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The fractional volatility model: An agent-based interpretation

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Abstract

Based on the criteria of mathematical simplicity and consistency with empirical market data, a model with volatility driven by fractional noise has been constructed which provides a fairly accurate mathematical parametrization of the data. Here, some features of the model are reviewed and extended to account for leverage effects. Using agent-based models, one tries to find which agent strategies and (or) properties of the financial institutions might be responsible for the features of the fractional volatility model.

Keywords: Fractional volatility; Agent-based models

1. Introduction

Classical Mathematical Finance has, for a long time, been based on the assumption that the price process of market securities may be approximated by geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB(t). \tag{1}$$

In liquid markets the autocorrelation of price changes decays to negligible values in a few minutes, consistent with the absence of long term statistical arbitrage. Geometric Brownian motion models this lack of memory, although it does not reproduce the empirical leptokurtosis. On the other hand, nonlinear functions of the returns exhibit significant positive autocorrelation. For example, there is volatility clustering, with large returns expected to be followed by large returns and small returns by small returns (of either sign). This, together with the fact that autocorrelations of volatility measures decline very slowly [1–3], has the implication that long memory effects should somehow be represented in the process and this is not included in the geometric Brownian motion hypothesis.

On the other hand, as pointed out by Engle [4], when the future is uncertain, investors are less likely to invest. Therefore uncertainty (volatility) would have to be changing over time. The conclusion is that a dynamical model for volatility is needed and σ in Eq. (1), rather than being a constant, becomes a process by itself. This idea led to many deterministic and stochastic models for the volatility (Refs. [5,6] and the references therein).

In a previous paper [7], using both a criteria of mathematical simplicity and consistency with market data, a stochastic volatility model was constructed, with volatility driven by fractional noise. It appears to be the minimal

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model consistent both with mathematical simplicity and the market data. This data-reconstructed model is different from the other stochastic volatility models that have been proposed in the literature. The model was used to compute the price return statistics and asymptotic behavior, which were compared with actual data. Deviations from the classical Black–Scholes result and a new option pricing formula were also obtained. The *fractional volatility model*, its predictions and comparison with data will be reviewed in Section 2. In addition, a new result concerning a specification of the model to account for the leverage effect is also included.

The fractional volatility model seems to be a reasonable *mathematical parametrization* of the market behavior. However, to obtain real economic understanding it is not sufficient to fit the data. One should also search for the mechanisms in the market that lead to the observed phenomena. No agent-based model can pretend to be the market itself, not even a realistic image of it. Nevertheless it may provide a surrogate model of the basic mechanisms at work there. Therefore, the idea in this paper is to use stylized agent-based market models and find out which features of these models correspond to the elements of the mathematical parametrization of the data.

2. The fractional volatility model

The basic hypothesis for the model construction were:

(H1) The log-price process $\log S_t$ belongs to a probability product space $\Omega \otimes \Omega'$ of which the first one, Ω, is the Wiener space and the second, Ω', is a probability space to be reconstructed from the data. Denote by $\omega \in \Omega$ and $\omega' \in \Omega'$ the elements (sample paths) in Ω and Ω' and by $\mathcal{F}_t$ and $\mathcal{F}'_t$ the σ-algebras in Ω and Ω' generated by the processes up to t. Then, a particular realization of the log-price process is denoted $\log S_t\left(\omega, \omega'\right)$. This first hypothesis is really not limitative. Even if none of the non-trivial stochastic features of the log-price were to be captured by Brownian motion, that would simply mean that S_t is a trivial function in Ω.

(H2) The second hypothesis is stronger, although natural. One assumes that, for each fixed ω', $\log S_t\left(\bullet, \omega'\right)$ is a square integrable random variable in Ω.

