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A laboratory scale fundamental time?

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Abstract The existence of a fundamental time (or fundamental length) has been conjectured in many contexts. However, the "stability of physical theories principle" seems to be the one that provides, through the tools of algebraic deformation theory, an unambiguous derivation of the stable structures that Nature might have chosen for its algebraic framework. It is well-known that c and $\hbar$ are the deformation parameters that stabilize the Galilean and the Poisson algebra. When the stability principle is applied to the Poincaré–Heisenberg algebra, two deformation parameters emerge which define two time (or length) scales. In addition there are, for each of them, a plus or minus sign possibility in the relevant commutators. One of the deformation length scales, related to non-commutativity of momenta, is probably related to the Planck length scale but the other might be much larger and already detectable in laboratory experiments. In this paper, this is used as a working hypothesis to look for physical effects that might settle this question. Phase-space modifications, resonances, interference, electron spin resonance and non-commutative QED are considered.

1 Introduction

The idea of modifying the algebra of the space-time components x_μ in such a way that they become non-commuting operators has appeared many times in the physical literature ([1–21], etc.). The aim of most of these proposals was to endow space-time with a discrete structure, to be able, for example, to construct quantum fields free of ultraviolet divergences. Sometimes a non-zero commutator is simply postulated, in some other instances the motivation is the formulation of field theory in curved spaces. String theories [22, 23] and quantum relativity [24, 25] have also provided hints concerning the non-commutativity of space-time at a fundamental level.

A somewhat different point of view has been proposed in [26, 27]. There the space-time non-commutative structure is arrived at through the application of the *stability of physical theories principle* (SPT). The rationale behind this principle is the fact that the parameters entering in physical theories are never known with absolute precision. Therefore, robust physical laws with a wide range of validity can only be those that do not change in a qualitative manner under a small change of parameters, that is, *stable* (or *rigid*) theories. The stable-model point of view originated in the field of non-linear dynamics, where it led to the notion of *structural stability* [28–30]. Later on, Flato [31] and Faddeev [32] have shown that the same pattern occurs in the fundamental theories of Nature, namely the transition from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as the replacement of two unstable theories by two stable ones. The stabilizing deformations lead, in the first case, from the Galilean to the Lorentz algebra and, in the second one, from the algebra of commutative phase space to the Moyal–Vey algebra (or equivalently to the Heisenberg algebra). The deformation parameters are $\frac{1}{c}$ (the inverse of the speed of light) and h (the Planck constant). Except for the isolated zero value, the deformed algebras are all equivalent for non-zero values of $\frac{1}{c}$ and h. Hence, relativistic mechanics and quantum mechanics might have been derived from the conditions for stability of two mathematical structures, although the exact values of the deformation parameters cannot be fixed by purely algebraic considerations. Instead, the deformation parameters are fundamental constants to be obtained from experiment and, in this sense, not only is deformation theory the theory of stable theories, it is also the theory that identifies the fundamental constants.

The SPT principle is related to the idea that physical theories drift towards simple algebras [33–35], because all simple algebras are stable, although not all stable algebras are simple.

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When the SPT principle is applied to the algebra of relativistic quantum mechanics (the Poincaré–Heisenberg algebra)

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} \\
&\qquad\qquad - M_{\mu\rho}\eta_{\nu\sigma}) \\
&[M_{\mu\nu}, p_\lambda] = i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}) \\
&[M_{\mu\nu}, x_\lambda] = i(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}) \\
&[p_\mu, p_\nu] = 0 \\
&[x_\mu, x_\nu] = 0 \\
&[p_\mu, x_\nu] = i\eta_{\mu\nu}\mathbf{1}
\end{aligned}
\tag{1}
$$

$\eta_{\mu\nu} = (1, -1, -1, -1)$, $c = \hbar = 1$, it leads [26] to

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} \\
&\qquad\qquad - M_{\mu\rho}\eta_{\nu\sigma}) \\
&[M_{\mu\nu}, p_\lambda] = i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}) \\
&[M_{\mu\nu}, x_\lambda] = i(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}) \\
&[p_\mu, p_\nu] = -i\frac{\epsilon'}{R^2}M_{\mu\nu} \\
&[x_\mu, x_\nu] = -i\epsilon\ell^2 M_{\mu\nu} \\
&[p_\mu, x_\nu] = i\eta_{\mu\nu}\Im \\
&[p_\mu, \Im] = -i\frac{\epsilon'}{R^2}x_\mu \\
&[x_\mu, \Im] = i\epsilon\ell^2 p_\mu \\
&[M_{\mu\nu}, \Im] = 0
\end{aligned}
\tag{2}
$$

The stabilization of the Poincaré–Heisenberg algebra has been further studied and extended in [36–38]. The essential message from (2) or from the slightly more general form obtained in [36] is that from the unstable Poincaré–Heisenberg algebra $\{M_{\mu\nu}, p_\mu, x_\nu\}$ one obtains a stable algebra with two deformation parameters ℓ and $\frac{1}{R}$. In addition there are two undetermined signs ϵ and ϵ' and the central element of the Heisenberg algebra becomes a non-trivial operator $\Im$. The existence of two continuous deformation parameters when the algebra is stabilized is a novel feature of the deformation point of view, which does not appear in other non-commutative space-time approaches. These deformation parameters may define two different length scales. Of course, once one of them is identified as a fundamental constant, the other will be a pure number.

Being associated to the non-commutativity of the generators of space-time translations, the parameter $\frac{1}{R}$ may be associated to space-time curvature and therefore might not be relevant for considerations related to the tangent space. It is, of course, very relevant for quantum gravity studies [38]. Already in the past, some authors [32], have associated the non-commutativity of translations to gravitational effects, the gravitation constant being the deformation parameter. Presumably then $\frac{1}{R}$ might be associated to the Planck length scale. However, ℓ, the other deformation parameter, defines a completely independent length scale which might be much closer to laboratory phenomena. This will be the working hypothesis to be explored in this paper. Therefore when $\frac{1}{R}$ is assumed to be very small the deformed algebra may be approximated by

$$
\begin{aligned}
&[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} \\
&\qquad\qquad - M_{\mu\rho}\eta_{\nu\sigma}) \\
&[M_{\mu\nu}, p_\lambda] = i(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}) \\
&[M_{\mu\nu}, x_\lambda] = i(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}) \\
&[p_\mu, p_\nu] = 0 \\
&[x_\mu, x_\nu] = -i\epsilon\ell^2 M_{\mu\nu} \\
&[p_\mu, x_\nu] = i\eta_{\mu\nu}\Im \\
&[p_\mu, \Im] = 0 \\
&[x_\mu, \Im] = i\epsilon\ell^2 p_\mu \\
&[M_{\mu\nu}, \Im] = 0
\end{aligned}
\tag{3}
$$

For future reference this algebra will be denoted $\mathcal{R}_{\ell,\infty}$. Notice that in relation to the more general deformation obtained in [36], we are also considering $\alpha_3 = 0$ (or $\beta = 0$ in [38]). The nature of the sign ϵ has physical consequences. If $\epsilon = +1$ time will have a discrete spectrum, whereas if $\epsilon = -1$ it is when one the space coordinates is diagonalized that discrete spectrum is obtained. In this sense if $\epsilon = +1$, ℓ might be called "the fundamental time" and "the fundamental length" if $\epsilon = -1$. In this paper one discusses consequences of both signs.

Notice that the non-commutativity of the space-time coordinates already implies that many notions currently used in the analysis of laboratory experiments become ill-defined. For example, because the space and the time coordinates cannot be simultaneously diagonalized, speed can only be defined in terms of expectation values,

$$
v^i_\psi = \frac{1}{\langle\psi_t, \psi_t\rangle}\frac{d}{dt}\langle\psi_t, x^i\psi_t\rangle
$$

ψ being a state with a small dispersion of momentum around a central value p. This would imply a deviation from $c(=1)$ of the "effective speed" of massless particles of order [39]

$$
\Delta v_\psi = -3\epsilon\ell^2\left(p^0\right)^2
$$

The deviation would be negative for $\epsilon = +1$ (ℓ a fundamental time) or positive for $\epsilon = -1$ (ℓ a fundamental length). General (non-commutative) geometry properties of the algebra (3) have been studied before [27] as well as some other consequences [40–44]. Here the emphasis will be on effects which might be detectable at the laboratory level, if the working hypothesis that ℓ defines a much larger scale than Planck's is true. In addition, some of the non-commutativity and time-discreteness effects that have been proposed in the past will be discussed, in particular to find out whether they are or not relevant as a test of the algebra (3).

