, in particular Eqs. (40)–(45), show that despite its apparent complex form, the marginal distribution formulation is a computationally effective way to obtain Lyapunov exponents, both classical and quantum. Classical and quantum exponents are seen to coincide for time-dependent local and non-local quadratic potentials. For non-quadratic potentials classical and quantum exponents are different. A characterization is obtained for the origin of the taming effect of quantum mechanics on classical chaos in the standard map. This is a step towards a rigorous characterization of this effect because it refers, as in the classical case, to the behavior of the Lyapunov exponents and not to indirect quantities like the energy growth.

## 2. Symplectic tomography of classical and quantum states

### 2.1. Classical mechanics

States in classical statistical mechanics are described by a function $\rho(q, p)$, the probability distribution function in $2n$-dimensional phase-space ($q \in R^n$, $p \in R^n$), with properties

$$\rho(q, p) \geq 0, \quad \int \rho(q, p)\, \mathrm{d}^n p = P(q), \quad \int \rho(q, p)\, \mathrm{d}^n q = \tilde{P}(p),$$

where $P(q)$ and $\tilde{P}(p)$ are the probability distributions for position and momentum (the *marginals* of $\rho$). The function $\rho(q, p)$ is normalized:

$$\int \rho(q, p)\, \mathrm{d}^n q\, \mathrm{d}^n p = 1. \tag{1}$$

We consider an observable $X(q, p)$, i.e., a function on the phase-space of the system. The inverse Fourier transform of the characteristic function $\langle \mathrm{e}^{\mathrm{i}k\cdot X}\rangle$ for any vector observable $X(q, p)$,

$$w(Y) = \frac{1}{(2\pi)^n} \int \langle \mathrm{e}^{\mathrm{i}k\cdot X}\rangle \mathrm{e}^{-\mathrm{i}k\cdot Y}\, \mathrm{d}^n k, \tag{2}$$

is a real non-negative function which is normalized, since

$$w(Y)=\int \rho(q,p)\delta^n(X(q,p)-Y)\,\mathrm{d}^nq\,\mathrm{d}^np, \tag{3}$$

and

$$\int w(Y)\,\mathrm{d}^nY=\int \rho(q,p)\,\mathrm{d}^nq\,\mathrm{d}^np=1. \tag{4}$$

As a classical analog of the quantum symplectic tomography observable, introduced in [29], we consider the following classical observable [33]:

$$X(q,p)=\mu \circledast q+\nu \circledast p, \tag{5}$$

where $\circledast$ denotes the componentwise product of vectors

$$(\mu \circledast q)_i=\mu_i q_i$$

and $\mu$ and $\nu$ are vector-valued real parameters. Together with

$$P(q,p)=\frac{1}{\nu}\circledast\mu\circledast q+\left(\frac{1}{\mu}+\mathbf{1}\right)\circledast p$$

(5) is a symplectic transformation [2] of the position and momentum observables. ($\mathbf{1}$ stands for the unit vector in $R^n$.)

The vector variable $X(q,p)$ may be interpreted as a coordinate of the system, when measured in a rotated and scaled reference frame in the classical phase-space. For the coordinate (5) in the transformed reference frame, we obtain from Eq. (2) the distribution function (the tomography map)

$$w(X,\mu,\nu)=\frac{1}{(2\pi)^n}\int \exp(-\mathrm{i}k\cdot(X-\mu\circledast q-\nu\circledast p))\rho(q,p)\,\mathrm{d}^nq\,\mathrm{d}^np\,\mathrm{d}^nk. \tag{6}$$

This function is homogeneous:

$$w(\lambda X,\lambda\mu,\lambda\nu)=|\lambda|^{-n}w(X,\mu,\nu), \tag{7}$$

and Eq. (6) has an inverse

$$\rho(q,p)=\frac{1}{(4\pi^2)^n}\int w(X,\mu,\nu)\exp[\mathrm{i}(X-\mu\circledast q-\nu\circledast p)\cdot\mathbf{1}]\,\mathrm{d}^nX\,\mathrm{d}^n\mu\,\mathrm{d}^n\nu. \tag{8}$$

Since the map

$$\rho(q,p)\Rightarrow w(X,\mu,\nu)$$

is invertible, the information contained in the distribution function $\rho(q,p)$ is equivalent to the information contained in the marginal distributions $w(X,\mu,\nu)$.

The Boltzmann evolution equation for the classical distribution function for a particle with mass, $m=1$ and potential, $V(q)$,

$$\frac{\partial\rho(q,p,t)}{\partial t}+p\cdot\nabla_q\rho(q,p,t)-\nabla_qV(q)\cdot\nabla_p\rho(q,p,t)=0, \tag{9}$$

[2] This is not, of course, the most general symplectic transformation. In general $X^i(x,p)=\mu^i_kx^k+\nu^i_kp^k$ with the corresponding expression for $P(x,p)$. Here we have considered the particular case where the tensors $\mu$ and $\nu$ are diagonal. This is the reason for the non-covariant look of our equations.

can be rewritten in terms of the marginal distribution $w(X, \mu, \nu, t)$,

$$\frac{\partial w}{\partial t} - \mu \cdot \nabla_\nu w - \nabla_x V(-\nabla_X^{-1} \circledast \nabla_\mu) \cdot (\nu \circledast \nabla_X w) = 0. \tag{10}$$

For the mean value of the position and momentum we have

$$\left\langle \begin{pmatrix} q \\ p \end{pmatrix} \right\rangle = \int \rho(q, p) \begin{pmatrix} q \\ p \end{pmatrix} \mathrm{d}^n q \, \mathrm{d}^n p = \mathrm{i} \int w(X, \mu, \nu) \mathrm{e}^{\mathrm{i} X \cdot \mathbf{1}} \begin{pmatrix} \nabla_\mu \\ \nabla_\nu \end{pmatrix} (\delta^n(\mu)\delta^n(\nu)) \, \mathrm{d}^n X \, \mathrm{d}^n \mu \, \mathrm{d}^n \nu. \tag{11}$$

With a change of variables $X \to \mu X$ and the homogeneity property $w(\mu X, \mu, 0) = \mu^{-n} w(X, \mathbf{1}, 0)$ one may, e.g., check the consistency of this definition for the mean value of the position

$$\langle q \rangle = \int w(X, \mathbf{1}, 0) X \, \mathrm{d}^n X. \tag{12}$$

By $\Pi_{\mathrm{cl}}(X, \mu, \nu, X', \mu', \nu', t_2, t_1)$ we denote the classical propagator that connects two marginal distributions at different times $t_0$ and $t\,(t > t_0)$:

$$w(X, \mu, \nu, t) = \int \Pi_{\mathrm{cl}}(X, \mu, \nu, X', \mu', \nu', t, t_0) w(X', \mu', \nu', t_0) \, \mathrm{d}^n X' \, \mathrm{d}^n \mu' \, \mathrm{d}^n \nu'. \tag{13}$$

The propagator satisfies the equation

$$\frac{\partial \Pi_{\mathrm{cl}}}{\partial t_2} - \mu \cdot \nabla_\nu \Pi_{\mathrm{cl}} - \nabla_x V(-\nabla_X^{-1} \circledast \nabla_\mu) \cdot (\nu \circledast \nabla_X \Pi_{\mathrm{cl}}) = 0 \tag{14}$$

with boundary condition

$$\lim_{t_2 \to t_1} \Pi_{\mathrm{cl}}(X, \mu, \nu, X', \mu', \nu', t_2, t_1) = \delta^n(X - X')\delta^n(\mu - \mu')\delta^n(\nu - \nu'). \tag{15}$$

