



Quantum control in infinite dimensions [☆]

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Abstract

Accurate control of quantum evolution is an essential requirement for quantum state engineering, laser chemistry, quantum information and quantum computing. Conditions of controllability for systems with a finite number of energy levels have been extensively studied. By contrast, results for controllability in infinite dimensions have been mostly negative, stating that full control cannot be achieved with a finite-dimensional control Lie algebra. Here we show that by adding a discrete operation to a Lie algebra it is possible to obtain full control in infinite dimensions with a small number of control operators.

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1. Introduction

To control the time evolution of quantum systems is an essential step in many new and old applications of quantum theory [1]. Among the fields requiring accurate control of quantum mechanical time evolution are quantum state engineering, cooling of molecular degrees of freedom, selective excitation, chemical reactions and quantum computing.

The first general results on controllability of quantum systems have been obtained by Huang, Tarn and

Clark (HTC) [2]. Subsequently an extensive amount of work has been done, mostly on establishing conditions and degrees of controllability for systems with a finite number of energy levels. By contrast, results for controllability in the infinite-dimensional Hilbert sphere $S_{\mathcal{H}}$ have been mostly negative, stating that full control in $S_{\mathcal{H}}$ cannot be achieved with a finite-dimensional control Lie algebra [2], a result similar to the one obtained by Ball, Marsden and Slemrod for classical control systems in Banach spaces [3].

It seemed therefore that control in infinite dimensions would require the use of infinite-dimensional Lie algebras. This is what happens in the proposal of Lloyd and Braunstein [4] for quantum computation over continuous variables. They propose, for the construction of a universal quantum computer, the successive application of quadratic Hamiltonians and a higher order one (for example, the Kerr Hamiltonian).

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By commutation all polynomial Hamiltonians are obtained. Hence, the control algebra, although finitely generated, is an infinite-dimensional Lie algebra.

The HTC no-go result is based on a local argument concerning the finite dimensionality of the local manifold generated by the unitary action of the finite-dimensional control group. Left open is the question of whether the local argument extends to the global action in the Hilbert sphere, which however is probably true. On the other hand the result does not apply to non-Lie groups, for example, a Lie group complemented by a discrete operation. This is the kind of situation that is explored in this Letter. As a physical motivation for the kind of control situation we deal with, consider a charged particle in a circle (a charged plane rotator) with Hilbert space $L^2(0, 2\pi)$ and free Hamiltonian

$$H_0 = -\frac{\partial^2}{\partial \varphi^2}$$

$\varphi \in [0, 2\pi)$, defined in the domain

$$D(H_0) = \{f \in AC^2; f(0) = f(2\pi)\},$$

where AC^2 stands for functions with absolutely continuous first derivative. The eigenstates of H_0 are $\{|k\rangle = e^{ik\varphi}; k \in \mathbb{Z}\}$ with eigenvalues k^2 .

An application of a magnetic field pulse corresponds to the unitary operator

$$U_+ = e^{i\varphi}$$

which shifts the eigenstates one level up

$$U_+|k\rangle = |k+1\rangle,$$

with inverse

$$U_+^{-1}|k\rangle = |k-1\rangle.$$

Because the energy level spacing is not uniform

$$\Delta E_k = k^2 - (k-1)^2 = 2k - 1$$

one may, by resonant and non-resonant excitation, make arbitrary $U(2)$ transformations between a particular pair of successive levels [5]. As it turns out (Section 2), these simple controls (namely U_+ , U_+^{-1} , $U(2)$) are sufficient for full controllability of this infinite-dimensional system.

2. Control in the $\ell^2(\mathbb{Z})$ —Hilbert sphere

Consider the space of double-infinite square-integrable sequences

$$a = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\} \in \ell^2(\mathbb{Z}),$$

$$|a| = \left(\sum_{-\infty}^{\infty} |a_k|^2 \right)^{1/2} < \infty,$$

with basis

$$e_k = \{\dots, 0, 0, 1_k, 0, 0, \dots\},$$

$$a = \sum_{-\infty}^{\infty} a_k e_k. \tag{1}$$

Define:

- (i) A linear operator U_+ acting as a shift on the basis states

$$U_+ e_k = e_{k+1}, \quad k \in \mathbb{Z}$$

and its inverse

$$U_+^{-1} e_k = e_{k-1}, \quad k \in \mathbb{Z}.$$

- (ii) Another linear operator Π

$$\Pi e_0 = e_1,$$

$$\Pi e_1 = e_0,$$

$$\Pi e_k = e_k, \quad k \in \mathbb{Z} \setminus \{0, 1\}.$$

Then

$$\Pi_n = U_+^n \Pi U_+^{-n}$$

acts as

$$\Pi_n e_n = e_{n+1},$$

$$\Pi_n e_{n+1} = e_n,$$

$$\Pi e_k = e_k, \quad k \neq n, n+1.$$

In the sequence $a = \sum_{-\infty}^{\infty} a_k e_k$ it exchanges a_n with a_{n+1} . Likewise

Lemma 1. Given $a \in \ell^2(\mathbb{Z})$, $k \in \mathbb{Z}$, $l \in \mathbb{Z}$, the linear operator $\Pi_{k,k+l}$ ($l \in \mathbb{N}$) defined by $\Pi_{k,k+l} = \Pi_k$ and

$$\Pi_{k,k+l} a = \Pi_k \Pi_{k+1} \cdots \Pi_{k+l-2} \Pi_{k+l-1} \cdots \Pi_{k+1} \Pi_k a, \tag{2}$$

for $l \geq 2$ exchanges the coefficients of e_k and e_{k+l} in (1), that is

$$\Pi_{k,k+l}a = a_{k+l}e_k + a_k e_{k+l} + \sum_{r \neq k,k+l} a_r e_r.$$

Theorem 1. Let $G(U_+, \Pi)$ stand for the group generated by U_+ , U_+^{-1} and Π . Then for any $0 \neq a \in \ell^2(\mathbb{Z})$ the linear span of $G(U_+, \Pi)a$ is dense in $\ell^2(\mathbb{Z})$.