In the recent past, most papers dealing with space-time non-commutativity start from the hypothesis

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{4}$$

$\theta_{\mu\nu}$ being a c-number antisymmetric tensor (for a review see [45]). Then, calculations are carried out by replacing the usual product of functions in space-time by the $*$-product

$$(f * g)(x) = f(x)e^{i\overleftarrow{\partial^\mu}\theta_{\mu\nu}\overrightarrow{\partial^\nu}}g(x) \tag{5}$$

A similar $*$-product formulation may be implemented for the algebra (2) by replacing in (5) $\theta_{\mu\nu}$ by the operator $M_{\mu\nu}$ and, to have a full $*$-product formulation, using the Moyal product for products of functions of coordinates and momenta. However, the situation is quite different from the one implied by (4) because $M_{\mu\nu}$ is an operator, not a constant tensor deformation parameter. Hence it does not lead to Lorentz violation, the deformed algebras (2)–(3) being consistent with preservation of Lorentz invariance. Therefore some of the tests proposed for (4) are not relevant for (3). In addition the deformed algebras introduce a new non-trivial operator $\Im$ which replaces the central element of the Heisenberg algebra. In particular this operator corresponds to an additional component in the most general connections compatible with (3) [27].

When the right-hand side of (4) is a c-number $\theta_{\mu\nu}$, with dimensions of length-squared, it may be roughly interpreted as the smallest patch of area in the $\mu\nu$-plane that one may consider to be observable, like $\hbar$ in $[x_i, p_j] = i\hbar\delta_{ij}$ may be interpreted as the smallest patch in phase space. However, here the right-hand side of the commutators is an operator and the interpretation is subtler.

The present paper is concerned with the discussion of effects which might lead to actual experimental tests if ℓ is not too small. It must be pointed out that some of these effects, as it will be referred to in the appropriate places, may have already been suggested by other authors. Nevertheless in most cases they are suggested in the framework of a simple time or space discreteness hypothesis, without the benefit of a full space-time algebra. For that reason some of the conclusions are different or more detailed.

Phase-space modifications, resonances, interference, electron spin resonance and non-commutative QED are considered. Finally, in the appendix, some explicit representations of the space-time algebra are collected, which are useful for the calculations.

2 Phase-space effects

2.1 Cross sections and particle multiplicity

If $\epsilon = +1$ there is a phase-space contraction effect at high energies. This was discussed in [42], being pointed out that it might relevant for the calculation of the GZK radius [46, 47]. From the calculations in [42], the conclusion was that, whereas the value of the GZK cutoff would not be much changed, the radius of the GZK sphere would increase, allowing for more nucleons from farther distances to reach earth at energies above 5×10^9 eV. For this effect to be detectable the fundamental time should not be smaller than 10^{-26} seconds.

Here, further consequences of the phase-space modification are studied. Both signs $\epsilon = +1$ and $\epsilon = -1$ are considered. In particular, if the conjecture that the scale ℓ is much larger than the Planck scale is true, such effects might already be observed at the energy of the existing colliders.

The modification of the density of states [42] is obtained by computing how many available states a particle of momentum p has, for example, in a scattering experiment. Once the direction of p is fixed, the problem becomes a 1-dimensional problem, which may be dealt with by a subalgebra $\{x^1, p^1, \Im\}$ of (3). Let $\epsilon = +1$ and define hyperbolic coordinates in the plane $(p^1, \Im)$

$$\begin{aligned} p^1 &= \frac{r}{\ell}\sinh\mu \\ \Im &= r\cosh\mu \end{aligned} \tag{6}$$

Then

$$\frac{\partial}{\partial\mu} = \frac{1}{\ell}\Im\frac{\partial}{\partial p^1} + \ell p^1\frac{\partial}{\partial\Im}$$

and comparing with the representation (81) one obtains

$$x^1 = i\ell\frac{\partial}{\partial\mu} \tag{7}$$

or, equivalently

$$\begin{aligned} x^1 &= x \\ p^1 &= \frac{r}{\ell}\sinh\left(\frac{\ell}{i}\frac{\partial}{\partial x}\right) \\ \Im &= r\cosh\left(\frac{\ell}{i}\frac{\partial}{\partial x}\right) \end{aligned} \tag{8}$$

For $r = 1$ and $\ell \to 0$, the classical result is obtained. Hereafter let us consider $r = 1$. In the x-basis the eigenvectors p of the momentum p^1 are $e^{ik_n x}$ which, with vanishing boundary conditions on a box, has eigenvalues

$$p_n = \frac{1}{\ell}\sinh\left(\frac{\pi}{L}n\ell\right) \tag{9}$$

corresponding to $k_n = \frac{\pi n}{L}$. Therefore the number of states with momenta smaller than p is

$$G_+^{1D}(p) = \frac{L}{\pi}\frac{1}{\ell}\sinh^{-1}(\ell p) \tag{10}$$

and the density of states is

$$g_+^{1D}(p) = \frac{dG_+^{1D}}{dp} = \frac{L}{\pi}\frac{1}{\sqrt{1+\ell^2 p^2}} \tag{11}$$

For three dimensions, considering the number of independent states with absolute momentum less than $|p|$

$$G_+^{3D}(|p|) = \frac{V}{6\pi^2}\frac{1}{\ell^3}\left(\sinh^{-1}(\ell|p|)\right)^3 \tag{12}$$

leads to a density of states

$$g_+^{3D}(|p|) = \frac{V}{2\pi^2}\frac{1}{\ell^2}\frac{(\sinh^{-1}(\ell|p|))^2}{\sqrt{1+\ell^2|p|^2}} \tag{13}$$

For $\epsilon = -1$ the appropriate coordinates in the plane $(p^1, \Im)$ are $p^1 = \frac{r}{\ell}\sin\theta$, $\Im = r\cos\theta$, $x^1 = i\ell\frac{\partial}{\partial\theta}$ and one would obtain the opposite effect, namely the factor $1/\sqrt{1-\ell^2 p^2})$. Therefore

$$g_-^{1D}(p) = \frac{L}{\pi}\frac{1}{\sqrt{1-\ell^2 p^2}} \tag{14}$$

$$g_-^{3D}(|p|) = \frac{V}{2\pi^2}\frac{1}{\ell^2}\frac{(\sin^{-1}(\ell|p|))^2}{\sqrt{1-\ell^2|p|^2}} \tag{15}$$

The conclusion is that for $\epsilon = +1$ there is a contraction of phase space increasing with energy and an expansion for $\epsilon = -1$, the cross sections being corrected by the new density of states (13) and (15). For $\epsilon = +1$ the suppression effect of the phase-space contraction on high energy reactions may be estimated by comparing the integral

$$I_N(\ell) = \int\cdots\int_0^\omega \frac{(\sinh^{-1}(\ell p_1))^2\,dp_1}{\ell^2\sqrt{1+\ell^2 p_1^2}}\cdots\frac{(\sinh^{-1}(\ell p_N))^2\,dp_N}{\ell^2\sqrt{1+\ell^2 p_N^2}} \times \max\left(\omega - \sum_{i=1}^N p_i, 0\right)$$

with $I_N(0)$. This estimates the suppression effect on an high energy final state integral for total energy ω neglecting masses. Changing variables one obtains

$$I_N(\ell) = \omega^3\int_0^1 \frac{(\sinh^{-1}(\beta x_1))^2\,dx_1}{\beta^2\sqrt{1+\beta^2 x_1^2}}\cdots\frac{(\sinh^{-1}(\beta x_N))^2\,dx_N}{\beta^2\sqrt{1+\beta^2 x_N^2}} \times \max\left(1 - \sum_{i=1}^N x_i, 0\right)$$

with $\beta = \omega\ell$. Figure 1 is a plot of the suppression function $S(\beta) = \frac{I_N(\ell)}{I_N(0)}$ for $N = 2, 3, 4$. One sees that the suppression effect decreases when the number of final particles increases. The phase-space suppression effect implies that if cross-section values found at low energies are used to predict the final states at higher energies, an increase in particle multiplicity will be found above the expected one.

For $\epsilon = -1$ the effect would be the opposite one, that is, a smaller multiplicity. One also sees that these effects will only become noticeable for $\beta \sim O(1)$. For example, an observation of the effects starting at around $\omega = 600$ GeV would imply $\ell \backsim O(10^{-27} s)$ or $\ell \backsim O(3\times 10^{-17}$ cm).

In (14) and (15) the argument of the square root would be negative for $p > 1/\ell$ leading to unphysical results. However, notice that the representation $p = \frac{r}{\ell}\sin\theta$ means that, in the $\epsilon = -1$ case, the existence of a fundamental length ℓ implies an upper bound r/ℓ for the momentum variable p. Choosing $r = 1$ the upper bound becomes $1/\ell$ and the argument of the square root can never become negative.