A mathematical consequence of hypothesis (H2) is that, for each fixed ω',

$$\frac{\mathrm{d}S_t}{S_t}\left(\bullet, \omega'\right) = \mu_t\left(\bullet, \omega'\right)\mathrm{d}t + \sigma_t\left(\bullet, \omega'\right)\mathrm{d}B(t) \tag{2}$$

where $\mu_t\left(\bullet, \omega'\right)$ and $\sigma_t\left(\bullet, \omega'\right)$ are processes in Ω. (Theorem 1.1.3 in Ref. [8])

Recall that if $\{X_t, \mathcal{F}_t\}$ is a process such that $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B(t)$, with μ_t and σ_t being $\mathcal{F}_t$-adapted processes, then

$$\begin{aligned} \mu_t &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{E(X_{t+\varepsilon} - X_t)|\mathcal{F}_t\} \\ \sigma_t^2 &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{E(X_{t+\varepsilon} - X_t)^2|\mathcal{F}_t\right\}. \end{aligned} \tag{3}$$

The process associated to the probability space Ω' could then be inferred from the data. According to (3), for each fixed ω' realization in Ω' one has

$$\sigma_t^2\left(\bullet, \omega'\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{E(\log S_{t+\varepsilon} - \log S_t)^2\right\}. \tag{4}$$

Each set of market data corresponds to a particular realization ω'. Therefore, assuming the realization to be typical, the σ_t^2 process may be reconstructed from the data by the use of (4). This data-reconstructed σ_t process was called the *induced volatility*.

Once several data sets were analyzed [7], the next step towards obtaining a mathematical characterization of the *induced volatility process* was to look for scaling properties. It turned out that neither $E\left|\sigma(t+\Delta) - \sigma(t)\right| \sim \Delta^H$ nor $E\left|\frac{\sigma(t+\Delta)-\sigma(t)}{\sigma(t)}\right| \sim \Delta^H$ were good hypothesis for the induced volatility process. It means that the induced volatility process itself is not self-similar.

Instead, using a standard technique to detect long-range dependencies [9], one computes the empirical integrated log-volatility and finds that it is well represented by a relation of the form

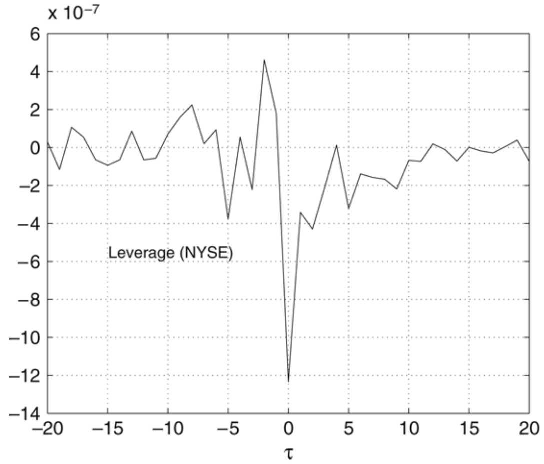

Fig. 1. Leverage for the NYSE one-day data in the period 1966–2000.

$$\sum_{n=0}^{t/\delta} \log \sigma\,(n\delta) = \beta t + R_\sigma(t) \tag{5}$$

the $R_\sigma(t)$ process possessing very accurate self-similar properties.

A nondegenerate process X_t, if it has finite variance, stationary increments and is self-similar ($\text{Law}(X_{at}) = \text{Law}(a^H X_t)$), must necessarily [10] have a covariance $\text{Cov}\,(X_s, X_t) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right) E\left(X_1^2\right)$ with $0 < H \leq 1$. The simplest process with these properties is a Gaussian process called fractional Brownian motion [11], with $\mathbb{E}\,[B_H(t)] = 0$, $\mathbb{E}\,[B_H(t)B_H(s)] = \frac{1}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}$ and, for $H > \frac{1}{2}$, a long-range dependence

Therefore, mathematical simplicity suggested the identification of the $R_\sigma(t)$ process with fractional Brownian motion, $R_\sigma(t) = kB_H(t)$. From the data, one obtains Hurst coefficients in the range 0.8–0.9. To obtain these values both the NYSE index and many of their individual companies were analyzed. Notice however that what is analyzed here is the Hurst exponent of the remainder $R_\sigma(t)$ process. Some other Hurst exponents reported in the literature refer to other processes (returns, volatility, etc).