### 2.2. *Quantum mechanics*

For quantum mechanics the construction is similar and the mathematical nature of the quantities that are constructed is the same, because it is a general fact that the inverse Fourier transform of a characteristic function is a positive distribution. The marginal distributions that are obtained are simply related to other well known quantum mechanical quantities. It was shown [29] that for the generic linear combination

$$X = \mu \circledast q + \nu \circledast p, \tag{16}$$

where $q$ and $p$ are the position and the momentum, the marginal distribution $w(X, \mu, \nu)$ (normalized in the variable $X$ and depending on two vector-valued real parameters $\mu$ and $\nu$) is related to the Wigner function $W(q, p)$. For $n$ degrees of freedom one has

$$w(X, \mu, \nu) = \int \exp[-\mathrm{i}k \cdot (X - \mu \circledast q - \nu \circledast p)] W(q, p) \frac{\mathrm{d}^n k \, \mathrm{d}^n q \, \mathrm{d}^n p}{(4\pi^2)^n}. \tag{17}$$

We see that Eq. (17) is formally identical to (6) of the classical case. For a pure state with wave function $\Psi(y)$, the marginal distribution would be [35]

$$w(X, \mu, \nu) = \frac{1}{(2\pi)^n |\nu_1 \ldots \nu_n|} \left| \int \Psi(y) \exp \mathrm{i} \sum_{j=1}^n \left( \frac{\mu_j y_j^2}{2\nu_j} - \frac{y_j X_j}{\nu_j} \right) \mathrm{d}^n y \right|^2. \tag{18}$$

Eq. (17) may be inverted and the Wigner function expressed in terms of the marginal distribution, like in the classical case of Eq. (8),

$$W(q,p)=\left(\frac{1}{2\pi}\right)^n\int w(X,\mu,\nu)\exp[\mathrm{i}(X-\mu\circledast q-\nu\circledast p)\cdot\mathbf{1}]\,\mathrm{d}^n\mu\,\mathrm{d}^n\nu\,\mathrm{d}^nX. \tag{19}$$

Therefore the usual quantum mechanical quantities can be reconstructed from the marginal distributions. These quantities (wave function and Wigner function) have a nature quite different from the classical quantities, however the marginals $w(X,\mu,\nu)$ are in both cases positive distributions with the same physical meaning.

As was shown in [31], for a system with Hamiltonian

$$H=\tfrac{1}{2}p^2+V(q) \tag{20}$$

the marginal distribution satisfies the quantum time-evolution equation

$$\frac{\partial w}{\partial t}-\mu\cdot\nabla_\nu w-\mathrm{i}\frac{1}{\hbar}\{V(-\nabla_X^{-1}\circledast\nabla_\mu-\mathrm{i}\tfrac{1}{2}\hbar\nu\circledast\nabla_X)-V(-\nabla_X^{-1}\circledast\nabla_\mu+\mathrm{i}\tfrac{1}{2}\hbar\nu\circledast\nabla_X)\}w=0 \tag{21}$$

which provides a dynamical characterization of quantum dynamics, alternative to the Schrödinger equation.

The evolution equation (21) can also be written in the form

$$\frac{\partial w}{\partial t}-\mu\cdot\nabla_\nu w-\nabla_x V(\tilde{q})\cdot(\nu\circledast\nabla_X w)+\frac{2}{\hbar}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\hbar}{2}\right)^{2n+1}\frac{\nabla_{i_1,\dots,i_{2n+1}}V(\tilde{q})}{(2n+1)!}(\nu\circledast\nabla_X)_{i_1}\dots(\nu\circledast\nabla_X)_{i_{2n+1}}w=0, \tag{22}$$

where $\tilde{q}$ stands for the operator

$$\tilde{q}=-\nabla_X^{-1}\circledast\nabla_\mu$$

and a sum over repeated indices is implied.

In Moyal's [37] formulation of quantum mechanics in phase-space, the transition from the classical to the quantum structure is a deformation of the Poisson algebra [27] with deformation parameter $\hbar$. In the symplectic tomography formulation, that we are describing, classical and quantum mechanics are described by the same set of positive probability distributions $w(X,\mu,\nu)$, the $\hbar$-deformation appearing only in the time-evolution equation (22).

For the propagator

$$w(X,\mu,\nu,t)=\int \Pi(X,\mu,\nu,X',\mu',\nu',t,t_0)w(X',\mu',\nu',t_0)\,\mathrm{d}^nX'\,\mathrm{d}^n\mu'\,\mathrm{d}^n\nu' \tag{23}$$

the equation is

$$\frac{\partial \Pi}{\partial t}-\mu\cdot\nabla_\nu \Pi-\nabla_x V(\tilde{q})\cdot(\nu\circledast\nabla_X \Pi)+\frac{2}{\hbar}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{\hbar}{2}\right)^{2n+1}\frac{\nabla_{i_1,\dots,i_{2n+1}}V(\tilde{q})}{(2n+1)!}(\nu\circledast\nabla_X)_{i_1}\cdots(\nu\circledast\nabla_X)_{i_{2n+1}}\Pi=0 \tag{24}$$

with boundary condition

$$\lim_{t\to t_0}\Pi(X,\mu,\nu,X',\mu',\nu',t,t_0)=\delta^n(X-X')\delta^n(\mu-\mu')\delta^n(\nu-\nu'). \tag{25}$$

## 3. Lyapunov exponents

### 3.1. Density formulation in classical mechanics

Lyapunov exponents and other ergodic invariants in the classical theory are usually formulated in terms of quantities related to trajectories in phase-space, like tangent maps, refinement of partitions, etc. [38]. Here, as a preparation for the formulation of Lyapunov exponents in quantum mechanics, using the marginal distributions $w(X, \mu, \nu)$, we explain briefly how these quantities may, in classical mechanics, be expressed as functionals of phase-space densities rather than in terms of trajectories. For more details we refer to [12].

A density in phase-space is a non-negative, normalized, integrable function, the space of densities being denoted by $D$:

$$D = \{\rho \in L^1 : \ \ \rho \geq 0, \|\rho\|_1 = 1\}, \tag{26}$$

where $D$ is the space of functions that, by the Radon–Nikodym theorem, characterize the measures that are absolutely continuous with respect to the underlying measure in phase-space. However, to define Lyapunov exponents by densities, it is necessary to restrict oneself to a subspace of *admissible densities* defined as follows.