2.2 The degeneracy pressure

That the phase-space volume modifications at high energies (contraction for $\epsilon = +1$, dilatation for $\epsilon = -1$) would also lead to statistical mechanics predictions was briefly mentioned in [41]. This might, in particular, have some consequences for models of dense star matter. Here I will analyze the modifications implied by the deformed algebra on the degeneracy pressure of a Fermi gas. Both the non-relativistic and the relativistic case will be analyzed. For a gas of non-relativistic particles the kinetic energy E is

$$E = \frac{p^2}{2m}$$

For $\epsilon = +1$, changing variables in (12) the density of states becomes

$$g_+^{3D}(E) = \frac{V}{4\pi^2}\frac{1}{\ell^2}\left(\sinh^{-1}(\ell\sqrt{2mE})\right)^2\frac{\sqrt{2m}}{\sqrt{E}\sqrt{1+2\ell^2 mE}}$$

From

$$N = \int_0^{E_F}(2s+1)g_+^{3D}(E)\,dE$$

at $T = 0$, one obtains

$$E_F = \frac{1}{2m\ell^2}\sinh\left(\frac{6\pi^2\ell^3 N}{(2s+1)V}\right)^{2/3}$$

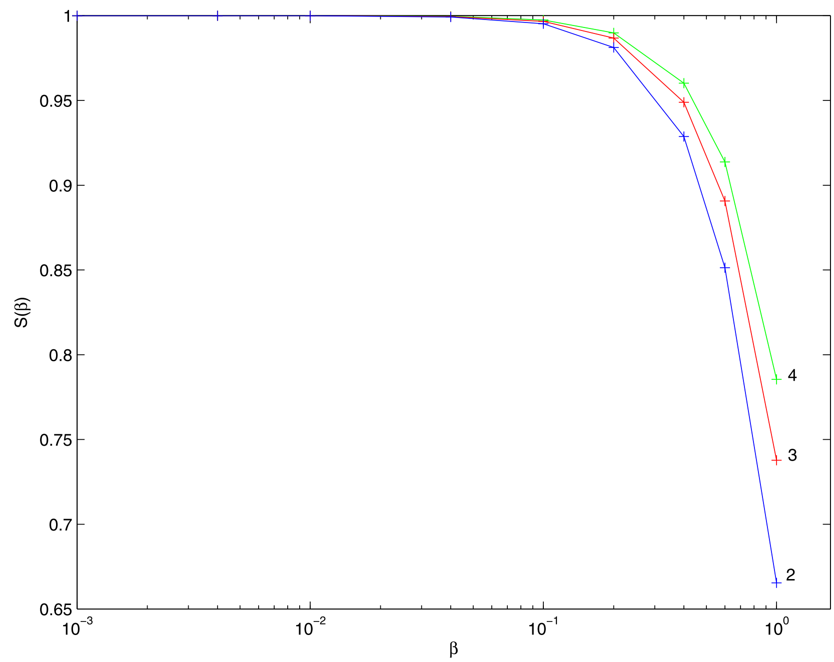


Fig. 1 Phase-space suppression function ($\epsilon = +1$) for 2, 3 and four final particles

In leading ℓ^2 order the energy and the pressure are

$$U(0) = \int_0^{E_F} (2s+1) E g_+^{3D}(E)\, dE \simeq (2s+1) \frac{V}{4\pi^2} (2m)^{3/2} \left\{ \frac{2}{5} E_F^{5/2} - \frac{10}{7} m\ell^2 E_F^{7/2} \right\}$$

$$P = -\left(\frac{\partial U(0)}{\partial V} \right)_{N,S} \simeq \frac{(2s+1)}{4\pi^2} (2m)^{3/2} \left\{ \frac{4}{15} E_F^{5/2} - \frac{4}{7} m\ell^2 E_F^{7/2} \right\}$$

leading to

$$P \simeq \frac{2}{3} \frac{U(0)}{V} + \frac{15}{14} \left(\frac{20\pi^2 m}{2s+1} \right)^{2/5} \ell^2 \left(\frac{U(0)}{V} \right)^{7/5}$$

In the relativistic case, which is the one that is relevant, for example, for neutron star matter, the total energy density per unit volume is

$$\rho = \int_0^{p_F} \left(p^2 + m^2 \right)^{1/2} \frac{1}{V} g_+^{3D}(p)\, dp$$

and the pressure

$$P = \frac{1}{3} \int_0^{p_F} \frac{p^2}{(p^2+m^2)^{1/2}} \frac{1}{V} g_+^{3D}(p)\, dp$$

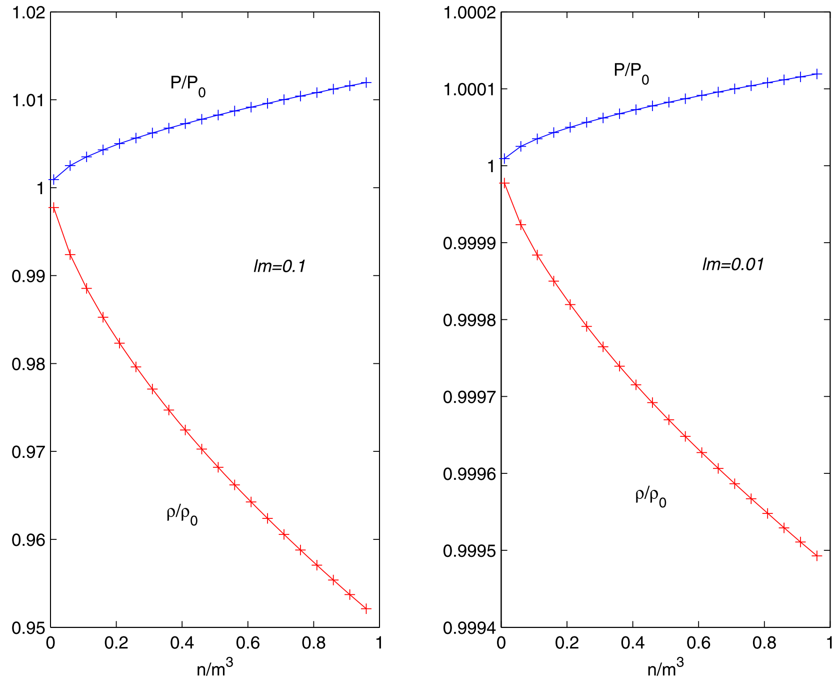
with $p_F = \frac{1}{\ell} \sinh(\frac{6\pi^2 \ell^3 N}{(2s+1)V})^{1/3}$. Writing ρ and P with the adimensional variables $m\ell$ and $\frac{N}{Vm^3} = \frac{n}{m^3}$

$$\rho_\ell\left(\ell m, \frac{n}{m^3} \right) = \frac{(2s+1)m^4}{2\pi^2} \times \int_0^{p_F/m} \frac{dx}{(\ell m)^2} \left(\sinh^{-1}(\ell m x) \right)^2 \times \frac{\sqrt{1+x^2}}{\sqrt{1+\ell^2 m^2 x^2}}$$

$$P_\ell\left(\ell m, \frac{n}{m^3} \right) = \frac{(2s+1)m^4}{6\pi^2} \times \int_0^{p_F/m} \frac{dx}{(\ell m)^2} \left(\sinh^{-1}(\ell m x) \right)^2 \times \frac{x^2}{\sqrt{(1+x^2)(1+\ell^2 m^2 x^2)}}$$

In Fig. 2 the ratios ρ_ℓ/ρ_0 and P_ℓ/P_0 are plotted for two ℓm values. One sees that the phase-space suppression ($\epsilon = +1$) implies a larger degeneracy pressure and a smaller total energy density for the same $n = \frac{N}{V}$. However, (in contrast with the effects that might be seen at high energy colliders, as discussed above) and for reasonable star matter densities the effect is probably too small to be observed. For example, with m the neutron mass, a matter density of 4×10^{14} g/cm^3 and $\ell = 10^{-26}$ s one has $\ell m = 1.35 \times 10^{-2}$ but only $\frac{n}{m^3} = 2.16 \times 10^{-3}$. One sees from Fig. 2 that for these values the effect is extremely small.

Fig. 2 Ratios ρ_ℓ/ρ_0 and P_ℓ/P_0 for two ℓm values



The corresponding results for the case $\epsilon = -1$ are obtained by replacing $(1 + \ell^2 m^2 x^2)$ by $(1 - \ell^2 m^2 x^2)$ and $\sinh^{-1}$ by $\sin^{-1}$ in the equations above.

3 Time quantization and resonances

Some years ago Ehrlich [49], finding a regularity in the resonance widths known at the time, conjectured that the widths might be quantized in multiples of some fundamental time unit. Later, the same author [50] using more recent data, pointed out that the quantization hypothesis of the resonance widths did not agree as well as before. Nevertheless, the conjecture has its merit and deserves to be checked within the present framework. If the time quantization has a direct bearing on resonance widths it should already be apparent in simple potential models. Consider the simple 1-dimensional potential displayed in Fig. 3.