Finally one obtained the following *fractional volatility model*

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B(t) \\ \log \sigma_t &= \beta + \frac{k}{\delta}\left\{B_H(t) - B_H(t-\delta)\right\} \end{aligned} \tag{6}$$

k is a volatility intensity parameter and δ is the observation time scale. Notice that the volatility is not driven by fractional Brownian motion but by fractional noise, naturally introducing an observation scale dependence.

A closed-form expression for the returns distribution and its asymptotic behavior is obtained [7], in good agreement with both one-day and high-frequency return distribution data. Also, new option pricing pricing formulas may be obtained from the model both in a simplified risk-neutral form or, more accurately, using fractional Malliavin calculus. The risk-neutral approximation has been compared [7] with the classical Black–Scholes (BS) result [12,13] by computing the implied volatility required in BS to reproduce the same results. The conclusion [7] is that, when compared to BS, it predicts a smile effect with the smile increasing as maturity approaches.

For the leverage effect a new result is obtained here, which implies a further refinement of the model. The following nonlinear correlation of the returns

$$L(\tau) = \left\langle |r(t+\tau)|^2\, r(t)\right\rangle - \left\langle |r(t+\tau)|^2\right\rangle \langle r(t)\rangle \tag{7}$$

is called *leverage* and the *leverage effect* is the fact that, for $\tau > 0$, $L(\tau)$ starts from a negative value whose modulus constantly decays to zero whereas for $\tau < 0$ it has almost negligible values. Fig. 1 shows $L(\tau)$ computed for the NYSE index one-day data in the period 1966–2000.

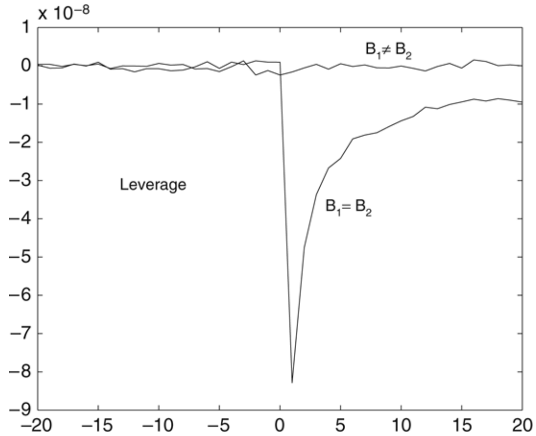

Fig. 2. Leverage in the fractional volatility model.

The leverage behavior of the fractional volatility model will now be examined. For this purpose it will be convenient to use the following integral representation of fractional Brownian motion [10].

$$B_H(t) = C\left\{\int_{-\infty}^{0}\left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right]\mathrm{d}B(s) + \int_0^t (t-s)^{H-\frac{1}{2}}\,\mathrm{d}B(s)\right\}. \quad (8)$$

Using this representation the fractional volatility model may be rewritten as

$$\begin{aligned} \mathrm{d}S_t &= \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}B^{(1)}(t) \\ \log\sigma_t &= \beta + k'\int_{-\infty}^{t}(t-s)^{H-\frac{3}{2}}\,\mathrm{d}B^{(2)}(s) \end{aligned} \quad (9)$$

where $B^{(1)}(s)$ and $B^{(2)}(s)$ may be different Brownian processes. In Fig. 2 one shows the leverage $L(\tau)$ computed for the model (9) with β and k' chosen to match the statistical parameters of the NYSE index. Both the $B^{(1)}(s) \neq B^{(2)}(s)$ and the $B^{(1)}(s) = B^{(2)}(s)$ cases are considered. One sees that for $B^{(1)}(s) \neq B^{(2)}(s)$ there is no leverage effect, whereas for $B^{(1)}(s) = B^{(2)}(s)$ an effect is found. Therefore, identifying the random generator of the log-price process with the stochastic integrator of the volatility, at least a part of the leverage effect is taken into account.