To each $\rho \in D$ we associate a square root, that becomes an element of an $L^2$ space. We then construct a Gelfand triplet

$$E^* \supset L^2 \supset E, \tag{27}$$

where $E$ is the space of functions of rapid decrease topologized by the family of semi-norms $\|x_\alpha \partial_\beta f\|_2$ and $E^*$ is its dual. Because $E$ is an algebra $f \in E$ implies $f^2 \in E$. Therefore for each $f$ such that $\|f\|_2 = 1$, $\rho = f^2$ is an admissible density. The restriction to such a subspace of admissible densities is necessary to be able to define Gateaux derivatives along generalized functions with point support. Gateaux derivatives along derivatives of the delta function for densities play the same role as the tangent map for trajectories. In this setting the Lyapunov exponent is [12]

$$\lambda_v = \lim_{t\to\infty} \frac{1}{t} \log \left\| -v^i D_{\partial_i \delta_x} \left( \int \mathrm{d}\mu(y) y P^t \rho(y) \right) \right\|, \tag{28}$$

$v \in R^{2n}$, $\|\cdot\|$ is the vector norm and the Gateaux derivative $D_{\partial_i \delta_x}$ operates in the argument of the functional, i.e., on the initial density

$$D_{\partial_i \delta_x} F(\rho(y)) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \{F(\rho(y) + \varepsilon \partial_i \delta(y-x)) - F(\rho(y))\}, \tag{29}$$

$\mu$ is the invariant measure in the support of which the Lyapunov exponent is being defined and $P^t$ is the operator of time-evolution for densities

$$P^t \rho(y, 0) = \rho(y, t). \tag{30}$$

A simple computation shows that the expression (28) is equivalent to the usual definition of Lyapunov exponent in terms of trajectories and the tangent map

$$\lambda_v = \lim_{t\to\infty} \frac{1}{t} \log \|DT_x^t v\|, \tag{31}$$

where $DT_x^t$ stands for the evolved tangent map applied to the vector $v$ at the phase-space point $x$. Here and in Eq. (28) $x$ and $y$ are phase-space vectors, i.e., in the notation of Section 2, $x = (q_x, p_x)$ and $y = (q_y, p_y)$.

According to Oseledec's theorem [39,40], for $\mu$ — almost every point $x$ there is a decreasing sequence of vector spaces

$$R^{2n} = E_1(x) \supset E_2(x) \supset \cdots \supset E_r = \{0\},$$

such that, by choosing the vector $v$ in $E_s(x)\backslash E_{s+1}(x)$, the $s$th Lyapunov exponent is obtained by the above calculation.

A similar construction is possible for the metric entropy. For the entropy, the notion that seems more appropriate for generalization to quantum mechanics [12,41] is the Brin–Katok local entropy [38], which for the classical case and for a compact metric space is equivalent to the Kolmogoroff–Sinai entropy. It is defined as follows:

$$B_\varepsilon(T,t,x) = \{y: \; d(T^\tau(x), T^\tau(y)) \le \varepsilon, 0 \le \tau \le t\},$$

where $B_\varepsilon(T,t,x)$ is the ball of phase-space points around $x$ that in the course of time-evolution do not separate by a distance larger than $\varepsilon$ up to time $t$. $T^\tau(x)$ is the image of $x$ after the time $\tau$ and $d(\cdot,\cdot)$ is the distance. The local entropy $h(T,x)$ measures the weighed (in the $\mu$-measure) rate of shrinkage in time of the ball $B_\varepsilon(T,t,x)$, namely

$$h(T,x) = \lim_{\varepsilon\to 0}\lim_{t\to\infty}\left\{-\frac{1}{t}\log\mu(B_\varepsilon(T,t,x))\right\}.$$

As in the case of the Lyapunov exponent, this quantity may be expressed as a functional of (admissible) densities by rewriting the ball $B_\varepsilon(T,t,x)$ as

$$B_\varepsilon(T,t,x) = \left\{y: \; \left|D_{(\delta_x-\delta_y)}\left(\int \mathrm{d}\mu(z) z P^\tau\rho(z)\right)\right| \le \varepsilon; 0 \le \tau \le t\right\}.$$

### 3.2. Classical and quantum Lyapunov exponents by marginal distributions

Let us now translate the equations of the preceding subsection in the tomographic framework discussed in Section 2. Initial densities are by the tomographic map mapped to initial tomographic densities by (6),

$$\rho(q,p) \to w(X,\mu,\nu,t=0) \equiv w(X,\mu,\nu). \tag{32}$$

To compute the Gateaux derivatives notice that the generalized density ($\in E^*$)

$$(v_1\cdot\nabla_q + v_2\cdot\nabla_p)\{\delta^n(q-q_0)\delta^n(p-p_0)\} \tag{33}$$

is mapped to the tomographic generalized density $w_\eta(\in E^*)$,

$$w_\eta(X,\mu,\nu) = ((v_1 \circledast \mu + v_2 \circledast \nu)\cdot\nabla_X)\delta^n(X-\mu q_0-\nu p_0). \tag{34}$$

According to Eq. (28), to compute the Lyapunov exponent one has to obtain the expectation value of a generic phase-space vector on the time-evolved perturbation of the initial density (33). Therefore

$$\lambda_v = \lim_{t\to\infty}\frac{1}{t}\log\left\| \int \mathrm{d}^n q\, \mathrm{d}^n p \begin{pmatrix} q \\ p \end{pmatrix} \mathcal{K}(q,p,q',p',t)(v_1\cdot\nabla_{q'} + v_2\cdot\nabla_{p'})\delta^n(q'-q_0)\delta^n(p'-p_0)\,\mathrm{d}^n q'\,\mathrm{d}^n p'\right\|, \tag{35}$$

where $\mathcal{K}(q,p,q',p',t)$ is the evolution kernel for densities

$$\rho(q,p,t) = \int \mathcal{K}(q,p,q',p',t)\rho(q',p')\,\mathrm{d}^n q'\,\mathrm{d}^n p'.$$

Notice that in Eq. (35) the integration is carried out over the flat phase-space measure $\mathrm{d}^n q\,\mathrm{d}^n p$. The result is equivalent to (28) for an invariant measure absolutely continuous with respect to $\mathrm{d}^n q\,\mathrm{d}^n p$. However the information and the dependence of the Lyapunov exponent on the invariant measure is carried by the choice of the initial point $(q_0, p_0)$. The set of Lyapunov exponents that is obtained by (35) is therefore the one that corresponds to the invariant measure on whose support $(q_0, p_0)$ lies.

Using (11), Eq. (35) may now be rewritten using marginal distributions

$$\lambda_v = \lim_{t\to\infty}\frac{1}{t}\log\left\| \begin{array}{c} \int \mathrm{d}^n X\,\mathrm{d}^n\mu\,\mathrm{d}^n\nu\,\mathrm{e}^{\mathrm{i}X\cdot\mathbf{1}}\left(\begin{pmatrix}\nabla_\mu\\ \nabla_\nu\end{pmatrix}\delta^n(\mu)\delta^n(\nu)\right)\Pi_{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',t,0) \\ ((v_1\circledast\mu' + v_2\circledast\nu')\cdot\nabla_{X'})\delta^n(X'-\mu' q_0 - \nu' p_0)\,\mathrm{d}X'^n\,\mathrm{d}\mu'^n\,\mathrm{d}\nu'^n \end{array}\right\|, \tag{36}$$

where $\Pi_{\mathrm{cl}}(X,\mu,\nu,X',\mu',\nu',t_2,t_1)$ is the classical propagator defined in (13)–(15).

Because (28) is equivalent to the usual definition of Lyapunov exponent, Eq. (36), being equivalent to (28), is also a correct expression for the classical Lyapunov exponent.

Now the transition to quantum mechanics is straightforward. Marginal distributions in classical and quantum mechanics satisfy formally identical expressions and have the same physical interpretation as probability densities. The only difference lies on the time-evolution which in classical mechanics obeys Eq. (10) and in quantum mechanics the $\hbar$-deformed equation (21). Therefore the Lyapunov exponent in quantum mechanics will also be given by Eq. (36), with however the classical propagator $\Pi_{\mathrm{cl}}$ replaced by the quantum propagator $\Pi$ for marginal distributions, defined in (23)–(25).

