STOCHASTIC SOLUTION OF A KPP-TYPE NONLINEAR FRACTIONAL DIFFERENTIAL EQUATION

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Abstract

A stochastic solution is constructed for a fractional generalization of the KPP (Kolmogorov, Petrovskii, Piskunov) equation. The solution uses a fractional generalization of the branching exponential process and propagation processes which are spectral integrals of Levy processes.

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1. Introduction: The notion of stochastic solution

The solutions of linear elliptic and parabolic equations, both with Cauchy and Dirichlet boundary conditions, have a probabilistic interpretation. These are classical results which may be traced back to the work of Courant, Friedrichs and Lewy [1] in the 1920's and became a standard tool in potential theory [2],[3]. For example, for the heat equation,

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) \qquad \text{with} \qquad u(0,x) = f(x), \qquad (1)$$

the solution may be written either as

$$u(t,x) = \frac{1}{2\sqrt{\pi}}\int \frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{2t}\right)f(y)\,dy, \qquad (2)$$

or as

$$u(t,x) = \mathbb{E}_x f(X_t), \tag{3}$$

where $\mathbb{E}_x$ means the expectation value, starting from x, of the process

$$dX_t = dW_t,$$

W_t being the Wiener process.

Eq.(1) is a *specification* of a problem whereas (2) and (3) are *solutions* in the sense that they both provide algorithmic means to construct a function satisfying the specification. An important condition for (2) and (3) to be considered as solutions is the fact that the algorithmic tools are independent of the particular solution, in the first case an integration procedure and in the second the simulation of a solution-independent process. This should be contrasted with stochastic processes constructed from a given particular solution, as has been done for example for the Boltzman equation [4].

In contrast with the linear problems, for nonlinear partial differential equations, explicit solutions in terms of elementary functions or integrals are only known in very particular cases. However, if a solution-independent stochastic process is found that (for arbitrary initial conditions) generates the solution in the sense of Eq.(3), a stochastic solution is obtained. In this way the set of equations for which exact solutions are known would be considerably extended.

The stochastic representations recently constructed for the Navier-Stokes equations ([5], [6], [7], [8]) and for the Vlasov-Poisson equations ([9], [10]) define solution-independent processes for which the mean values of some functionals are solutions to these equations. Therefore, they are exact *stochastic solutions*.

In the stochastic solutions one deals with a process that starts from the point where the solution is to be found, a functional being then computed along the whole sample path or until it reaches a boundary. In all cases one needs to average over many independent sample paths to obtain a expectation value of the functional. The localized and parallelizable nature of the solution construction is clear. Provided some differentiability conditions are satisfied, the process also handles equally well simple or very complex boundary conditions.

The stochastic solutions also provide an intuitive characterization of the physical phenomena, relating nonlinear interactions with cascading processes. By the study of exit times from a domain they also sometimes provide access to quantities that cannot be obtained by perturbative methods, see [11].

One way to construct stochastic solutions is based on a probabilistic interpretation of the Picard series. The differential equation is written as an integral equation which is rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series. This method is used in this paper to obtain a stochastic solution of a nonlinear partial differential equation, which is a fractional version of the Kolmogorov-Petrovskii-Piskunov (KPP) equation, see [12].

2. A fractional nonlinear partial differential equation

We consider the following equation

$$
{}_tD_*^\alpha u(t,x) = \frac{1}{2}\, {}_xD_\theta^\beta u(t,x) + u^2(t,x) - u(t,x).
\tag{4}
$$

We use the same notations as in the study of the linear problem in [13]. here ${}_tD_*^\alpha$ is a Caputo derivative of order α,

$$
{}_tD_*^\alpha f(t) =
\begin{cases}
\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-m}} & m-1 < \alpha < m, \\[2ex]
\dfrac{d^m}{dt^m} f(t) & \alpha = m,
\end{cases}
\tag{5}
$$

with m integer, and ${}_xD_\theta^\beta$ is a Riesz-Feller derivative defined through its Fourier symbol by

$$
\mathcal{F}\left\{{}_xD_\theta^\beta f(x)\right\}(k) = -\psi_\beta^\theta(k)\,\mathcal{F}\{f(x)\}(k),
\tag{6}
$$

with $\psi_\beta^\theta(k) = |k|^\beta\, e^{i(\mathrm{sign}k)\theta\pi/2}$.