3. Two agent-based studies

Many factors play a role in a real market. To take into account all the factors in a model is neither possible nor, in many cases, illuminating. The objective is to isolate some of the more relevant mechanisms that presumably play a role in the market and, by stripping the model from other (inessential?) complications, to exhibit and understand the *purified* effect of these factors. As in other branches of science, the splitting apart of the dynamical components of a phenomena, may improve its understanding [14].

Two stylized models are considered. In the first the traders strategies play a determinant role. In the second the determinant effect is the limit-order book dynamics, the agents having a random nature.

3.1. Agent strategies and market impact

A market model with either random self-adapted strategies or fixed strategies was studied in detail in Ref. [15]. It was found that the dominance of two types of strategies was to a large extent determined by the initial conditions. Different types of return statistics corresponded to the relative importance of either “value investors” or “technical traders”. The occurrence of market bubbles also corresponded to situations where technical trader strategies were well represented.

Here, that model will be used for comparison purposes with the fractional volatility parametrization. The basic ingredients of the model are summarized below:

One considers a set of investors playing *against* the market, that is, they have some effect on an existing market that is influenced by other factors (other investors and general economic effects). This assumption implies that in addition to the impact of this group of investors on the market, the other factors are represented by a stochastic process. Therefore

$$z_{t+1} = f(z_t, \omega_t) + \eta_t \tag{10}$$

represents the change in the log-price ($z_t = \log p_t$) with ω_t being the total investment made by the group of traders and η_t the stochastic process that represents all the other factors. No conservation law is assumed for the total amount of *stock* s and *cash* m detained by the group of traders. If p_t is the price of the traded asset at time t, the purpose of the investors is to have an increase, as large as possible, of the total wealth $m_t + p_t \times s_t$ at the expense of the rest of the market. The collective variable is z and each investor payoff at time t is $\Delta_t^{(i)} = \left(m_t^{(i)} + p_t \times s_t^{(i)}\right) - \left(m_0^{(i)} + p_0 \times s_0^{(i)}\right)$.

Market impact

The following market impact function is used

$$z_{t+1} - z_t = \frac{\omega_t}{\lambda_0 + \lambda_1 |\omega_t|^{\alpha}} + \eta_t. \tag{11}$$

This price impact function was first proposed in Ref. [15] with $\alpha = \frac{1}{2}$. For small orders it recovers the log-linear approximation and for very large orders (and $\alpha = \frac{1}{2}$) Zhang's square root law.

The agent strategies

Two main types of informations are taken into account by the investors, namely the difference (misprice) between price and perceived value v_t

$$\xi_t - z_t = \log(v_t) - \log(p_t) \tag{12}$$

and the variation in time of the price (the price trend)

$$z_t - z_{t-1} = \log(p_t) - \log(p_{t-1}). \tag{13}$$

Consider now a non-decreasing function $f(x)$ such that $f(-\infty) = 0$ and $f(\infty) = 1$. (For example $f_1(x) = \theta(x)$ or $f_2(x) = \frac{1}{1+\exp(-\beta x)}$.) The information about misprice and price trend is coded on a four-component vector γ

$$\gamma_t = \begin{pmatrix} f(\xi_t - z_t) f(z_t - z_{t-1}) \\ f(\xi_t - z_t)(1 - f(z_t - z_{t-1})) \\ (1 - f(\xi_t - z_t)) f(z_t - z_{t-1}) \\ (1 - f(\xi_t - z_t))(1 - f(z_t - z_{t-1})) \end{pmatrix}. \tag{14}$$

The strategy of each investor is also a four-component vector $\alpha^{(i)}$ with entries -1, 0, or 1. -1 means to sell, 1 means to buy and 0 means to do nothing. Hence, at each time, the investment of agent i is $\alpha^{(i)} \cdot \gamma$. A *fundamental* (value-investing strategy) that buys when the price is smaller than the value and sells otherwise would be $\alpha^{(i)} = (1, 1, -1, -1)$ and a pure *trend-following* strategy would be $\alpha^{(i)} = (1, -1, 1, -1)$. In this setting the total number of possible strategies is $3^4 = 81$. The strategies will be labelled by numbers $n^{(i)} = \sum_{k=0}^{3} 3^k \left(\alpha_k^{(i)} + 1\right)$. Therefore the fundamental strategy is strategy no. 72 and the pure trend-following one is no. 60.

