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A fractional calculus interpretation of the fractional volatility model

R. Vilela Mendes

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Abstract Based on criteria of mathematical simplicity and consistency with empirical market data, a model with volatility driven by fractional noise has been constructed which provides a fairly accurate mathematical parametrization of the data. Here, the model is formulated in terms of a fractional integration of stochastic processes.

Keywords Fractional calculus · Dynamics of markets · Stochastic models

1 Introduction

Among the man-made dynamical processes, a most interesting one is the market process. Interesting not only because its role and impact in the economy has considerably changed in the course of time, but also because rather than being a featureless random process as the "efficient market hypothesis" might imply, it displays a number of nontrivial structures. In liquid markets, the autocorrelation of price changes decays to negligible values in a few minutes, consistent with the absence of long term statistical arbitrage. However, nonlinear functions of the returns exhibit significant positive autocorrelation and long memory effects. There is volatility clustering with large returns expected to be followed by large returns and small returns by small returns. In particular, the existence of long memory effects excludes the simple geometric Brownian motion model still used in most mathematical finance calculations. On the other hand, the interplay of short linear correlation and long nonlinear correlations makes it hard to model the process either by a Gibbs measure or even by chains with complete connections [1]. Using a function of the returns as a metric, one also finds a small geometric dimension for the markets [2] and considerable geometrical changes during market crisis [3].

Recently [4], using both a criteria of mathematical simplicity and consistency with market data, a stochastic volatility model was constructed with volatility driven by fractional noise. It appears to be the minimal model consistent both with mathematical simplicity and the market data. The model has been used to compute the price return statistics, asymptotics and deviations from the classical Black–Scholes result. Extensive comparison of this *fractional volatility model* with market data (both daily and high-frequency data) on what concerns scaling properties and statistics of returns is included in [4]. It was also compared with agent-based models [5].

R. Vilela Mendes (✉)
Centro de Matemática e Aplicações Fundamentais,
Complexo Interdisciplinar, Universidade de Lisboa,
Av. Prof. Gama Pinto 2, 1649-003 Lisbon, Portugal
e-mail: vilela@cii.fc.ul.pt

R. Vilela Mendes
Universidade Técnica de Lisboa, Lisbon, Portugal

The fractional volatility model is briefly described in Sect. 2, with some emphasis on the leverage effect. Then in Sect. 3, the fractional calculus interpretation of the model is developed. The model with leverage, when formulated with two stochastic integrals of the same Brownian motion may be considered as an illustration of Podlubny's beautiful geometric interpretation of fractional integration [6].

In a series of papers, several authors [7–12] have used the continuous-time-random-walk (CTRW) model, first introduced by Montroll and Weiss [13] in statistical mechanics to describe the probability distribution of returns at random times $p(x,t)$ ($x=$ log-price). With a factorized joint density and a memory function with power law decay, the master equation of the CTRW model becomes an equation of the Kolmogorov–Feller type [14] with the time derivative replaced by a (Caputo) fractional time derivative. In the diffusion limit [10], this equation becomes a fractional diffusion equation with a Riesz space-fractional derivative. The model has been compared with data in future prices [8] and high-frequency stock prices [9]. From the conceptually point of view, the use of fractional diffusion equations to describe the market fluctuations has the problem of not distinguishing between the almost memoryless nature of the linear correlation of returns and the memory properties of nonlinear functions. Nevertheless, the CTRW model is not directly comparable with the fractional volatility model studied here because in its present version, waiting times play no essential role.

In the fractional volatility model, fractional calculus comes into play through integration of a white noise process with normal integration for the price process and (right-sided) Riemann–Liouville fractional integration for the volatility. In this way, the different memory qualities of the processes are put into evidence.

2 The fractional volatility model and leverage

The basic hypothesis for the model construction is that the log-price process $\log S_t(\omega,\omega')$ belongs to a probability product space $\Omega\otimes\Omega'$ with $\omega\in\Omega$ and $\omega'\in\Omega'$. The first one, Ω, is the Wiener space and the second, Ω', is a probability space to be reconstructed from the data. Furthermore, one assumes that for each fixed ω', $\log S_t(\bullet,\omega')$ is a square integrable random variable in Ω. This implies [15] that

$$\frac{dS_t}{S_t}(\bullet,\omega')=\mu_t(\bullet,\omega')\,dt+\sigma_t(\bullet,\omega')\,dB(t)$$

where $\mu_t(\bullet,\omega')$ and $\sigma_t(\bullet,\omega')$ are processes in Ω.