Let the wave functions in regions 1 and 2 be

$$\Psi^{(1)} = A\sin(kx)$$
$$\Psi^{(2)} = e^{-ik^{(2)}x} + Be^{ik^{(2)}x} \tag{16}$$

Quantized time would correspond to $\epsilon = +1$, therefore, using the representation (8),

$$x = x$$
$$p = -i\frac{r}{\ell}\sin\left(\ell\frac{d}{dx}\right) \tag{17}$$
$$\Im = r\cos\left(\ell\frac{d}{dx}\right)$$

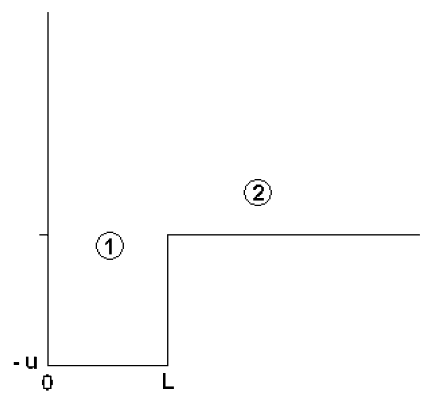


Fig. 3 A simple 1-dimensional potential

the momentum p associated to a wave number k is

$$p = \frac{r}{\ell}\sinh(k\ell) \tag{18}$$

Using the matching conditions at $x = L$ one obtains

$$A = \frac{2e^{-ik^{(2)}L}}{\sin(kL) + \frac{ik}{k^{(2)}}\cos(kL)} \tag{19}$$

and

$$B = e^{-i2k^{(2)}L}\left\{\frac{2\sin(kL)}{\sin(kL) + \frac{ik}{k^{(2)}}\cos(kL)} - 1\right\} \tag{20}$$

with $k^{(2)}$ obtained from k by the matching of the energy in regions 1 and 2. For a non-relativistic approximation and

$r=1$ it is

$$\cosh\bigl(2k^{(2)}\ell\bigr)=\cosh(2k\ell)-4mu\ell^2 \tag{21}$$

and in the relativistic case

$$\cosh\bigl(2k^{(2)}\ell\bigr)=1+\bigl(\sqrt{\cosh(2k\ell)-1+2m_0^2\ell^2}-\sqrt{2}\ell u\bigr)^2 - 2m_0^2\ell^2 \tag{22}$$

The resonances are associated to the complex zeros of $\sin(kL)+\frac{ik}{k^{(2)}}\cos(kL)$, that is using (21), to the zeros of the function

$$F(k)=\cosh(2k\ell)-4mu\ell^2-\cos\left(\frac{2k\ell}{\tan(kL)}\right)$$

In Fig. 4 the location of the zeros of $F(k)$ are plotted for $m=u=L=1$, $\ell=0.01$ and $\ell=0.5$.

One sees no evidence for the width of the resonances being quantized in multiples of ℓ. Rather, the widths and separation of the resonances is related to the geometry of the problem. However, what one notices is that as ℓ approaches the scale of the problem, the resonances become extremely wide being, in practice, undetectable in the scattering amplitudes. This is illustrated in Fig. 5 where the amplitude of A is plotted.

Therefore the only effect to be expected is that as soon as one deals with phenomena close to the scale of ℓ only few or no resonances will be observed. Hadronic resonances being of order 10^{-24} s, this establishes an upper limit $\ell\lesssim 10^{-25}$ s.

4 Phases and interference

In the algebra (3) a complete momentum-space description is possible with the variables $(t,p^0,p^1,p^2,p^3,\Im)$ in the commuting basis (t,p^1,p^2,p^3) with the representation $(\epsilon=+1)$

$$\begin{aligned} x^0&=t\\ p^0&=\frac{i}{\ell}r\sinh\left(\ell\frac{d}{dt}\right)\\ \Im&=r\cosh\left(\ell\frac{d}{dt}\right)\end{aligned} \tag{23}$$

Because of the non-commutativity of time with the space coordinates, the simpler approach is to consider in each case the eigenvalues of the Hamiltonian and then to obtain the time evolution of each eigenstate.

From the equation

$$p^0\psi=H\psi \tag{24}$$

one obtains, in first approximation,[1] the time evolution of an eigenstate by

$$p^0\psi_E=E\psi_E \tag{25}$$

$$\frac{i}{\ell}r\sinh\left(\ell\frac{d}{dt}\right)\psi_E =\frac{i}{2\ell}r\bigl(\psi_E(t+\ell)-\psi_E(t-\ell)\bigr)=E\psi_E \tag{26}$$

that is, the (momentum-space) Schrödinger-like equation becomes a finite-difference equation, as has been conjectured by several authors [48]. Here, however, it is a direct consequence of the algebra (3). For the case $\epsilon=-1$ the corresponding equation would be

$$\frac{1}{2\ell}r\bigl(\psi_E(t+i\ell)-\psi_E(t-i\ell)\bigr)=E\psi_E \tag{27}$$

The solution of (26) is

$$\psi_E(t)=\exp\left\{-\frac{it}{\ell}\sin^{-1}\left(\frac{\ell E}{r}\right)\right\}\psi_E(0) \tag{28}$$

The inverse trigonometric nature of exponentials of this type, have been interpreted by some authors as meaning that in the quantized time case there is an upper bound $\frac{1}{\ell}$ for the energy. This is an unwarranted conclusion because the most general representation of the subalgebra $\{p_0,x_0,\Im\}$ allows for the arbitrary factor r. Therefore the maximum energy (of stationary states) would be $\frac{r}{\ell}$ and not $\frac{1}{\ell}$. Notice that as soon as one considers also the space coordinates, r is a variable needed for the consistency of the representation with the commutation relations (3) (see the appendix). In any case, the fact that there would not be any stationary eigenstates $(|\psi_E(t)|=|\psi_E(0)|,\forall t)$ for $E>\frac{r}{\ell}$ does not mean that the spectrum of p^0 has an upper bound.

Consider now the interference pattern of two eigenstates

$$\begin{aligned}\bigl|\psi_{E_1}(t)+\psi_{E_2}(t)\bigr|^2 &=\bigl|\psi_{E_1}(t)\bigr|^2+\bigl|\psi_{E_2}(t)\bigr|^2\\ &\quad+2\psi_{E_1}(0)\psi_{E_2}(0)\cos\left\{\frac{t}{\ell}\left[\sin^{-1}\left(\frac{\ell E_1}{r}\right)-\sin^{-1}\left(\frac{\ell E_2}{r}\right)\right]\right\}\end{aligned} \tag{29}$$

[1] As found in [39], the time shift generator has corrections of order ℓ^2. Therefore this Schrödinger-like equation is only an approximation. However, because in this section one only wants to explore the effects of the replacement of derivatives by finite-differences, this approximation will suffice.

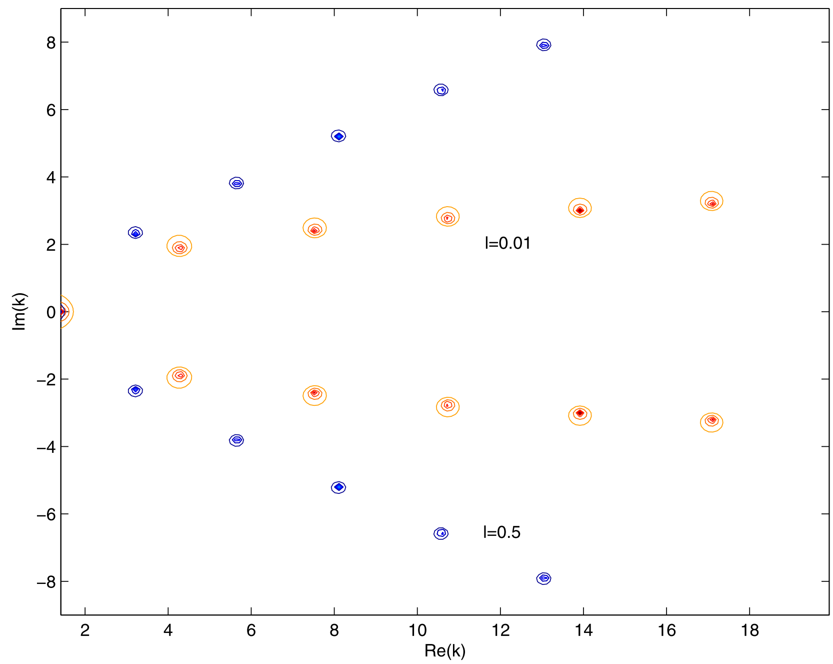


Fig. 4 The complex zeros of the function $F(k)$

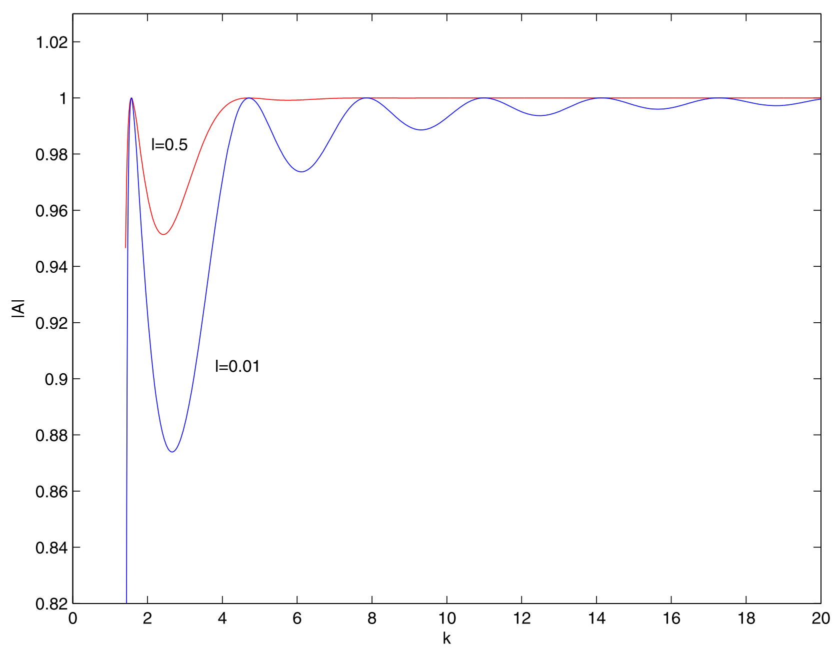


Fig. 5 The amplitude $|A|$ for $\ell = 0.01$ and 0.5

For a small enough difference Δ between two large energies, the oscillation frequency ω of the interference would be largely affected by the non-commutative structure. Let