## 4. Hilbert space expression for the quantum Lyapunov exponent

As we will see in Section 5, Eq. (36) provides an efficient way to compute the Lyapunov exponent. However, for comparison with other approaches, it is useful to translate Eq. (36) in the Hilbert space quantum mechanical formalism.

To simplify the notation, we consider $n = 1$, i.e., a two-dimensional phase-space. Generalization to the $n$-dimensional case is straightforward. To write the quantum Lyapunov exponent (36) in terms of the evolution operator acting in the Hilbert space of states, we use the following equation [34] that relates the tomographic propagator to Hilbert space Green's functions:

$$\Pi(X,\mu,\nu,X',\mu',\nu',t) = \frac{1}{4\pi^2}\int k^2 G\left(a+\frac{k\nu}{2},y,t\right)G^*\left(a-\frac{k\nu}{2},z,t\right) \times\delta(y-z-k\nu')\exp\left[\mathrm{i}k\left(X'-X+\mu a-\mu'\frac{y+z}{2}\right)\right]\mathrm{d}k\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}a.$$

Then, the quantum Lyapunov exponent expressed in terms of Green's functions is

$$\lambda_v = \lim_{t\to\infty}\frac{1}{t}\int \mathrm{d}a\,\mathrm{d}z\,\mathrm{e}^{-2\mathrm{i}p_0 z}G^*(a,z,t)v_1\left[\mathrm{i}p_0 G(a,2q_0-z,t) - \frac{\partial G}{\partial z}(a,2q_0-z,t)\right] +\mathrm{i}v_2(q_0-z)G(a,2q_0-z,t).$$

Using a complete set of wave functions $\psi_n(x,t)$ this expression may be rewritten as

$$\lambda_v = \lim_{t\to\infty}\frac{1}{t}\ln\left|\int \mathrm{d}a\,\mathrm{d}z\,\mathrm{e}^{-2\mathrm{i}p_0 z}a\sum_{n,m}\psi_n^*(a,t)\psi_n(z,t)\{\mathrm{i}[v_1 p_0 + v_2(q_0-z)][\psi_m(a,t)\psi_m^*(2q_0-z,t)] +v_1\psi_m(a,t)\psi_m^{*\prime}(2q_0-z,t)\}\right|.$$

In Refs. [7–11] a quantum characteristic exponent has been defined, in Hilbert space, by using the time-evolution of the expectation values on a $\delta'$-perturbed wave function. This definition may be compared with the present marginal probability construction. Under an $\varepsilon\delta'(x-q_0)$ perturbation of the wave function, the density matrix changes, in leading order, by

$$\Delta(\psi(x)\psi(x'))=\varepsilon\{\delta'(x-q_0)\psi^*(x')+\psi(x)\delta'(x'-q_0)\}. \tag{37}$$

On the other hand the marginal probability perturbation

$$(v_1\mu+v_2\nu)\delta'(X-\mu q_0-\nu p_0)$$

induces a perturbation of the density matrix

$$\Delta\rho(x,x')=-\mathrm{e}^{\mathrm{i}p_0(x-x')}\{v_1\delta'(q_0-\tfrac{1}{2}(x+x'))+\mathrm{i}v_2(x-x')\delta(q_0-\tfrac{1}{2}(x+x'))\}. \tag{38}$$

Comparing (37) and (38) one sees that, for the calculation of the Lyapunov exponent, they coincide on the diagonal terms $(x=x')$, but are different on the non-diagonal terms. In particular, the marginal density perturbation is not a pure state perturbation and cannot be reproduced by the perturbation of a single wave function.

## 5. Examples: kicked systems on the line and on the circle

### 5.1. One-dimensional systems with time-dependent potentials

We consider here one-dimensional systems with time-dependent potentials defined by the Hamiltonian

$$H=\tfrac{1}{2}p^2+V(q,t). \tag{39}$$

For these systems, the Lyapunov exponent expression (36) is

$$\lambda_v=\lim_{t\to\infty}\frac{1}{t}\log\left\|\int \mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\,\mathrm{e}^{\mathrm{i}X}\left(\begin{pmatrix}\partial_\mu\\ \partial_\nu\end{pmatrix}\delta(\mu)\delta(\nu)\right)F(X,\mu,\nu,t)\right\|, \tag{40}$$

where $F(X,\mu,\nu,t)$ is the time-evolved perturbation, namely

$$F(X,\mu,\nu,t)=\int \Pi(X,\mu,\nu,X',\mu',\nu',t,0)(v_1\mu'+v_2\nu')\delta'(X'-\mu'q_0-\nu'p_0)\,\mathrm{d}X'\,\mathrm{d}\mu'\,\mathrm{d}\nu'. \tag{41}$$

Passing to the Fourier transform

$$G(k,\mu,\nu,t)=\frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}kX}F(X,\mu,\nu,t)\,\mathrm{d}X, \tag{42}$$

one obtains

$$\lambda_v=\lim_{t\to\infty}\frac{1}{t}\log\left\|\int \mathrm{d}\mu\,\mathrm{d}\nu\left(\begin{pmatrix}\partial_\mu\\ \partial_\nu\end{pmatrix}\delta(\mu)\delta(\nu)\right)G(1,\mu,\nu,t)\right\|=\lim_{t\to\infty}\frac{1}{t}\log\left\|\begin{pmatrix}G^{(2)}(1,0,0,t)\\ G^{(3)}(1,0,0,t)\end{pmatrix}\right\|, \tag{43}$$

where by $G^{(2)}$ and $G^{(3)}$ we denote the derivatives in the second and third arguments and $G(k,\mu,\nu,t)$ is a solution of the equation

$$\frac{\partial G}{\partial t}-\mu\frac{\partial G}{\partial \nu}-\mathrm{i}k\nu\partial_q V\left(-\frac{1}{\mathrm{i}k}\frac{\partial}{\partial\mu}\right)G+\frac{2}{\hbar}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)!}\left(\mathrm{i}k\frac{\nu\hbar}{2}\right)^{2n+1}\frac{\partial^{2n+1}}{\partial q^{2n+1}}V\left(-\frac{1}{\mathrm{i}k}\frac{\partial}{\partial\mu}\right)G=0, \tag{44}$$

with initial condition

$$G(k,\mu,\nu,t)=-\frac{\mathrm{i}k}{2\pi}(v_1\mu+v_2\nu)\mathrm{e}^{\mathrm{i}k(q_0\mu+p_0\nu)}. \tag{45}$$

Therefore, the computation of the Lyapunov exponents, both classical and quantum, reduces to the study of the large time limit of the solutions of Eq. (44). Also the simple expression (43) shows that, despite its apparently complex form, Eq. (36) is a computationally efficient way to obtain the Lyapunov exponent.

We now study several time-dependent (kicked) potentials.