Proof. It is sufficient to show that $b \perp G(U_+, \Pi)a$ implies $b = 0$. Suppose $b = e_k$ for some $k \in \mathbb{Z}$. Since $a \neq 0$ there is $l \in \mathbb{N} \cup \{0\}$ such that at least one of the numbers a_{k+l} or a_{k-l} is different from zero. Then $(b, \Pi_{k,k+l}a) = a_{k+l}$ or $(b, \Pi_{k-l,k}a) = a_{k-l}$, a contradiction. Similarly if both a and b are terminating sequences, that is, sequences with a finite number of non-zero elements.

Suppose now that b is terminating but a is not. Let $b_k = 0$ for $|k| > N'$. Then $\exists N \leq N'$ such that $(b, a) = \sum_{-N}^N b_k^* a_k = 0$ and either $b_N^* a_N \neq 0$ or $b_{-N}^* a_{-N} \neq 0$. Then there is l such that $a_{N+l} \neq a_N$ or $a_{-N-l} \neq a_{-N}$. Hence $(b, \Pi_{N,N+l}a) = \sum_{-N}^{N-1} b_k^* a_k + b_N^* a_{N+l} \neq 0$ or $(b, \Pi_{N,-N-l}a) = \sum_{-N}^{N-1} b_k^* a_k + b_N^* a_{-N-l} \neq 0$, a contradiction. Similarly for a terminating and b non-terminating.

If neither a nor b terminates, then there are pairs $a_k \neq a_l$ and $b_m \neq b_n$. With appropriate $g, g' \in G(U_+, \Pi)$ we obtain

$$(b, ga) = b_m^* a_k + b_n^* a_l + b_k^* a_m + b_l^* a_n + \sum_{r \neq k,l,m,n} b_r^* a_r = 0,$$

$$(b, g'a) = b_n^* a_k + b_m^* a_l + b_k^* a_m + b_l^* a_n + \sum_{r \neq k,l,m,n} b_r^* a_r = 0.$$

Hence $b_m^* a_k + b_n^* a_l = b_n^* a_k + b_m^* a_l$, which is possible only if either $b_m = b_n$ or $a_k = a_l$, a contradiction. \square

Now instead of the Π operator we consider a $U(2)$ group operating in the linear space spanned by e_0 and e_1 and as the identity on $\ell^2(\mathbb{Z}) \ominus \{e_0, e_1\}$. In particular $\Pi \in U(2)$.

Theorem 2. For any $a \in \ell^2(\mathbb{Z})$, $|a| = 1$, the set $G(U_+, U(2))a$ is dense in the $\ell^2(\mathbb{Z})$ –Hilbert sphere.

Lemma 2. Suppose a is a terminating normalized sequence. Then, there is $g \in G(U_+, U(2))$ such that $ge_0 = a$.

Proof. Let

$$a = a_{-N}e_{-N} + \dots + a_0e_0 + \dots + a_Ne_N.$$

By $U(2)$ transformations in the $\{e_0, e_1\}$ subspace and use of the $\Pi_{k,k+l}$ operators (Eq. (2)) one constructs with operators $g_i \in G(U_+, U(2))$ the following sequence

$$\begin{aligned} g_1e_0 &= x_1e_0 + a_{-N}e_{-N} = \alpha_1, \\ g_2\alpha_1 &= x_2e_0 + a_{-N+1}e_{-N+1} + a_{-N}e_{-N} = \alpha_2, \\ &\vdots \end{aligned}$$

$$g_{2N}\alpha_{2N-1} = x_{2N}e_0 + \sum_{-N}^N a_k e_k = \alpha_{2N},$$

$$g_{2N+1}\alpha_{2N} = a.$$

Finally

$$g_{2N+1}g_{2N} \dots g_2g_1e_0 = a. \quad \square$$

Proof of Theorem 2. Consider $a, b \in \ell^2(\mathbb{Z})$ with $|a| = |b| = 1$. Choose ε and N such that

$$\alpha = \left| \sum_{-N}^N a_k e_k \right| > 1 - \varepsilon,$$

$$\beta = \left| \sum_{-N}^N b_k e_k \right| > 1 - \varepsilon.$$

By the Lemma 2 there are $g, g' \in G(U_+, U(2))$ such that

$$g \sum_{-N}^N a_k e_k = \alpha e_0,$$

$$g'(\alpha e_0) = \frac{\alpha}{\beta} \sum_{-N}^N b_k e_k.$$

Hence

$$|b - g'ga| \leq 2\varepsilon + \left| 1 - \frac{\alpha}{\beta} \right| \leq 3\varepsilon. \quad \square$$

In conclusion: given any initial state $0 \neq a \in \ell^2(\mathbb{Z})$ it is possible by the unitary action of an element in $G(U_+, U(2))$ to approach as closest as desired any other state b in $\ell^2(\mathbb{Z})$ with the same norm.

Any infinite-dimensional separable Hilbert space is isomorphic to ℓ^2 [6]. Therefore the results have a large degree of generality. However, depending on the Hilbert space realization for each concrete infinite-dimensional quantum system, the control operators discussed here may or may not be easy to implement.

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