$$\frac{E_1}{r} = \xi + \Delta$$

$$\frac{E_2}{r} = \xi$$

Then, the oscillating frequency in (29) is

$$\omega = \frac{1}{\ell}\left[\sin^{-1}\bigl(\ell(\xi+\Delta)\bigr) - \sin^{-1}(\ell\xi)\right]$$

$$\simeq \Delta + \frac{\ell^2}{6}\bigl(3\xi^2\Delta + 3\xi\Delta^2 + \Delta^3\bigr)$$

which for small Δ leads to a correction $\Delta(1+\frac{1}{2}\ell^2\xi^2)$.

As in phase-space suppression (or dilation) and particle multiplicity effects (Sect. 2), observation of this correction depends on the product $\ell\xi$. For instance $\ell\xi \sim O(1)$ for $\ell \sim 10^{-27}$ s and $\xi \sim 300$ GeV.

5 Electron spin resonance

Consider an (unpaired) electron interacting with a magnetic field, for which one considers only its spin degree of freedom. In the basis where $\chi_+(t)$ and $\chi_-(t)$ are the up and down spin states the Hamiltonian is

$$H = \frac{g}{2}\mu_B \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \tag{30}$$

Let $B_z = B_0$ be fixed and B_x, B_y time-dependent. The time dependence for a massless field ϕ is obtained from

$$\left((p^0)^2 - |p|^2\right)\phi = 0 \tag{31}$$

Use the commuting basis (t, p^1, p^2, p^3) and assume ϕ to be an eigenstate of momentum

$$|p|^2\phi = k^2\phi$$

Then from (82) and (83) it follows that Eq. (31) becomes

$$-\frac{\gamma^2}{\ell^2}\sinh^2\left(\ell\frac{\partial}{\partial t}\right)\phi_+ = k^2\phi_+ \tag{32}$$

for $\epsilon = +1$ and

$$-\frac{\gamma^2}{\ell^2}\sin^2\left(\ell\frac{\partial}{\partial t}\right)\phi_- = k^2\phi_- \tag{33}$$

for $\epsilon = -1$, with solutions (setting $\gamma = 1$ which is simply a momentum unit)

$$\phi_+(t) = \phi_+(0)\exp\left(\pm\frac{it}{\ell}\sin^{-1}(\ell k)\right) \tag{34}$$

$$\phi_-(t) = \phi_-(0)\exp\left(\pm\frac{it}{\ell}\sinh^{-1}(\ell k)\right) \tag{35}$$

the main modification of the non-commutative structure being that k is no longer the frequency of the massless matter wave,

$$\omega_+ = \frac{1}{\ell}\sin^{-1}(\ell k)$$

$$\omega_- = \frac{1}{\ell}\sinh^{-1}(\ell k)$$

Consider now a field

$$B_x = b\cos(\omega t); \qquad B_y = b\sin(\omega t); \qquad B_z = B_0 \tag{36}$$

Defining

$$\omega_1 = \frac{g}{2}\mu_B b; \qquad \omega_0 = \frac{g}{2}\mu_B B_0 \tag{37}$$

from

$$p^0\chi(t) = H\chi(t) \tag{38}$$

one obtains ($\epsilon = +1$)

$$\begin{aligned} \frac{i}{2\ell}\left\{\chi_+(t+\ell) - \chi_+(t-\ell)\right\} &= \omega_0\chi_+(t) + \omega_1 e^{-i\omega_+ t}\chi_-(t) \\ \frac{i}{2\ell}\left\{\chi_-(t+\ell) - \chi_-(t-\ell)\right\} &= -\omega_0\chi_-(t) + \omega_1 e^{i\omega_+ t}\chi_+(t) \end{aligned} \tag{39}$$

Replacing $\chi_-(t)$, taken from the first equation, on the second one obtains with

$$\chi_+(t) = \chi_+(0)\exp(i\lambda t) \tag{40}$$

the characteristic equation

$$\begin{aligned} &\frac{1}{2\ell^2}\left\{\cos\left(\ell(\omega + 2\lambda)\right) - \cos(\ell\omega)\right\} \\ &\quad = \frac{\omega_0}{\ell}\left\{\sin\left(\ell(\omega+\lambda)\right) - \sin(\ell\lambda)\right\} - \omega_0^2 - \omega_1^2 \end{aligned} \tag{41}$$

which for $\ell = 0$ reduces to

$$\lambda^2 + \omega_+\lambda + \omega_0\omega_+ - \omega_0^2 - \omega_1^2 = 0 \tag{42}$$

with solution

$$\lambda_\pm^{(0)} = -\frac{\omega_+}{2} \pm \sqrt{\omega_1^2 + \left(\omega_0 - \frac{\omega_+}{2}\right)} \tag{43}$$

To obtain the leading ℓ^2 corrections to this result one finds from (41)

$$\left.\frac{d\lambda}{d\ell}\right|_{\ell=0} = 0; \qquad \left.\frac{d\lambda}{d\ell^2}\right|_{\ell=0} = \frac{\omega_0((\omega_+ + \lambda)^3 - \lambda^3)}{6(\omega_+ + 2\lambda)} \tag{44}$$

Therefore, in order ℓ^2

$$\lambda_\pm^{(1)} = \lambda_\pm^{(0)} + \ell^2\frac{\omega_0((\omega_+ + \lambda_\pm^{(0)})^3 - \lambda_\pm^{(0)3})}{6(\omega_+ + 2\lambda_\pm^{(0)})} \tag{45}$$

Let the initial conditions at $t = 0$ be

$$\chi(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then from

$$\chi_+(t) = Ae^{i\lambda_+ t} + Be^{i\lambda_- t} \tag{46}$$

$B = -A$ and

$$\chi_-(0) = -\frac{A}{\ell\omega_1}\left\{\sin(\ell\lambda_+) - \sin(\ell\lambda_-)\right\} = 1 \tag{47}$$

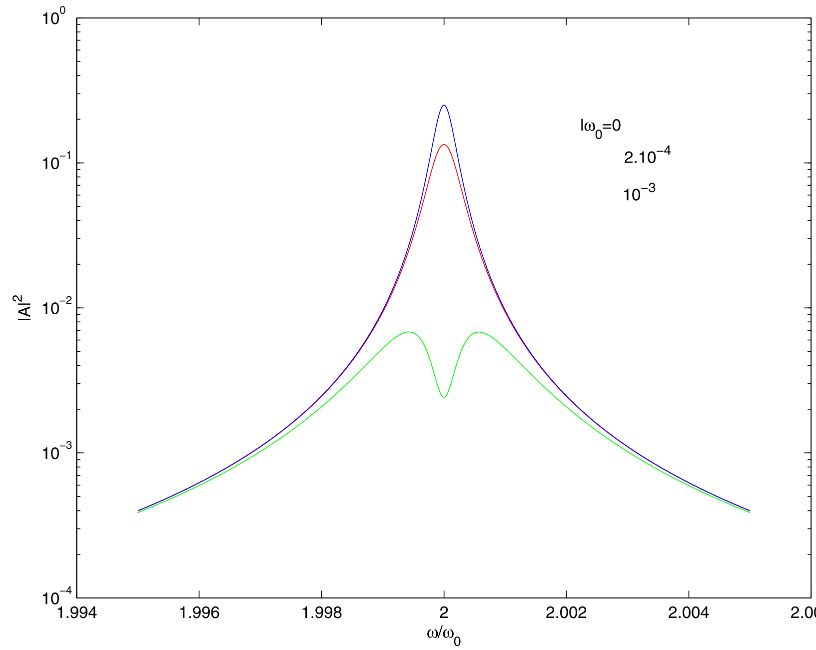


Fig. 6 Absorption spectrum for several $\ell\omega_0$ values

leading to

$$|A| = \frac{\omega_1}{\frac{1}{\ell}\{\sin(\ell\lambda_+) - \sin(\ell\lambda_-)\}} \tag{48}$$

If the energy of the electromagnetic field at frequency ω_+ is dissipated by relaxation processes the absorbed energy would be proportional to $|A|^2$. Figure 6 illustrates, for several $\ell\omega_0$ values, the kind of deviations in the absorption spectrum that would be observed. Similar results are obtained for the $\epsilon = -1$ case.