### 5.2. *Harmonic kicks on the line*

Here the potential is

$$V(q)=\frac{\gamma\alpha}{\pi}\frac{q^2}{2}\sum_{n=-\infty}^{\infty}\delta(t-n). \tag{46}$$

This system belongs to the class of time-dependent quadratic systems [33] and a solution may be found for the general case

$$V(q)=\alpha(t)\tfrac{1}{2}q^2. \tag{47}$$

Eq. (44) reduces to

$$\frac{\partial G}{\partial t}-\mu\frac{\partial G}{\partial \nu}+\alpha(t)\nu\frac{\partial G}{\partial \mu}=0. \tag{48}$$

The $\hbar$-deformed part of Eq. (44) disappears and we obtain the (expected) result that in this case classical and quantum results coincide. Eq. (48) has the solution

$$G(k,\mu,\nu,t)=G\left(k,\frac{\mu}{2}(\varepsilon+\varepsilon^*)+\frac{\nu}{2}(\dot\varepsilon+\dot\varepsilon^*),\frac{\nu}{2\mathrm{i}}(\dot\varepsilon-\dot\varepsilon^*)+\frac{\mu}{2\mathrm{i}}(\varepsilon-\varepsilon^*),0\right) \tag{49}$$

the function $\varepsilon(t)$ being a solution of

$$\ddot\varepsilon(t)+\alpha(t)\varepsilon(t)=0, \tag{50}$$

with initial conditions

$$\varepsilon(0)=1,\qquad \dot\varepsilon(0)=\mathrm{i}. \tag{51}$$

For the calculation of the Lyapunov exponent, the function $G(k,\cdot,\cdot,0)$ is the one given in Eq. (45).

For the kicked case in (46) the function $\varepsilon(t)$ is obtained by establishing a matrix recurrence relation. Between times $t_{n-1}$ and $t_n$ we denote the function $\varepsilon(t)$ by $\varepsilon_n(t)$. Then

$$\varepsilon_n(t)=a_n+b_nt, \tag{52}$$

and the following matrix recurrence is obtained

$$\begin{pmatrix}a_n\\ b_n\end{pmatrix}=\begin{pmatrix}1+\dfrac{\gamma\alpha}{\pi}n & \dfrac{\gamma\alpha}{\pi}n^2\\ -\dfrac{\gamma\alpha}{\pi} & 1-\dfrac{\gamma\alpha}{\pi}n\end{pmatrix}\begin{pmatrix}a_{n-1}\\ b_{n-1}\end{pmatrix} \tag{53}$$

with initial condition

$$a_0 = 1, \qquad b_0 = \mathrm{i}. \tag{54}$$

The recurrence relation (53) for the coefficients $a_n, b_n$ yields the map

$$\begin{pmatrix} \mathcal{E}(t_{n+1}) \\ \dot{\mathcal{E}}(t_{n+1}) \end{pmatrix} = \begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\dfrac{\gamma\alpha}{\pi} & 1-\dfrac{\gamma\alpha}{\pi} \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix} \tag{55}$$

for the position and momentum just after each kick.

The Floquet matrix in (55) has two eigenvalues

$$\lambda_{1,2} = 1 - \frac{\gamma\alpha}{2\pi} \pm \sqrt{\frac{(\gamma\alpha)^2}{4\pi^2} - \frac{\gamma\alpha}{\pi}}. \tag{56}$$

Substituting in Eq. (43) one concludes that for $z = \gamma\alpha/\pi < 4$ the Lyapunov exponent vanishes and that for $z = \gamma\alpha/\pi > 4$ there is one positive Lyapunov exponent

$$\lambda = \ln|1 - \tfrac{1}{2}z - \sqrt{\tfrac{1}{4}z^2 - z}|, \tag{57}$$

This is just the classical result and also the quantum result for another definition of quantum exponent [7,11], already discussed in Section 4. The positive Lyapunov exponent corresponds simply to the situation where the Floquet operator spectrum is transient absolutely continuous [42]. However, the next example corresponds to a classically chaotic system which when quantized yields an absolutely continuous quasi-energy spectrum [43]. This suggests that it might be an example of genuine quantum chaos. The Lyapunov exponent analysis supports this conclusion.

### 5.3. *The configurational quantum cat*

The configurational quantum cat is a system with four-dimensional phase-space, for which the configuration space dynamics resembles the classical Arnold cat [44]. It describes a charged particle constrained to move in the unit square with periodic boundary conditions, under the influence of time-dependent electromagnetic pulses. It may be associated to the Hamiltonians [7,43]

$$H_1 = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + x_2 p_1 + x_1 p_2 \sum_{n\in Z} \delta(t - n\tau), \tag{58}$$

or

$$H_1 = \tfrac{1}{2}p_1^2 + \tfrac{1}{2}p_2^2 + (x_2 p_1 + (x_1 + x_2) p_2) \sum_{n\in Z} \delta(t - n\tau). \tag{59}$$

A similar model may be constructed [7] by considering only the kick part and defining the quantum theory directly by the Floquet operator. To deal with the Hamiltonians (58) or (59) by the tomographic formalism, we need to extend it to non-local potentials.

Let a non-local Hamiltonian be written as

$$H(x, p) = T(p) + V(x) + I(x, p), \tag{60}$$

where $T(p)$ is the kinetic energy, $V(x)$ the local potential energy and $I(x, p)$ the symmetrized position and momentum-dependent interaction. Then, the equation for the density operator $\rho$

$$\dot{\rho} + \mathrm{i}(H\rho - \rho H) = 0 \tag{61}$$

becomes, in the tomographic representation

$$\dot{w}(\mathbf{X}, \vec{\mu}, \vec{\nu}, t) + \mathrm{i}H(\hat{\mathbf{x}}^{(1)}, \hat{\mathbf{p}}^{(1)})w(\mathbf{X}, \vec{\mu}, \vec{\nu}, t) - \mathrm{i}H(\hat{\mathbf{x}}^{(2)}, -\hat{\mathbf{p}}^{(2)})w(\mathbf{X}, \vec{\mu}, \vec{\nu}, t) = 0, \tag{62}$$

where, for $n$ degrees of freedom $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, $\vec{\nu} = (\nu_1, \nu_2, \ldots, \nu_n)$, and the components of the vector-operators $\hat{\mathbf{x}}^{(1),(2)}$, $\hat{\mathbf{p}}^{(1),(2)}$, act on $w(\mathbf{X}, \vec{\mu}, \vec{\nu}, t)$ as follows:

$$\hat{x}_k^{(1)} = -\left(\frac{\partial}{\partial X_k}\right)^{-1} \frac{\partial}{\partial \mu_k} + \frac{\mathrm{i}}{2}\nu_k \frac{\partial}{\partial X_k} = \hat{x}_k^{(2)*}, \qquad \hat{p}_k^{(1)} = -\frac{\mathrm{i}}{2}\mu_k \frac{\partial}{\partial X_k} - \left(\frac{\partial}{\partial X_k}\right)^{-1} \frac{\partial}{\partial \nu_k} = -\hat{p}_k^{(2)*}. \tag{63}$$

For the propagator $\Pi(X, \mu, \nu, X', \mu', \nu', t)$ corresponding to Eq. (62) one obtains an equation where the quantum contributions are explicitly expressed by a series in powers of $\hbar$. To do this, we first introduce some notation.