The process $\sigma_t(\bullet,\omega')$ is then reconstructed searching for its scaling properties from the market data. This data-reconstructed σ_t process was called the *induced volatility*. Finally, the following *fractional volatility model* is obtained

$$\begin{aligned} dS_t &= \mu S_t\,dt+\sigma_t S_t\,dB(t)\\ \log\sigma_t &= \beta+\frac{k}{\delta}\{B_H(t)-B_H(t-\delta)\} \end{aligned} \tag{1}$$

k being a volatility intensity parameter, δ the observation time scale, and H an Hurst parameter found to lie in the range 0.8–0.9. Notice that the volatility is not driven by fractional Brownian motion but by fractional noise, naturally introducing an observation scale dependence.

The following nonlinear correlation of the returns

$$L(\tau)=\langle|r(t+\tau)|^2r(t)\rangle-\langle|r(t+\tau)|^2\rangle\langle r(t)\rangle \tag{2}$$

is called *leverage* and the *leverage effect* is the fact that for $\tau>0$, $L(\tau)$ starts from a negative value whose modulus constantly decays to zero whereas for $\tau<0$, it has almost negligible values. In the form of (1), the volatility process σ_t affects the log-price, but is not affected by it. Therefore, in its simplest form, the fractional volatility model contains no leverage effect.

Leverage may, however, be implemented in the model in a simple way. For this purpose, it will be convenient to use the following integral representation of fractional Brownian motion [16].

$$\begin{aligned} B_H(t)=C\Big\{&\int_{-\infty}^{0}\big[(t-s)^{H-\frac12}-(-s)^{H-\frac12}\big]\,dB(s)\\ &+\int_0^t(t-s)^{H-\frac12}\,dB(s)\Big\} \end{aligned} \tag{3}$$

Using this representation, the fractional volatility model may be rewritten as

$$\begin{aligned} dS_t &= \mu S_t\,dt+\sigma_t S_t\,dB^{(1)}(t)\\ \log\sigma_t &= \beta+k'\int_{-\infty}^{t}(t-s)^{H-\frac32}\,dB^{(2)}(s) \end{aligned} \tag{4}$$

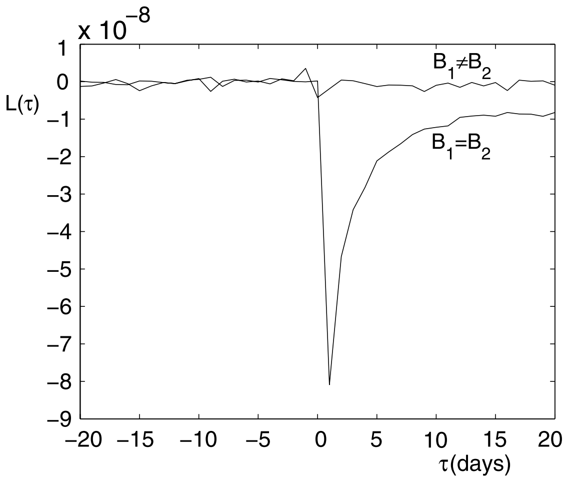

Fig. 1 Leverage in the fractional volatility model comparing the $B^{(1)}(s) \neq B^{(2)}(s)$ and the $B^{(1)}(s) = B^{(2)}(s)$ cases

where $B^{(1)}(s)$ and $B^{(2)}(s)$ may be different Brownian processes. Figure 1 shows the leverage $L(\tau)$ computed for the model (4) with β and $k^{'}$ chosen to match the statistical parameters of the NYSE index daily values in the period 1966–2000. Both the $B^{(1)}(s) \neq B^{(2)}(s)$ and the $B^{(1)}(s) = B^{(2)}(s)$ cases are considered. One sees that for $B^{(1)}(s) \neq B^{(2)}(s)$, there is no leverage effect, whereas for $B^{(1)}(s) = B^{(2)}(s)$, an effect is found. Therefore, identifying the random generator of the log-price process with the stochastic integrator of the volatility, at least a part of the leverage effect is taken into account.