6 Non-commutative QED

The general construction of gauge fields as Lie algebra-valued connections on the deformed algebra (3) was sketched in Ref. [27]. Here, I will review those results and construct the electromagnetic field in the operator context. Then an operator symbol (star-product) formulation will be developed which is useful for practical calculations.

6.1 Non-commutative space-time and the electromagnetic field

To deal with gauge theories in a non-commutative context, it essential to use the tools of non-commutative geometry where one starts from a non-commutative C^*-algebra and uses the same correspondence as in the commutative case to characterize the geometric properties of the non-commutative space. In particular, a differential algebra may be defined either by duality from the derivations of the algebra, when a sufficient number of derivations is available, or directly from the triple $(H, \pi(C^*), D)$, where $\pi(C^*)$ is a representation of the C^* algebra in the Hilbert space H and D is the Dirac operator. In the latter case the commutator with the Dirac operator is used to obtain the one-forms. The algebra $\mathcal{R}_{\ell,\infty}$ has enough derivations to allow for a direct construction of the differential algebra [27].

The derivations of the algebra $\mathcal{R}_{\ell,\infty}$ (3) are the inner derivations plus a dilation $\mathcal{D}$ ($[\mathcal{D}, P_\mu] = P_\mu$; $[\mathcal{D}, \Im] = \Im$; $[\mathcal{D}, M_{\mu\nu}] = [\mathcal{D}, x_\mu] = 0$).

$$\mathrm{Der}\{\Re_{\ell,\infty}\} = \{X_\mu, M_{\mu\nu}, P_\mu, \Im, \mathcal{D}\} \tag{49}$$

Because in the construction of the differential algebra, the derivations corresponding to $\frac{1}{i}P_\mu$ and $\frac{1}{i\ell}\Im$ play a special role, they are denoted by the symbols ∂_μ and ∂_4 to emphasize their role as elements of $\mathrm{Der}\{\Re_{\ell,\infty}\}$ rather than elements of the enveloping algebra[2] $U_\Re = \{X_\mu, M_{\mu\nu}, P_\mu, \Im, \Im^{-1}, 1\}$ of $\Re_{\ell,\infty}$. The action on the generators is

$$\begin{aligned}
&\partial_\mu(X_\nu) = \eta_{\mu\nu}\Im \\
&\partial_4(X_\mu) = \ell P_\mu \\
&\partial_\sigma(M_{\mu\nu}) = \eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu \\
&\partial_\mu(P_\nu) = \partial_\mu(\Im) = \partial_\mu(1) = 0 \\
&\partial_4(M_{\mu\nu}) = \partial_4(P_\mu) = \partial_4(\Im) = \partial_4(1) = 0
\end{aligned} \tag{50}$$

[2] For the generation of the enveloping algebra one adds the operators $\{\Im^{-1}, 1\}$ and their powers. Because $\Im$ is a small deformation of 1, $\Im^{-1}$ is well defined. The commutation relations with $\Im^{-n}$ are easily obtained from the vanishing of all commutators with $\Im\Im^{-1}$. For example $[X_\mu, \Im^{-1}] = -i\ell^2 P_\mu \Im^{-2}$.

The set of derivations $\{\partial_\mu, \partial_4\}$ is the minimal set that contains the usual ∂_μ's, is maximal abelian and is action closed on the coordinate operators, in the sense that the action of ∂_μ on x_ν leads to the operator $\Im$ that corresponds to ∂_4 and conversely. Denoting by V the complex vector space of derivations spanned by $\{\partial_\mu, \partial_4\}$, the algebra of differential forms $\Omega(U_\Re)$ is now constructed from the complex $C(V, U_\Re)$ of multilinear antisymmetric mappings from V to $U_\Re$. For an explicit construction of $\Omega(U_\Re)$ one may use a basis of 1-forms $\{\theta^\mu, \theta^4\}$ defined by

$$\theta^a(\partial_b) = \delta^a_b, \quad a, b \in (0, 1, 2, 3, 4) \tag{51}$$

The elements θ^a of the 1-form basis do not coincide with dx_μ. Actually

$$dX_\mu = \eta_{\nu\mu}\Im\theta^\nu + \ell P_\mu\theta^4 \tag{52}$$

Although the operators associated to the coordinates are just the four X_μ, $\mu \in (0, 1, 2, 3)$, (no extra dimension in the set of physical coordinates) one sees that an additional degree of freedom appears in the set of derivations which, by duality, leads to an additional degree of freedom in the exterior algebra. Therefore quantum fields that are connections may pick up additional components.

To define gauge fields in this setting consider a right $U_\Re$-module generated by 1.

$$E = \{1a;\ a \in U_\Re\} \tag{53}$$

A connection is a mapping $\nabla : E \to E \otimes \Omega^1(U_\Re)$ such that

$$\nabla(\chi a) = \chi da + \nabla(\chi)a \tag{54}$$

$\chi \in E$, $a \in U_\Re$. For each derivation $\delta_i \in V$ the connection defines a mapping $\nabla_{\delta_i} : E \to E$. Because of Eq. (54), knowing how the connection acts on the algebra unit 1, one has the complete action. Define

$$\nabla(1) \doteq A = A_i\theta^i, \quad A_i \in U_\Re \tag{55}$$

A gauge transformation will be a unitary element ($U^*U = 1$) acting on E. Such unitary elements exist in the C^*-algebra formed from the elements of the enveloping algebra by the standard techniques. Let $\phi \in E$ be a scalar field. Then

$$\nabla(\phi) = d\phi + \nabla(1)\phi \tag{56}$$

Acting on $\nabla(\phi)$ with a unitary element

$$\begin{aligned} U\nabla(\phi) &= U\,d\big(U^{-1}U\phi\big) + U\nabla(1)U^{-1}U\phi \\ &= d(U\phi) + \big\{U\big(dU^{-1}\big) + U\nabla(1)U^{-1}\big\}U\phi \\ &= \nabla'(U\phi) \end{aligned} \tag{57}$$

Therefore the gauge field transformation under a gauge transformation is

$$\nabla(1) \to U\big(dU^{-1}\big) + U\nabla(1)U^{-1} \tag{58}$$

The non-commutativity of $U_\Re$ prevents the vanishing of the second term.

The connection is extended to a mapping $E \otimes \Omega(U_\Re) \to E \otimes \Omega(U_\Re)$ by

$$\nabla(\phi\alpha) = \nabla(\phi)\alpha + \phi\,d\alpha \tag{59}$$

$\phi \in E$ and $\alpha \in \Omega(U_\Re)$. Computing $\nabla^2(1)$

$$\begin{aligned} \nabla^2(1) &= \nabla\big(1A_i\theta^i\big) = \nabla(1A_i)\theta^i + 1A_i\,d\theta^i \\ &= 1\,dA_i\theta^i + \nabla(1)A_i\theta^i + 1A_i\,d\theta^i \\ &= \partial_j(A_i)\theta^j \wedge \theta^i + A_jA_i\theta^j \wedge \theta^i \end{aligned} \tag{60}$$

Therefore, given an electromagnetic potential $A = A_i\theta^i$ ($A_i \in U_\Re$) the corresponding electromagnetic field is $F_{ij}\theta^i \wedge \theta^j$ where

$$F_{ij} = \partial_i(A_j) - \partial_j(A_i) + [A_i, A_j] \tag{61}$$

$F_{ij} \in U_\Re$. Unlike the situation in commutative space-time, the commutator term does not vanish and pure electromagnetism is no longer a free theory, because of the quadratic terms in F_{ij}. Also the indices in the connections (55) and gauge fields (61) run over $(0, 1, 2, 3, 4)$, which resulted from the most natural choice for the differential algebra basis.

To construct an action for the electromagnetic field consider a diagonal metric $\eta_{ab} = (1, -1, -1, -1, 1)$ and construct

$$G = G_{knl}\theta^k \wedge \theta^n \wedge \theta^l \tag{62}$$

where $G_{knl} = \epsilon^{ij}_{\cdot\cdot knl}F_{ij} \in U_\Re$. The action S_A is obtained from the trace of $F \wedge G$

$$S_A = \mathrm{Tr}\big\{F_{ab}F^{ab}\big\} = \mathrm{Tr}\big\{F_{\mu\nu}F^{\mu\nu} + 2F_{4\mu}F^{4\mu}\big\} \tag{63}$$

$\mu, \nu \in (0, 1, 2, 3)$.