Let $n$-vectors $\mathbf{x}$ and $\mathbf{p}$ be described by one $2n$-vector $Q_\alpha, \alpha = 1, 2, \ldots, 2n$, with components $(p_1, p_2, \ldots, p_n, x_1, x_2, \ldots, x_n)$, i.e.,

$$Q_1 = p_1, \quad Q_2 = p_2, \quad Q_n = p_n, \quad Q_{n+1} = x_1, \quad Q_{2n} = x_n.$$

Let us also define the operator-vector $\tilde{\mathbf{Q}}$ with components $\tilde{Q}_\alpha (\alpha = 1, 2, \ldots, 2n)$

$$\tilde{Q}_1 = -\left(\frac{\partial}{\partial X_1}\right)^{-1} \frac{\partial}{\partial \nu_1}, \quad \tilde{Q}_2 = -\left(\frac{\partial}{\partial X_2}\right)^{-1} \frac{\partial}{\partial \nu_2}, \quad \ldots, \quad \tilde{Q}_n = -\left(\frac{\partial}{\partial X_n}\right)^{-1} \frac{\partial}{\partial \nu_n}$$
$$\tilde{Q}_{n+1} = -\left(\frac{\partial}{\partial X_1}\right)^{-1} \frac{\partial}{\partial \mu_1}, \quad \tilde{Q}_{n+2} = -\left(\frac{\partial}{\partial X_2}\right)^{-1} \frac{\partial}{\partial \mu_2}, \quad \ldots, \quad \tilde{Q}_{2n} = -\left(\frac{\partial}{\partial X_n}\right)^{-1} \frac{\partial}{\partial \mu_n},$$

and a $2n$-vector $\mathbf{d}$ with the components

$$d_1 = -\frac{\mu_1}{2}\frac{\partial}{\partial X_1}, \quad d_2 = -\frac{\mu_2}{2}\frac{\partial}{\partial X_2}, \quad \ldots, \quad d_n = -\frac{\mu_n}{2}\frac{\partial}{\partial X_n},$$
$$d_{n+1} = \frac{\nu_1}{2}\frac{\partial}{\partial X_1}, \quad d_{n+2} = \frac{\nu_2}{2}\frac{\partial}{\partial X_2}, \quad \ldots, \quad d_{2n} = \frac{\nu_n}{2}\frac{\partial}{\partial X_n}.$$

Then, the equation for the propagator of Eq. (62) is

$$\dot{\Pi} - \frac{2}{\hbar}\sin\left(\hbar\mathbf{d}\frac{\partial}{\partial \mathbf{Q}}\right) H(\mathbf{Q})|_{\mathbf{Q}\to\tilde{\mathbf{Q}}}\Pi = 0, \tag{64}$$

where

$$\mathbf{d}\frac{\partial}{\partial \mathbf{Q}} \equiv \sum_{\alpha=1}^{2n} d_\alpha \frac{\partial}{\partial Q_\alpha}.$$

Using the series expansion for $\sin\alpha$ and separating the classical-limit term from the quantum corrections one obtains

$$\dot{\Pi} - 2\mathbf{d}\frac{\partial}{\partial \mathbf{Q}} H(\mathbf{Q})|_{\mathbf{Q}\to\tilde{\mathbf{Q}}}\Pi - \frac{2}{\hbar}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\hbar\mathbf{d}\frac{\partial}{\partial \mathbf{Q}}\right)^{2n+1} H(\mathbf{Q})|_{\mathbf{Q}\to\tilde{\mathbf{Q}}}\Pi = 0. \tag{65}$$

In particular one sees that for quadratic interactions (local or non-local) the quantum evolution is formally identical to the classical one. In the configurational quantum cat we have a two degree of freedom Hamiltonian

$$H(\mathbf{x},\mathbf{p},t)=H_0(\mathbf{x},\mathbf{p})+H_k(\mathbf{x},\mathbf{p})\sum_{n=-\infty}^{\infty}\delta(t-n), \tag{66}$$

where both $H_0$ and $H_k$ are quadratic forms in the position and momentum operators. To write the functions explicitly, we define a four-vector

$$\mathbf{Q}=\begin{pmatrix}\mathbf{p}\\ \mathbf{x}\end{pmatrix} \tag{67}$$

and symmetric 4×4-matrices $B_0$ and $B_k$.

Then the Hamiltonians $H_0$ and $H_k$ are taken in the form

$$H_0=\tfrac{1}{2}\mathbf{Q}B_0\mathbf{Q},\qquad H_k=\tfrac{1}{2}\mathbf{Q}B_k\mathbf{Q}. \tag{68}$$

The system with the Hamiltonian (66) has four linear integrals of motion [45]

$$\mathbf{I}(t)=\Lambda(t)\mathbf{Q}, \tag{69}$$

where the symplectic matrix satisfies the equation

$$\dot{\Lambda}(t)=\Lambda\Sigma B(t), \tag{70}$$

with a 4×4-matrix $\Sigma$ with identity 2×2-blocks

$$\Sigma=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{71}$$

and

$$B(t)=B_0+B_k\sum_{n=-\infty}^{\infty}\delta(t-n). \tag{72}$$

The initial condition for the 4×4-matrix $\Lambda(t)$ is

$$\Lambda(0)=1. \tag{73}$$

The Floquet solution to (70) has the form

$$\Lambda(1_+)=\mathrm{e}^{\Sigma B_0}\,\mathrm{e}^{\Sigma B_k} \tag{74}$$

and for $n$ kicks

$$\Lambda(n_+)=\Lambda^n(1_+)=(\mathrm{e}^{\Sigma B_0}\,\mathrm{e}^{\Sigma B_k})^n. \tag{75}$$

The equation for the propagator is

$$\dot{\Pi}+\mathrm{i}\left[H_0(\hat{\mathbf{x}}^{(1)},\hat{\mathbf{p}}^{(1)})+H_k(\hat{\mathbf{x}}^{(1)},\hat{\mathbf{p}}^{(1)})\sum_{n=-\infty}^{\infty}\delta(t-n)\right]\Pi \tag{76}$$

$$-\mathrm{i}\left[H_0(\hat{\mathbf{x}}^{(2)},-\hat{\mathbf{p}}^{(2)})+H_k(\hat{\mathbf{x}}^{(2)},-\hat{\mathbf{p}}^{(2)})\sum_{n=-\infty}^{\infty}\delta(t-n)\right]\Pi=0, \tag{77}$$

the vector-operators being

$$\hat{\mathbf{x}}^{(1)} = \left( -\left[\frac{\partial}{\partial X_1}\right]^{-1} \frac{\partial}{\partial \mu_1} + \frac{\mathrm{i}}{2}\nu_1 \frac{\partial}{\partial X_1}, -\left[\frac{\partial}{\partial X_2}\right]^{-1} \frac{\partial}{\partial \mu_2} + \frac{\mathrm{i}}{2}\nu_2 \frac{\partial}{\partial X_2} \right), \tag{78}$$

$$\hat{\mathbf{p}}^{(1)} = \left( -\frac{\mathrm{i}}{2}\mu_1 \frac{\partial}{\partial X_1} - \left[\frac{\partial}{\partial X_1}\right]^{-1} \frac{\partial}{\partial \nu_1}, -\frac{\mathrm{i}}{2}\mu_2 \frac{\partial}{\partial X_2} - \left[\frac{\partial}{\partial X_2}\right]^{-1} \frac{\partial}{\partial \nu_2} \right),$$

and

$$\hat{\mathbf{x}}^{(2)} = \hat{\mathbf{x}}^{(1)*}, \qquad \hat{\mathbf{p}}^{(2)} = -\hat{\mathbf{p}}^{(1)*}.$$

The equation for the Fourier component $G(\mathbf{k}, \vec{\mu}, \vec{\nu}, t)$ used to compute the Lyapunov exponent is the same as Eq. (76) with

$$\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2} \to \mathrm{i}k_1, \mathrm{i}k_2.$$

Eq. (76) can be integrated by

$$\Pi(\mathbf{X}, \vec{\mu}, \vec{\nu}, n) = \Pi(\mathbf{X}, \vec{\mu}_\Lambda, \vec{\nu}_\Lambda, 0), \tag{79}$$

where the parameters $\vec{\mu}_\Lambda, \vec{\nu}_\Lambda$ are expressed in terms of $\vec{\mu}$ and $\vec{\nu}$ by the matrix product rule

$$(\vec{\nu}_\Lambda, \vec{\mu}_\Lambda) = (\vec{\nu}, \vec{\mu})\Lambda^{-1}(n_+), \tag{80}$$

where $\Lambda^{-1}(n_+)$ is given by (75).