An evolution dynamics may be implemented in the model in the following way. After a number r of time steps, s agents copy the strategy of the s best performers and, at the same time, have some probability to mutate that strategy. This evolution aims at attaining the goal of improving gains, while at the same time allowing for some renewal of the strategies. The percentage of each strategy changes in time and one may find whether some of them become dominating or stable and when this may occur.

The model may be run with different initial conditions and with or without evolution of the strategies. In particular, when the model is run with evolution [15] it is found out that that the asymptotic steady-state behavior is somewhat

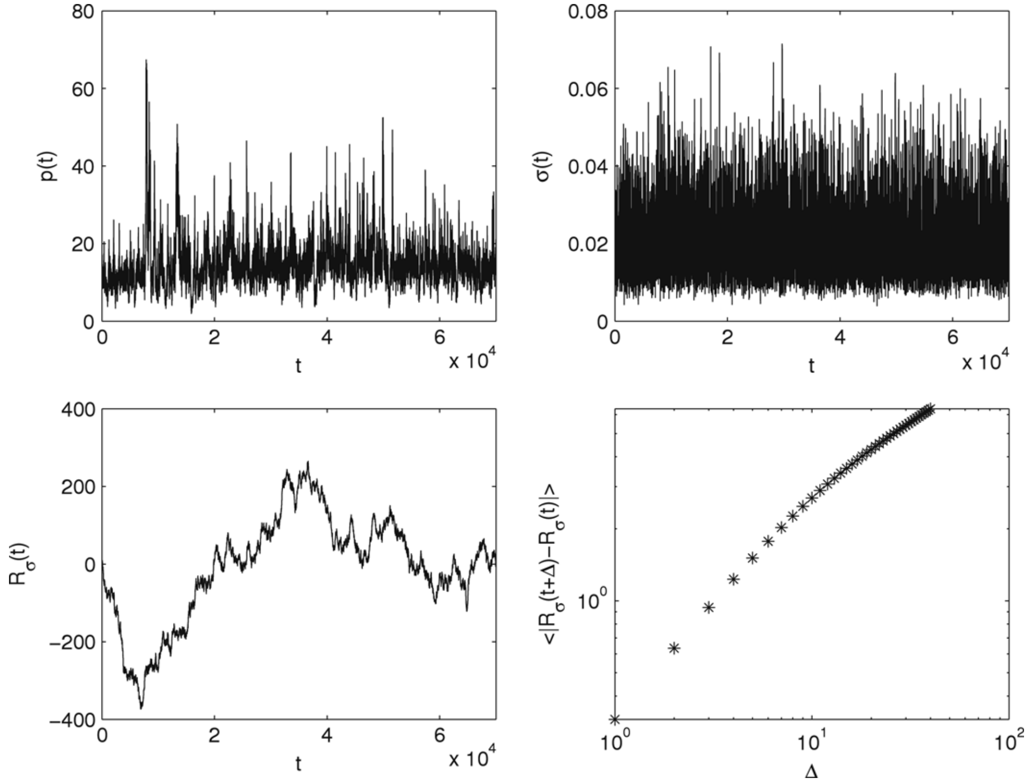

Fig. 3. Price, volatility and $R_\sigma(t)$ with equal amounts of fundamental and trend-following traders.

dependent on the initial conditions. This market model has been explored for many different mixtures of initial strategies (random or fixed) without or with evolution, that is, letting the strategies evolve according to their past gains or losses. In no case convincing results were obtained, that are both compatible with the returns distribution and the volatility behavior observed in the market.