3 The fractional calculus interpretation

The relation of fractional Brownian motion to fractional calculus is easily established [17] starting from the integral representation (3) which one rewrites as

$$B_H(t) = \frac{1}{C(H)} \int_{\mathbb{R}} \left\{ (t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right\} dB(s) \tag{5}$$

the equality meaning "equality in distribution" and

$$C(H) = \frac{\Gamma(H+\frac{1}{2})}{(\Gamma(2H+1)\sin \pi H)^{1/2}} \tag{6}$$

The (right-sided) Riemann–Liouville fractional integral on the line is [18]

$$\begin{aligned} \left(I_-^{\alpha}\phi\right)(s) &= \frac{1}{\Gamma(\alpha)} \int_s^{\infty} \phi(u)(u-s)^{\alpha-1}\, du \\ &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)(u-s)_+^{\alpha-1}\, du \end{aligned} \tag{7}$$

and the (left-sided) Riemann–Liouville fractional integral

$$\begin{aligned} \left(I_+^{\alpha}\phi\right)(s) &= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{s} \phi(u)(s-u)^{\alpha-1}\, du \\ &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(u)(s-u)_+^{\alpha-1}\, du \end{aligned} \tag{8}$$

For sufficiently regular functions, the following fractional integration-by-parts formula holds

$$\int_{\mathbb{R}} \phi(s)\left(I_+^{\alpha}\psi\right)(s)\, ds = \int_{\mathbb{R}} \left(I_-^{\alpha}\phi\right)(s)\psi(s)\, ds \tag{9}$$

Now, from the fact that

$$\left(I_-^{\alpha}\mathbf{1}_{[a,b)}\right)(s) = \frac{1}{\Gamma(\alpha+1)} \left\{ (b-s)_+^{\alpha} - (a-s)_+^{\alpha} \right\} \tag{10}$$

one obtains

$$B_H(t) = \frac{\Gamma(H+\frac{1}{2})}{C(H)} \int_{\mathbb{R}} \left(I_-^{H-\frac{1}{2}} \mathbf{1}_{[o,t)} \right)(s)\, dB(s) \tag{11}$$

The derivative $\overset{\bullet}{B}(s)$ of Brownian motion is not a regular stochastic process. However, it is a well-defined generalized process in the framework of white noise analysis [19] as an infinite-dimensional generalized function in the Hida space (S^*). Therefore, in the white-noise analysis setting, it makes sense to write

$$B(t) = \int^{t} \overset{\bullet}{B}(s)\, ds \tag{12}$$

Using this in (11) and carrying to the $(S), (S^*)$ dual spaces the integration by parts formula, one might rewrite (11) as

$$B_H(t) = \frac{\Gamma(H+\frac{1}{2})}{C(H)} \int_0^{t} \left(I_+^{H-\frac{1}{2}} \overset{\bullet}{B} \right)(s)\, ds \tag{13}$$

Notice that although the integral on s is from 0 to t, the calculation of the Riemann–Liouville integral is between $-\infty$ and s.

One may now rewrite the fractional volatility model using the integral formulas discussed before. From (1) and (11), one obtains

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma_t S_t\,dB^{(1)}(t) \\ \log\sigma_t &= \beta + \frac{k}{\delta}\frac{\Gamma(H+\frac{1}{2})}{C(H)} \\ &\quad \times \int_{\mathbb{R}} \left(I_-^{H-\frac{1}{2}} \mathbf{1}_{[t-\delta,t)}\right)(s)\,dB^{(2)}(s) \end{aligned} \tag{14}$$

and in the small δ limit, using (13), the result is

$$\begin{aligned} dS_t &= \mu S_t\,dt + \sigma_t S_t \overset{\bullet}{B}{}^{(1)}\,dt \\ \log\sigma_t &= \beta + \frac{\Gamma(H+\frac{1}{2})}{C(H)}\left(I_+^{H-\frac{1}{2}} \overset{\bullet}{B}{}^{(2)}\right)(t) \end{aligned} \tag{15}$$

4 Remarks

(1) The stochastic model (4), rewritten in the framework of fractional calculus, puts in evidence the nature of the memory properties of returns and volatility. One sees that the stochastic parts of the log-price and the log-volatility are both driven by a stochastic integration of white noise. The difference is that whereas for the log-price, this is a regular integration, for the log-volatility, it is a fractional integration. In analogy with Podlubny's [6] geometrical interpretation of fractional integration, when $B^{(1)}(s) = B^{(2)}(s)$ one may look at the role of the unique driving Brownian process as the same process projecting two distinct shadows on the log-price and on the volatility.

(2) When modeling long-memory processes by fractional differential equations, one has to decide which type of fractional derivative to use, whether Riemann–Liouville or Caputo or Riesz–Feller. Riesz–Feller is useful for a stochastic interpretation in terms of Lèvy processes and Caputo is thought to be the more appropriate one to define physical boundary conditions. Nevertheless, it is also possible, in some cases, to give a physical meaning to initial conditions in terms of Riemann–Liouville derivatives [20].

In this paper, however, the relevant mathematical object is fractional Brownian motion (fBM) and the relation between fBM and the Riemann–Liouville integrals ((11) and (13)) is established without ambiguity.