For the Hamiltonians $H_1$ and $H_2$ of Eqs. (58) and (59) the matrices $B_0$ and $B_k$ are

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad B_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for $H_1$ and

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad B_k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

for $H_2$.

For the model [7] where only the kick contributions are kept in the Hamiltonian one has

$$B_0 = 0,$$

and

$$B_k = \frac{\ln(1+\omega)}{\omega+2} \begin{pmatrix} 0 & \mathcal{L}(\omega) \\ \mathcal{L}(\omega) & 0 \end{pmatrix},$$

the $2 \times 2$-matrix being

$$\mathcal{L}(\omega) = \begin{pmatrix} -\omega & \dfrac{2(1+\omega)}{\omega} \\ 2\omega & \omega \end{pmatrix},$$

with $\omega = \frac{1}{2}(1+\sqrt{5})$.

For this model the matrix $\Lambda^{-1}(n_+)$ in (80) is given by

$$\Lambda^{-1}(n_+) = \begin{pmatrix} \tilde{f}^n & 0 \\ 0 & \tilde{f}^{-n} \end{pmatrix},$$

where

$$\tilde{f}^n = \begin{pmatrix} \omega^{-2n+1} + \omega^{2n-1} & -\omega^{-2n} + \omega^{2n} \\ -\omega^{-2n} + \omega^{2n} & \omega^{-2n+1} + \omega^{2n+1} \end{pmatrix}.$$

For all three models Eq. (76) is the same for classical and quantum motion. Therefore the Lyapunov exponent must be the same in the classical and quantum cases. In particular, as is known from the classical case [7], there is a positive Lyapunov exponent, namely

$$\lambda = \ln \omega^2.$$

### 5.4. The standard map

The standard map is a case where the phenomena of wave function localization is believed to have a taming effect on chaos. The Lyapunov exponent analysis gives a characterization of how this taming effect comes about.

The Hamiltonian is

$$H = \frac{p^2}{2} + \gamma \cos(q) \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{81}$$

the configuration space being now the circle, $q \in S^1$. This system describes a particle rotating in a ring and subjected to periodic kicks. It has been extensively used in studies of quantum chaos [46–50] and has even been tested experimentally with ultra-cold atoms trapped in a magneto-optic trap [51].

From (44) the equation to be solved now is

$$\frac{\partial G}{\partial t} - \mu \frac{\partial G}{\partial \nu} - \frac{\gamma}{\hbar} \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \sin\left(\frac{\hbar}{2}\nu\right) \{G(1, \mu + 1, \nu, t) - G(1, \mu - 1, \nu, t)\} = 0, \tag{82}$$

where we have specialized to the value $k = 1$ because this is the only $k$-value needed to compute the Lyapunov exponent (43). Notice that we have used here the same tomographic transformations that were described in Section 2 for functions on the line. This is justified by considering all functions as defined not in $S^1$ but in the suspension of $S^1$.

From (82) one sees that between any two kicks the function propagates freely, namely

$$G(1, \mu, \nu, t_0) \to G(1, \mu, \nu, t_{1_-}) = G(1, \mu, \nu + \mu\tau, t_0), \tag{83}$$

and at the time of the kick a quantity is added that is proportional to a finite difference (in $\mu$):

$$G(1, \mu, \nu, t_{1_+}) = G(1, \mu, \nu, t_{1_-}) + \tfrac{1}{2}\gamma f(\nu)\{G(1, \mu + 1, \nu, t_{1_-}) - G(1, \mu - 1, \nu, t_{1_-})\}, \tag{84}$$

where, for the classical case

$$f(\nu) = \nu, \tag{85}$$

and for the quantum case

$$f(\nu) = \frac{2}{\hbar} \sin\left(\frac{\hbar}{2}\nu\right). \tag{86}$$

To compute the Lyapunov exponent we need the evolution of the derivatives (in $\mu$ and $\nu$) of $G$ at $\mu=\nu=0$. From (83) and (84) one obtains the following iteration for the derivatives:

$$G^{(2)}(1,0,0,t+1)=G^{(2)}(1,0,0,t)+G^{(3)}(1,0,0,t),$$
$$G^{(3)}(1,0,0,t+1)=G^{(2)}(1,0,0,t)+\tfrac{1}{2}\gamma(G(1,1,\tau,t)-G(1,-1,-\tau,t)). \tag{87}$$

Let us first consider the classical case ($\hbar=0$ , $f(\nu)=\nu$ and $\gamma>0$). Let $\tau=1$ and $q_0=p_0=0$ in the initial condition (45). Then, one obtains the following recursion for the derivatives of $G$ at $\mu=\nu=0$:

$$G^{(2)}(1,0,0,n+1)=G^{(2)}(1,0,0,n)+G^{(3)}(1,0,0,n),$$
$$G^{(3)}(1,0,0,n+1)=\gamma G^{(2)}(1,0,0,n)+(1+\gamma)G^{(3)}(1,0,0,n), \tag{88}$$

which has the solution

$$G^{(2)}(1,0,0,n)=A_n(z)v_1+B_n(z)v_2,\qquad G^{(3)}(1,0,0,n)=C_n(z)v_1+D_n(z)v_2 \tag{89}$$

with $z=2+\gamma$ and

$$A_n(z)=U_{n-1}\left(\frac{z}{2}\right)-U_{n-2}\left(\frac{z}{2}\right),\qquad B_n(z)=\frac{1}{z-2}C_n(z),$$
$$C_n(z)=U_n\left(\frac{z}{2}\right)-2U_{n-1}\left(\frac{z}{2}\right)+U_{n-2}\left(\frac{z}{2}\right),\qquad D_n(z)=U_n\left(\frac{z}{2}\right)-U_{n-1}\left(\frac{z}{2}\right), \tag{90}$$

where $U_n(z)=(\sin((n+1)\cos^{-1}z))/(\sin(\cos^{-1}z))$ is a Chebyshev polynomial.

For the Lyapunov exponent one obtains in this case

$$\lambda=\ln|1+\tfrac{1}{2}\gamma+\sqrt{\tfrac{1}{4}\gamma^2+\gamma}|, \tag{91}$$

a result similar to the harmonic kicks on the line. One sees that as long as $\gamma>0$ the exponent $\lambda$ in Eq. (91) is always positive. This results from the choice made for the phase-space point ($p_0=q_0=0$) where the marginal distribution receives the singular perturbation (34). If instead we had chosen ($p_0=0$ and $q_0=\pi$) in the initial condition (45), one sees easily by a change of coordinates in the Hamiltonian that this is equivalent to replace $\gamma$ by $-\gamma$. Then the Lyapunov exponent $\lambda$ in Eq. (91) is positive only for $\gamma>4$. As discussed at length in the next section, this only means that it is the phase-space point $(p_0,q_0)$ that defines the measure for which the Lyapunov exponent is computed. Hence, for the measure that supports the hyperbolic point ($p_0=0$, $q_0=0$) the exponent is always positive, whereas for sufficiently small $\gamma>0$ the exponent for the measure that supports the elliptic point ($p_0=0$, $q_0=\pi$) is negative.