To discuss matter fields one needs spinors, and an appropriate set of γ matrices to contract the derivations ∂_a. A massless action term for spinor matter fields may be written

$$S_\psi = i\overline{\psi}\gamma^a\partial_a\psi \tag{64}$$

where $a \in (0, 1, 2, 3, 4)$, $\gamma^a = (\gamma^0, \gamma^1, \gamma^2, \gamma^3, i\gamma^5)$ and ψ is a field in a projective module $E_\psi \subset U_\Re^{\otimes 4}$. It follows from the properties of the derivations that this term is Lorentz invariant. Notice that although the set $\{M_{\mu\nu}, X_\mu\}$ has a $O(2,3)$ structure, it is only the $O(1,3)$ part that is a symmetry group. Coupling the fermions to the gauge field

$$S_\psi = \overline{\psi}i\gamma^a(\partial_a + igA_a)\psi \tag{65}$$

One sees that the fermions may be coupled to the connection A_a without having to introduce new degrees of freedom in the fermion sector.

Given the connection A_μ as a member of the enveloping algebra $U_\Re$ it may be decomposed into a set of operator eigenvalues of the momenta $\{P_\mu\}$ with c-number coefficients $A_\mu(k)$. One has

$$\left[P_\mu, e^{-\frac{i}{2}k_\nu\{X^\nu,\Im^{-1}\}_+}\right] = k_\mu e^{-\frac{i}{2}k_\nu\{X^\nu,\Im^{-1}\}_+} \tag{66}$$

Then

$$A_\mu = \int d^4k \left\{A_\mu(k)e^{-\frac{i}{2}k_\nu\{X^\nu,\Im^{-1}\}_+} + A_\mu^\dagger(k)e^{\frac{i}{2}k_\nu\{X^\nu,\Im^{-1}\}_+}\right\} \tag{67}$$

For the electromagnetic field $F_{\mu\nu}$ in (61) one has to compute

$$\left[e^{-\frac{i}{2}k_\nu\{X^\nu,\Im^{-1}\}_+}, e^{-\frac{i}{2}q_\mu\{X^\mu,\Im^{-1}\}_+}\right] \tag{68}$$

which in leading ℓ^2-order is

$$i\frac{\ell^2}{2}e^{-\frac{i}{2}(k_\nu+q_\nu)\{X^\nu,\Im^{-1}\}_+}(k_\nu q_\mu - q_\nu k_\mu)\Im^{-2}\Sigma^{\nu\mu} \tag{69}$$

$\Sigma^{\nu\mu}$ being a spin operator

$$\Sigma^{\nu\mu} = M^{\nu\mu} - \left\{X^\nu\Im^{-1}, P^\mu\right\}_+ + \left\{X^\mu\Im^{-1}, P^\nu\right\}_+ \tag{70}$$

Then

$$\begin{aligned} F_{\mu\nu}(k) &= -i\big(k_\mu A_\nu(k) - k_\nu A_\mu(k)\big) \\ &\quad + i\ell^2\left\{\int d^4q\, A_\mu(k-q)A_\nu(q)(k-q)_\sigma q_\varepsilon \Sigma^{\sigma\varepsilon}\right. \\ &\quad \left. + \int d^4q\, A_\mu(k+q)A_\nu^\dagger(q)(k+q)_\sigma q_\varepsilon \Sigma^{\varepsilon\sigma}\right\} \end{aligned} \tag{71}$$

the last term in (71) being the momentum-space image of the non-commuting $[A_i, A_j]$ in (61). One sees that non-commutative pure QED is not a free theory, having non-trivial 3- and 4-photon vertices of order ℓ^2 which are spin-dependent.

6.2 An operator symbol formulation

An algebra of non-commuting operators may be represented in a space of functions with a modified (star) product. The general context of this formulation is described in Appendix B. Here a star product is found which reproduces the non-commutative features of the space-time algebra. The non-commuting algebra that is being represented is

$$\begin{aligned} &[\widehat{P}_\mu, \widehat{P}_\nu] = 0 \\ &[\widehat{X}_\mu, \widehat{X}_\nu] = -i\epsilon\ell^2 M_{\mu\nu} \\ &[\widehat{P}_\mu, \widehat{X}_\nu] = i\eta_{\mu\nu}\widehat{\Im} \\ &[\widehat{P}_\mu, \widehat{\Im}] = 0 \\ &[\widehat{X}_\mu, \widehat{\Im}] = i\epsilon\ell^2\widehat{P}_\mu \end{aligned} \tag{72}$$

where the hat symbols are meant to emphasize the non-commuting operator nature of $\{\widehat{P_\mu}, \widehat{X_\mu}, \widehat{\Im}\}$. These are going to be represented by functions $\{p_\mu, x_\mu, \Im\}$ with a star product

$$\begin{aligned} &G(p,x,\Im) * H(p,x,\Im) \\ &\quad = G\exp\left\{\frac{i}{2}\big(\overleftarrow{\partial_p^\mu}\eta_{\mu\nu}\Im\overrightarrow{\partial_x^\nu} - \overleftarrow{\partial_x^\mu}\eta_{\mu\nu}\Im\overrightarrow{\partial_p^\nu}\big)\right. \\ &\qquad \left. - \frac{i\epsilon\ell^2}{2}\big(\overleftarrow{\partial_x^\mu}M_{\mu\nu}\overrightarrow{\partial_x^\nu} + \overleftarrow{\partial_x^\mu}P_\mu\overrightarrow{\partial_\Im} - \overleftarrow{\partial_\Im}P_\mu\overrightarrow{\partial_x^\mu}\big)\right\}H \end{aligned} \tag{73}$$

From this one obtains for the electromagnetic field

$$\begin{aligned} F_{\mu\nu}(x) &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \\ &\quad + A_\mu(x)e^{-\frac{i\epsilon\ell^2}{2}\overleftarrow{\partial_x^\sigma}M_{\sigma\rho}\overrightarrow{\partial_x^\rho}}A_\nu(x) \\ &\quad - A_\nu(x)e^{-\frac{i\epsilon\ell^2}{2}\overleftarrow{\partial_x^\sigma}M_{\sigma\rho}\overrightarrow{\partial_x^\rho}}A_\mu(x) \end{aligned} \tag{74}$$

in ℓ^2-order

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\epsilon\ell^2\partial^\sigma A_\mu(x)\partial^\rho A_\nu(x)M_{\sigma\rho} \tag{75}$$

and in momentum space

$$\begin{aligned} F_{\mu\nu}(k) &= -i\big(k_\mu A_\nu(k) - k_\nu A_\mu(k)\big) \\ &\quad + i\epsilon\ell^2\left\{\int d^4q\, A_\mu(k-q)A_\nu(q)(k-q)^\sigma q^\rho\right. \\ &\quad \left. + \int d^4q\, A_\mu(k+q)A_\nu^\dagger(q)(k+q)^\sigma q^\rho\right\}M_{\sigma\rho} \end{aligned} \tag{76}$$

In the context of the non-commuting phase-space structure defined in (72) the orbital angular momentum would be represented by $\widehat{X}_\mu\widehat{P}_\nu - \widehat{X}_\nu\widehat{P}_\mu$. Therefore it makes sense to interpret $M_{\mu\nu}$ as the spin operator and one obtains the same $F_{\mu\nu}(k)$ structure as before.

For the photon-spinor interactions, by minimal coupling one has

$$\overline{\psi}(x)(D_\mu - m)\gamma^\mu\psi(x) \tag{77}$$

with

$$D_\mu \psi(x) = \partial_\mu \psi(x) - i A_\mu(x) * \psi(x)$$

In conclusion: one has 3-photon vertices of order ℓ^2 and 4-photon vertices of order ℓ^4. The 3-photon coupling (in ℓ^2order) is

$$\begin{aligned}
&\epsilon \ell^2 (2\pi)^4 \delta^4(p_1 + p_2 + p_3)\{g^{\mu_1\mu_2} p_1^{\mu_3} p_3^\sigma p_2^\rho M_{\sigma\rho} \\
&\quad - g^{\mu_1\mu_2} p_2^{\mu_3} p_1^\sigma p_3^\rho M_{\sigma\rho} + c.p.\}
\end{aligned} \tag{78}$$

$c.p.$ meaning cyclic permutations of $\{1, 2, 3\}$ (refer to Fig. 7 for notation), the photon-spinor coupling is

$$e(2\pi)^4 \delta^4(p - p' - k)(\gamma^\mu)_{\alpha\beta} \left\{1 + \frac{\epsilon \ell^2}{2} p'^\sigma M_{\sigma\rho} p^\rho\right\} \tag{79}$$

and the bare propagators are unchanged. All non-commuting contributions have momentum and spin dependence.

The 3- and 4-photon vertices lead to new one-loop contributions, see Fig. 8 for the 2- and 3-point functions.

In particular, the new coupling (79)

$$-\frac{e\epsilon\ell^2}{2} \bar{\psi}(p') k^\sigma \gamma^\mu \frac{i}{2} [\gamma_\sigma, \gamma_\rho] p^\rho \psi(p)$$

implies the existence of extra spin-dependent contributions in spinor scattering which are enhanced at large p and k.