For a simulation without evolution, with a fixed 50% of fundamental strategies (no. 72) and 50% of trend-following ones (no. 60), one sees a large number of bubbles and crashes in the price evolution and the price increments distribution has fat tails. Because this case is the one where the returns statistics is closer to the actual market data, it is here further analyzed to see whether it also displays the other features of the fractional volatility model. From typical simulation runs one computes $\sigma_t^2 = \frac{1}{|T_0-T_1|}\text{var}(\log p_t)$, $\sum_{n=0}^{t/\delta} \log\sigma(n\delta) = \beta t + R_\sigma(t)$ and $|R_\sigma(t+\Delta) - R_\sigma(t)|$. Fig. 3 shows a typical plot of the price process $p(t)$, the volatility, $R_\sigma(t)$ and $\mathbb{E}\{|R_\sigma(t+\Delta) - R_\sigma(t)|\}$ obtained from the model with equal amounts of *fundamental* $(1, 1, -1, -1)$ and *trend-following* $(1, -1, 1, -1)$ agents and no evolution. One notices the lack of scaling behavior of $R_\sigma(t)$ with an asymptotic exponent 0.55, denoting the lack of memory of the volatility process. This might already be evident from the time behavior of $R_\sigma(t)$ in the lower left plot. Also, although the returns have fat tails in this case, they are of different shape from those observed in the market data. Similar conclusions are obtained with other combinations of agent strategies. In conclusion: It seems that the features of the fractional volatility model are not easily captured by a choice of strategies in an agent-based model. Notice however that what the fractional volatility model parametrizes is the bulk of the market data, that is, the behavior of the market in normal days. The agents reactions and strategies are very probably determinant during market crisis and market bubbles.

3.2. A limit-order book dynamics model

Here one considers a limit-order book where *asks* and *bids* arrive at random on a window $[p - w, p + w]$ around the current price p. Every time a *buy* order arrives it is fulfilled by the closest non-empty ask slot, the new current price being determined by the value of the ask that fulfills it. If no ask exists when a buy order arrives it goes to a

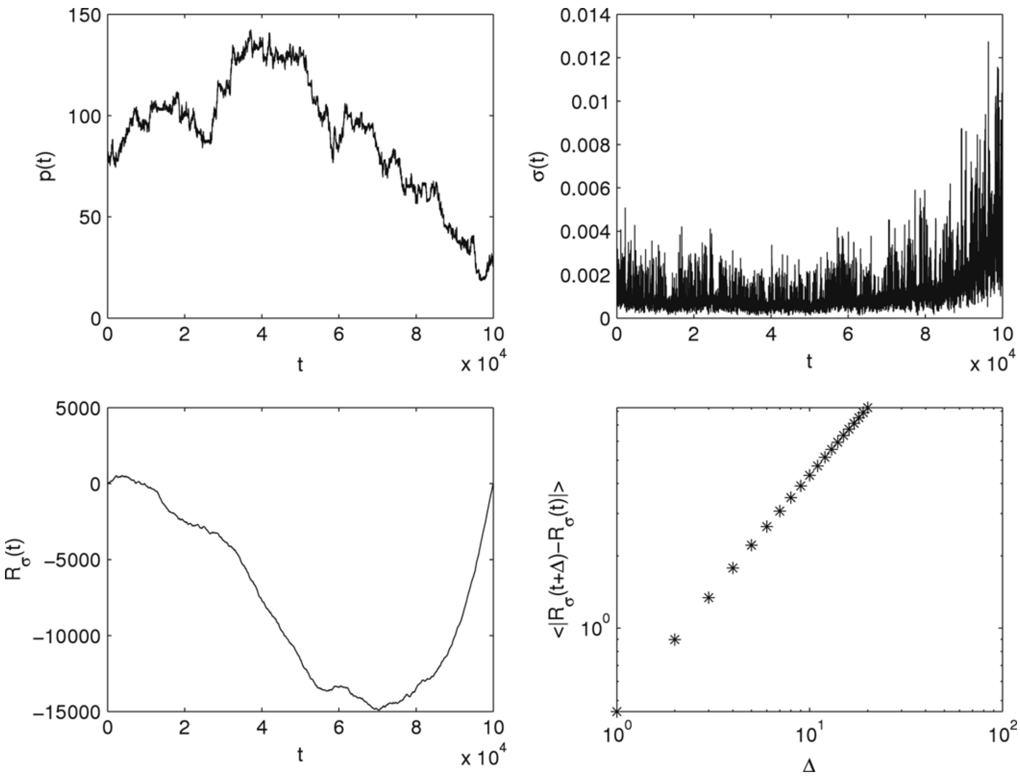

Fig. 4. Price, volatility and $R_\sigma(t)$ in the limit-order model.