For the quantum case ($\hbar\neq 0$) let us consider an initial condition $G(1,\mu,\nu,0)=\mu+\nu$ (corresponding to $p_0=0$, $q_0=0$, $v_1=v_2=1$) and $\tau=1$. According to Eq. (87), all one needs to compute the Lyapunov exponent is the time-evolution of $G(1,1,1,t)$. For this purpose we set up a matrix recursion for the evolution equations (83) and (84). Define the following matrices:

$$M_0=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad M_+=\begin{pmatrix}1&1&0\\0&1&0\\1&1&1\end{pmatrix},\quad M_-=\begin{pmatrix}1&1&0\\0&1&0\\-1&-1&-1\end{pmatrix}, \tag{92}$$

and vectors

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$$

where $\alpha$ counts the number of $\mu$'s, $\beta$ the number of $\nu$'s and $\gamma$ is a simple number. Then with Tr denoting the sum of the elements in a vector and

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad y_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{93}$$

the initial condition is $G(1, 1, \tau, 0) = \mathrm{Tr}(x_0)$ and the function $f(\nu) = f(\mathrm{Tr}(y_0))$.

On arbitrary functions of three-dimensional vectors, the operators $K_0, K_+, K_-$ act on the arguments by the matrices $M_0, M_+, M_-$:

$$K_i g(x) = g(M_i x). \tag{94}$$

Then

$$G(1, 1, 1, n) = \mathrm{Tr}\{(K_0 + \tfrac{1}{2}\gamma f(y_0)[K_+ - K_-])^n x_0\}, \tag{95}$$

where it is understood that the power of the operator is fully expanded before the Tr operation is applied to each one of the vector arguments. When this expansion is made, one obtains an expression of the form

$$G(1, 1, 1, n) = n + 2 + \mathrm{Tr}\left\{\sum_{k=1}^{n} \left(\frac{\gamma}{2}\right)^k \sum_{i=1}^{2^{k-1}n!/k!(n-k)!} c_i f\left(\prod M_{i_1} y_0\right) \dots f\left(\prod M_{i_k} y_0\right)\right\}. \tag{96}$$

The products of $M$ matrices in the arguments of $f(\cdot)$ contain a variable number of factors, from 1 to $k$. However for each term, a different combination of products will appear. For $\hbar \neq 0$ the function $f(\nu)$ is proportional to a sine and, if $\hbar\tau/4\pi$ is irrational, the coefficient of each $\gamma^k$ behaves like a sum of random variables of zero mean. Therefore each coefficient averages to zero and

$$G(1, 1, 1, n) \sim n + 2. \tag{97}$$

Large fluctuations are however to be expected in view of the large number of terms in the sums for large $n$. From (87), the result (97) now implies

$$G^{(2)}(1, 0, 0, n) \sim 1 + n(1 - \gamma) + \tfrac{1}{2}\gamma(\tfrac{1}{2}n(n + 1) + \tfrac{1}{6}n(n + 1)(2n + 1)),$$
$$G^{(3)}(1, 0, 0, n) \sim 1 + \gamma(\tfrac{1}{2}(n + 1)(n + 2) - 1).$$

For large $n$, $\log G^{(2)}(1, 0, 0, n) \sim 3 \log n$ and $\log G^{(2)}(1, 0, 0, n) \sim 2 \log n$ and the Lyapunov exponent vanishes.

The situation we have been studying ($p_0 = q_0 = 0$ in the initial perturbation) corresponds to the (hyperbolic) case where the classical Lyapunov exponent is positive for any $\gamma$. We see here clearly the taming effect of quantum mechanics on classical chaos and its dynamical origin. It results from the replacement in the evolution equation of the linear function $f(\nu) = \nu$ by $f(\nu) = 2/\hbar \sin((\hbar/2)\nu)$. This in turn is a consequence of the replacement of the classical Boltzman equation by the quantum evolution equation (22), or in algebraic terms, by the replacement of the ordinary product by the Moyal–Vey product in the non-commutative quantum phase-space.

In this model, the origin of the taming effect of quantum mechanics on classical chaos, is traced back to the existence, in the $\hbar$-deformed equation (44), of infinitely many terms in the series which add up to a bounded function in $\nu$. How general this mechanism is, for other quantum systems, is an open question. In any case the taming effect of quantum mechanics, obtained here for the standard map, is more accurate than previous discussions of the same system, because it refers to the behavior of the Lyapunov exponent rather than to indirect chaos symptoms, like the energy growth or diffusion behavior.

## 6. Remarks and conclusions

(1) The method developed in this paper, for the quantum Lyapunov exponents, provides a fairly unambiguous construction of these quantities, in the sense that classical and quantum exponents have the same functional form. The difference lies only on the time-evolution laws for the propagators.

The dynamical evolution laws of the marginal distributions obtained by the tomographic map are apparently more complex than the familiar Schrödinger equation. However, for the computation of the Lyapunov exponents, they provide a fairly efficient computational scheme.

Quantum mechanics is widely believed to have a taming effect on classical chaos. However, most discussions are of a qualitatively nature and fail to identify the conditions under which the taming effect is expected to occur and those in which it will not occur. This is a very relevant question in view of the fact that for local quadratic potentials, the quantum behavior differs very little from the classical one and genuine examples of quantum chaos with bounded configuration space are known, like the four-dimensional or configurational quantum cat [7,43].

In the standard map, studied in Section 5, it is clear that the suppression of chaos is directly related both to the nature of the potential and the analytical structure of the series in Eq. (44). This operational series, when acting on the potential, convert an unbounded function into a bounded function in $\nu$ (the symplectic parameter conjugate to $p$). The structure of the series corresponds to the structure of the Moyal bracket and the way the quadratic potential (and presumably other polynomial potentials) avoid the suppression effect, is by truncating the action of the Moyal bracket to a finite number of cocycles.

The fact that for non-polynomial interactions all derivatives of the potential intervene in the quantum evolution means by an analyticity argument that the future evolution of any local perturbation depends strongly on what is going on at all other points. This interference between quantum "trajectories" is probably the decisive factor that determines the nature of the quantum modifications of classical chaos.

(2) A second important question concerns the support properties of the Lyapunov exponents that have been constructed. In classical mechanics, Lyapunov exponents are ergodic invariants. This means that they are defined in the support of some measure. In the construction (both classical and quantum) developed in Section 3, the Lyapunov is obtained from a singular perturbation of the $X$-coordinate at the point $\mu q_0 + \nu p_0$ for each pair $(\mu, \nu)$. For the classical case, the interpretation is clear. From the point of view of measures in phase-space, it means that one is constructing the Lyapunov exponent that corresponds to the measure whose support contains the point $(q_0, p_0)$.

Measures on classical phase-space may be interpreted as measures on the joint spectrum of the (commuting) operators $q$ and $p$. Therefore in the classical case a measure $\mu$ plays the double role of a probability measure in phase-space and a spectral measure for the dynamical operators. For the quantum case, however, $q$ and $p$ do not commute and there is no joint spectrum for these operators. Then, instead of one measure playing a double role we have two:

- One is the state that is perturbed. This is the analog of the classical probability measure, because states are the non-commutative analogs of Borel measures.
- The other is the spectral measure of the operator $X$, the perturbation acting, for each pair $(q_0, p_0)$, at the point $\mu q_0 + \nu p_0$ of the spectrum.

In conclusion the interpretation of the quantum Lyapunov exponent as an ergodic invariant requires two measures, a state and a spectral measure. In the classical case the two measures coincide.

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