In most star-product formulations of non-commutative space-time the right-hand side of the commutators is a constant matrix of ordinary c-numbers (see Eq. (4)), implying violation of Lorentz invariance. An exception is the paper of Carlson and Carone [51] where $\theta_{\mu\nu}$ is promoted to an operator with the right tensor properties under the Lorentz group. The algebra is, however, quite different from the one in this paper, because, in particular, that operator commutes with the coordinates. The Carson–Carone algebra is in fact a rescaling and contraction of the $\epsilon = -1$ case. In this process the fundamental length ℓ becomes zero and the non-commutative structure is only kept by a rescaling of $\theta_{\mu\nu}$. As a result, the implications for non-commutative QED are different from those in this paper. In addition to the vanishing of the fundamental length, a basic difference in the Carson–Carone approach is that $\theta_{\mu\nu}$, not being a physically identifiable operator, it is integrated over to obtain the new vertex contributions. This is what leads to the vanishing of the 3-photon vertices. In our case the corresponding operator is $M_{\mu\nu}$ which acts on the spin of the particles.

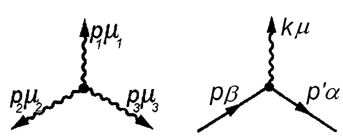


Fig. 7 One and three-photon vertices

Fig. 8 One-loop diagrams

7 Remarks and conclusions

(1) The most relevant point of the stability approach to non-commutative space-time is the emergence of two deformation parameters, which might define different length scales. This led to the conjecture that one of them might be much larger than the Planck length and therefore already detectable by contemporary experimental means.

(2) The effects explored in this paper are rather conservative in the sense that they explore well-known physical observables. Other similar consequences of the non-commutative structure, already mentioned before [40], follow from the non-vanishing right-hand side of the double commutators

$$\begin{aligned}
&[[p_\mu, x_\nu], x_\alpha] = \epsilon \ell^2 \eta_{\mu\nu} p_\alpha \\
&[[x_\mu, x_\nu], x_\alpha] = \epsilon \ell^2 (\eta_{\nu\alpha} x_\mu - \eta_{\mu_\alpha} x_\nu)
\end{aligned}$$

(3) A more speculative aspect of the non-commutative structure concerns the physical relevance of the extra derivation ∂_4 described in Sect. 6.1. This includes new fields associated to gauge interactions which may lead to effective mass terms for otherwise massless particles (see [27] for more details)

Appendix A: Representations of the deformed algebra and its subalgebras

For explicit calculations of the consequences of the non-commutative space-time algebra (2) (with $\epsilon' = 0$) it is useful to have at our disposal functional representations of this structure. Such representations on the space of functions defined on the cone C^4 ($\epsilon = -1$) or $C^{3,1}$ ($\epsilon = +1$) have been described in [27]. Here one collects a few other useful representations of the full algebra and some subalgebras.

1. As differential operators in a 5-dimensional commutative manifold $M_5 = \{\xi_\mu\}$ with metric $\eta_{aa} = (1, -1, -1, -1, \epsilon)$

$$\begin{aligned} p_\mu &= i\frac{\partial}{\partial \xi^\mu} \\ \Im &= 1 + i\ell\frac{\partial}{\partial \xi^4} \\ M_{\mu\nu} &= i\left(\xi_\mu\frac{\partial}{\partial \xi^\nu} - \xi_\nu\frac{\partial}{\partial \xi^\mu}\right) \\ x_\mu &= \xi_\mu + i\ell\left(\xi_\mu\frac{\partial}{\partial \xi^4} - \epsilon\xi^4\frac{\partial}{\partial \xi^\mu}\right) \end{aligned} \tag{80}$$

2. Another global representation is obtained using the commuting set $(p^\mu, \Im)$, namely

$$\begin{aligned} x_\mu &= i\left(\epsilon\ell^2 p_\mu\frac{\partial}{\partial \Im} - \Im\frac{\partial}{\partial p^\mu}\right) \\ M_{\mu\nu} &= i\left(p_\mu\frac{\partial}{\partial p^\nu} - p_\nu\frac{\partial}{\partial p^\mu}\right) \end{aligned} \tag{81}$$

3. Representations of subalgebras

Because of non-commutativity only one of the coordinates can be diagonalized. Here, consider the restriction to one space dimension, namely the algebra of $\{p^0, \Im, p^1, x^0, x^1\}$.

For $\epsilon = +1$ define hyperbolic coordinates in the plane $(p^1, \Im)$ and polar coordinates in the plane $(p^0, \Im)$. Then, it follows from (81) that

$$\begin{aligned} p^1 &= \frac{r}{\ell}\sinh\mu \\ p^0 &= \frac{\gamma}{\ell}\sin\theta \\ \Im &= r\cosh\mu = \gamma\cos\theta \\ x^1 &= i\ell\frac{\partial}{\partial\mu} \\ x^0 &= -i\ell\frac{\partial}{\partial\theta} \end{aligned} \tag{82}$$

For $\epsilon = -1$ with polar coordinates in the plane $(p^1, \Im)$ and hyperbolic coordinates in the plane $(p^0, \Im)$,

$$\begin{aligned} p^1 &= \frac{r}{\ell}\sin\theta \\ p^0 &= \frac{\gamma}{\ell}\sinh\mu \\ \Im &= \gamma\cosh\mu = r\cos\theta \\ x^1 &= i\ell\frac{\partial}{\partial\theta} \\ x^0 &= -i\ell\frac{\partial}{\partial\mu} \end{aligned} \tag{83}$$

Appendix B: Operator symbol formulation

Let $\widehat{A}$ be an operator in a Hilbert space $\mathcal{H}$ and $\widehat{U}(\vec{x})$, $\widehat{D}(\vec{x})$ two families of operators called *dequantizers* and *quantizers*, respectively, such that

$$\mathrm{Tr}\{\widehat{U}(\vec{x})\widehat{D}(\vec{x}')\} = \delta(\vec{x} - \vec{x}') \tag{84}$$

The labels $\vec{x}$ (with components $x_1, x_2, \dots, x_n$) are coordinates in a linear space V where the functions (operator symbols) are defined. Some of the coordinates may take discrete values. For them the delta function in (84) should be understood as a Kronecker delta. Provided the property (84) is satisfied, one defines the *symbol of the operator* $\widehat{A}$ by the formula

$$f_A(\vec{x}) = \mathrm{Tr}\{\widehat{U}(\vec{x})\widehat{A}\}, \tag{85}$$

assuming the trace to exist. In view of (84), one has the reconstruction formula

$$\widehat{A} = \int f_A(x)\widehat{D}(\vec{x})\,d\vec{x} \tag{86}$$

The role of quantizers and dequantizers may be exchanged. Then

$$f_A^d(\vec{x}) = \mathrm{Tr}\{\widehat{D}(\vec{x})\,\widehat{A}\} \tag{87}$$

is called the dual symbol of $f_A(\vec{x})$ and the reconstruction formula is

$$\widehat{A} = \int f_A^d(x)\widehat{U}(\vec{x})\,d\vec{x} \tag{88}$$

Symbols of operators can be multiplied using the star-product kernel as follows:

$$f_A(\vec{x}) \star f_B(\vec{x}) = \int f_A(\vec{y}) f_B(\vec{z}) K(\vec{y}, \vec{z}, \vec{x})\,d\vec{y}\,d\vec{z} \tag{89}$$

the kernel being

$$K(\vec{y}, \vec{z}, \vec{x}) = \mathrm{Tr}\{\widehat{D}(\vec{y})\widehat{D}(\vec{z})\widehat{U}(\vec{x})\} \tag{90}$$

The star product is associative,

$$\big(f_A(\vec{x}) \star f_B(\vec{x})\big) \star f_C(\vec{x}) = f_A(\vec{x}) \star \big(f_B(\vec{x}) \star f_C(\vec{x})\big) \tag{91}$$

this property corresponding to the associativity of the product of operators in Hilbert space.

With the dual symbols the trace of an operator may be written in integral form

$$\mathrm{Tr}\{\widehat{A}\widehat{B}\} = \int f_A^d(\vec{x}) f_B(\vec{x})\,d\vec{x} = \int f_B^d(\vec{x}) f_A(\vec{x})\,d\vec{x} \tag{92}$$

For two different symbols $f_A(\vec{x})$ and $f_A(\vec{y})$ corresponding, respectively, to the pairs $(\widehat{U}(\vec{x}), \widehat{D}(\vec{x}))$ and

$(\widehat{U}_1(\vec{y}), \widehat{D}_1(\vec{y}))$, one has the relation

$$f_A(\vec{x}) = \int f_A(\vec{y}) K(\vec{x}, \vec{y})\, d\vec{y} \tag{93}$$

with intertwining kernel

$$K(\vec{x}, \vec{y}) = \mathrm{Tr}\big\{\widehat{D}_1(\vec{y})\widehat{U}(\vec{x})\big\} \tag{94}$$

This general formulation of operators, as operator symbols in a space of functions with a star product, is useful in many other contexts, for example in signal processing [52].

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