cumulative register to wait to be fulfilled. The symmetric process occurs when a *sell* order arrives, the new price being the bid that buys it. Because the window around the current price moves up and down, asks and bids that are too far away from the current price are automatically eliminated. Sell and buy orders, asks and bids all arrive at random. The only parameters of the model are the width w of the limit-order book and the size n of the asks and bids, the sell and buy orders being normalized to one.

The model was run for different widths w and order sizes n and, for comparison with the fractional volatility model, one computes as before σ_t^2, $\sum_{n=0}^{t/\delta} \log \sigma(n\delta) = \beta t + R_\sigma(t)$ and $|R_\sigma(t+\Delta) - R_\sigma(t)|$.

Although the exact values of the statistical parameters depend on w and n, the statistical nature of the results seems to be essentially the same. Fig. 4 shows typical plots of the price process $p(t)$, the volatility, $R_\sigma(t)$ and $\mathbb{E}\{|R_\sigma(t+\Delta) - R_\sigma(t)|\}$ obtained for $n = 2$ and the limit-order book divided into $2w + 1 = 21$ discrete price slots with $\Delta p = 0.1$. The scaling properties of $R_\sigma(t)$ are quite evident from the lower right plot in the figure, the Hurst coefficient being 0.96. Fig. 5 shows the correlation and the pdf of the one-time returns. From these results one concludes that the main statistical properties of the market data (fast decay of the linear correlation of the returns, non-Gaussianity and volatility memory) are already generated by the dynamics of the limit-order book with random behavior of the agents. This implies, as pointed out by some authors that in the past have considered limit-order book models [16–19], that a large part of the market statistical properties (in normal business-as-usual days) depends more on the nature of the price fixing financial institutions than on particular investor strategies.

4. Conclusions

(a) The fractional volatility model provides a reasonable mathematical parametrization of the bulk market data, that is, it captures the behavior of the market in business-as-usual trading days.

(b) A small modification of the original model, identifying the random generator of the log-price process and the integrator of the volatility process, also describes, at least, a part of the leverage effect.

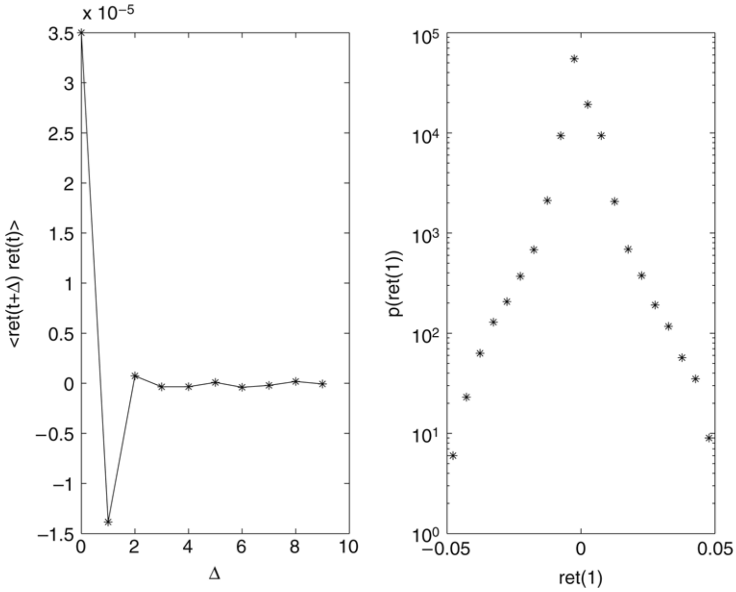

Fig. 5. Linear correlation and pdf of returns in the limit-order model.

(c) The market statistical behavior in normal days seems to be more influenced by the nature of the financial institutions (the double auction process) than by the traders' strategies. Specific trader strategies and psychology should however play a role on market crisis and bubbles.

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