(3) Market data, either daily or high-frequency, are discrete processes. The same is true for most experimentally collected data. When modeling such data by a continuous process, one is either assuming that some underlying continuous process is generating the discretely observed data or that the continuous interpolation is unique. This last situation has a solid mathematical basis if, in addition, one restricts the model to some classes of functions, for example, band-limited for Shannon's reconstruction or almost-periodic in the case of random sampling [21].

(4) This paper uses a relation between fractional calculus and stochastic processes as expressed in (11) and (13), which seems to be the more appropriate one for the market model that is considered. However, the market process contains some other stochastic features, in particular, a random distribution of waiting times, for which other probabilistic interpretations [22–24] of fractional calculus might be useful.

References

1. Vilela Mendes, R., Lima, R., Araújo, T.: A process-reconstruction analysis of market fluctuations. Int. J. Theor. Appl. Finance **5**, 797–821 (2002)
2. Vilela Mendes, R., Araújo, T., Louçã, F.: Reconstructing an economic space from a market metric. Phys. A **323**, 635–650 (2003)
3. Araújo, T., Louçã, F.: The geometry of crashes. A measure of the dynamics of stock market crises. Quant. Finance **7**, 63–74 (2007)
4. Vilela Mendes, R., Oliveira, M.J.: A data-reconstructed fractional volatility model. Economics (2008). http://www.economics-ejournal.org/economics/discussionpapers/2008-22. arXiv:math/0602013
5. Vilela Mendes, R.: The fractional volatility model: An agent-based interpretation. Phys. A **387**, 3987–3994 (2008)
6. Podlubny, I.: Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. **5**, 367–386 (2002)
7. Scalas, E., Gorenflo, R., Mainardi, F.: Fractional calculus and continuous-time finance. Phys. A **284**, 376–384 (2000)
8. Mainardi, F., Raberto, M., Gorenflo, R., Scalas, E.: Fractional calculus and continuous-time finance II: The waiting-time distribution. Phys. A **287**, 468–481 (2000)
9. Raberto, M., Scalas, E., Mainardi, F.: Waiting-times and returns in high-frequency financial data: an empirical study. Phys. A **314**, 749–755 (2002)
10. Gorenflo, R., Mainardi, F., Scalas, E., Raberto, M.: Fractional calculus and continuous-time finance III: the diffusion limit. In: Kohlmann, M., Tang, S. (eds.) Mathematical Finance, pp. 171–180. Birkhäuser, Basel (2001)

11. Scalas, E., Gorenflo, R., Mainardi, F., Meerschaert, M.M.: Speculative option valuation and the fractional diffusion equation. In: Sabatier, J., Tenreiro Machado, J. (eds.) Proceedings of the IFAC Workshop on Fractional Differentiation and its Applications, Bordeaux (2004)
12. Meerschaert, M.M., Scalas, E.: Coupled continuous time random walks in finance. Phys. A **370**, 114–118 (2006)
13. Montroll, E.W., Weiss, G.H.: Random walks on lattices II. J. Math. Phys. **6**, 167–181 (1965)
14. Saichev, A.I., Zaslavsky, G.M.: Fractional kinetic equations: Solutions and applications. Chaos **7**, 753–764 (1997)
15. Nualart, D.: The Malliavin Calculus and Related Topics. Springer, Berlin (1995)
16. Embrechts, P., Maejima, M.: Selfsimilar Processes. Princeton University Press, Princeton (2002)
17. Pipiras, V., Taqqu, M.S.: Fractional calculus and its connections to fractional Brownian motion. In: Doukhan, P., Oppenheim, G., Taqqu, M.S. (eds.) Theory and Applications of Long-Range Dependence, pp. 165–201. Birkhäuser, Basel (2003)
18. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon Breach, London (1993)
19. Hida, T., Kuo, H.-H., Potthoff, J., Streit, L.: White Noise—An Infinite Dimensional Calculus. Kluwer, Dordrecht (1993)
20. Heymans, N., Podlubny, I.: Physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives. Rheol. Acta **45**, 765–771 (2006)
21. Collet, P.: Sampling almost periodic functions with random probes of finite density. Proc. R. Soc. Lond. A **452**, 2263–2277 (1996)
22. Machado, J.A.T.: A probabilistic interpretation of fractional-order differentiation. Fract. Calc. Appl. Anal. **6**, 73–80 (2003)
23. Stanislavsky, A.A.: Probabilistic interpretation of the integral of fractional order. Theor. Math. Phys. **138**, 418–431 (2004)
24. Grigolini, P., Rocco, A., West, B.: Fractional calculus as a macroscopic manifestation of randomness. Phys. Rev. E **59**, 2603–2613 (1999)