Eq. (4) is a fractional version of the KPP equation, studied by probabilistic means by McKean [14]. Physically it describes a nonlinear diffusion with growing mass and in our fractional generalization it would represent the same phenomenon taking into account memory effects in time and long range correlations in space.

As outlined in the introduction, the first step towards a probabilistic formulation is the rewriting of Eq. (4) as an integral equation including the initial conditions. For this purpose we take the Fourier transform ($\mathcal{F}$) in space and the Laplace transform ($\mathcal{L}$) in time obtaining

$$
\begin{aligned}
s^{\alpha}\widetilde{\widehat{u}}(s,k)(s,k) \ = \ & s^{\alpha-1}\widehat{u}\left(0^{+},k\right) - \frac{1}{2}\psi_{\beta}^{\theta}(k)\widetilde{\widehat{u}}(s,k) - \widetilde{\widehat{u}}(s,k) \\
& + \int_{0}^{\infty} dt\, e^{-st}\mathcal{F}\left(u^{2}(t,x)\right),
\end{aligned} \tag{7}
$$

where

$$
\widehat{u}(t,k) = \mathcal{F}(u(t,x)) = \int_{-\infty}^{\infty} e^{ikx}u(t,x)\,dx,
$$

$$
\widetilde{u}(s,x) = \mathcal{L}(u(t,x)) = \int_{0}^{\infty} e^{-st}u(t,x)\,dt.
$$

This equation holds for $0 < \alpha \le 1$ or for $0 < \alpha \le 2$ with $\frac{\partial}{\partial t}u\left(0^{+},x\right) = 0$. Solving for $\widetilde{\widehat{u}}(s,k)$ one obtains an integral equation

$$
\widetilde{\widehat{u}}(s,k) = \frac{s^{\alpha-1}}{s^{\alpha}+1+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\widehat{u}\left(0^{+},k\right) + \int_{0}^{\infty} dt\frac{e^{-st}}{s^{\alpha}+1+\frac{1}{2}\psi_{\beta}^{\theta}(k)}\mathcal{F}\left(u^{2}(t,x)\right). \tag{8}
$$

Taking the inverse Fourier and Laplace transforms (see e.g. [15]), we have

$$
u(t,x) = E_{\alpha,1}\left(-t^{\alpha}\right)\int_{-\infty}^{\infty} dy\,\mathcal{F}^{-1}\left(\frac{E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)t^{\alpha}\right)}{E_{\alpha,1}\left(-t^{\alpha}\right)}\right)
$$

$$
\times\ (x-y)\,u\left(0^{+},y\right) + \int_{0}^{t} d\tau\,(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)
$$

$$
\int_{-\infty}^{\infty} dy\,\mathcal{F}^{-1}\left(\frac{E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)(t-\tau)^{\alpha}\right)}{E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)}\right)(x-y)\,u^{2}(\tau,y), \tag{9}
$$

where $E_{\beta,\rho}$ is the generalized Mittag-Leffler function (see e.g. [15])

$$
E_{\alpha,\rho}(z) = \sum_{j=0}^{\infty}\frac{z^{j}}{\Gamma(\alpha j+\rho)}.
$$

We define the following propagation kernel

$$
G_{\alpha,\rho}^{\beta}(t,x) = \mathcal{F}^{-1}\left(\frac{E_{\alpha,\rho}\left(-\left(1+\frac{1}{2}\psi_{\beta}^{\theta}(k)\right)t^{\alpha}\right)}{E_{\alpha,\rho}\left(-t^{\alpha}\right)}\right)(x) \tag{10}
$$

and from the normalization relation,

$$E_{\alpha,1}\left(-t^{\alpha}\right) + \int_0^t d\tau \, (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right) = 1,$$

we may interpret $E_{\alpha,1}\left(-t^{\alpha}\right)$ and $(t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-(t-\tau)^{\alpha}\right)$, respectively as a survival probability up to time t and as the probability density for the branching at time τ in a branching process B_{α}. It is a fractional generalization of an exponential process. This provides a probabilistic sampling of the Picard series obtained by iteration of Eq. (9). The solution is therefore obtained by the expectation of the exit values of the following process:

Starting at time zero, a particle lives according to the process B_{α}. At the branching time τ the initial particle dies and two new particles are born at the dying point. The process continues in the same way with independent evolution of each one of the newborn particles. At time t the solution is obtained as a functional of the n existing particles at time t, namely as the product of the initial condition propagated from the point where each one of the n particles is at time t up to the initial position:

$$u(t,x) = \mathbb{E}_x\left(\varphi_1 \varphi_2 \cdots \varphi_n\right), \tag{11}$$

with

$$\begin{aligned}
\varphi_i &= \int dy_1^{(i)} dy_2^{(i)} \cdots dy_{k-1}^{(i)} dy_k^{(i)} G_{\alpha,\alpha}^{\beta}\left(\tau_1, x - y_1^{(i)}\right) G_{\alpha,\alpha}^{\beta}\left(\tau_2, y_1^{(i)} - y_2^{(i)}\right) \cdots \\
&\quad \cdots G_{\alpha,\alpha}^{\beta}\left(\tau_{k-1}, y_{k-2}^{(i)} - y_{k-1}^{(i)}\right) G_{\alpha,1}^{\beta}\left(\tau_k, y_{k-1}^{(i)} - y_k^{(i)}\right) u\left(0^+, y_k^{(i)}\right), \tag{12}
\end{aligned}$$

with $\sum_{j=1}^k \tau_j = t$, $k-1$ being the number of branchings leading to particle i. Notice that the last propagator in (12) is different from the others.

Because of the normalization of the probabilities in the process B_{α}, the probability of each one of the products in (11) corresponds to the weight of the corresponding term in the Picard series. Therefore the expectation value exists whenever the Picard series converges.

The solution (11) is not yet a purely stochastic solution because it involves both the expectation value over the process B_{α} and a multiple integration of the initial condition with the propagation kernels $G_{\alpha,1}^{\beta}$ and $G_{\alpha,\alpha}^{\beta}$. To obtain a purely stochastic solution we notice that, for $0 < \alpha \leq 1$, the propagation kernels satisfy the conditions to be the Green's functions of stochastic processes in $\mathbb{R}$ (see the Appendix).

We denote the processes associated to $G_{\alpha,1}^{\beta}(t,x)$ and $G_{\alpha,\alpha}^{\beta}(t,x)$, respectively by $\Pi_{\alpha,1}^{\beta}$ and $\Pi_{\alpha,\alpha}^{\beta}$. Therefore the process leading to the solution is as described before with all the particles until the last branching propagating

according to the process $\Pi_{\alpha,\alpha}^{\beta}$ and the last ones (that sample the initial condition) propagating by the process $\Pi_{\alpha,1}^{\beta}$. At time t the particle coordinates $x + \xi_i$ are recorded and the solution is given by

$$u(t, x) = \mathbb{E}_x \left(u(0^+, x + \xi_1) u(0^+, x + \xi_2) \cdots u(0^+, x + \xi_n) \right). \qquad (13)$$

Eq. (13) is a stochastic solution of (4) and our main result is summarized as follows:

THEOREM. *The nonlinear fractional partial differential equation (4), with $0 < \alpha \le 1$, has a stochastic solution given by (13), the coordinates $x + \xi_i$ in the arguments of the initial condition obtained from the exit values of a propagation and branching process, the branching being ruled by the process B_α and the propagation by $\Pi_{\alpha,1}^{\beta}$ for the particles that reach time t and by $\Pi_{\alpha,\alpha}^{\beta}$ for all the remaining ones.*

A sufficient condition for the existence of the solution is

$$\left| u(0^+, x) \right| \le 1. \qquad (14)$$

REMARKS:

1) The condition $|u(0^+, x)| \le 1$ imposes a finite value for all contributions to the multiplicative functional. However, the solution may exist under more general conditions, namely when the decreasing value of the probability of higher order products in (13) compensates the growth of the powers of the initial condition.

2) The stochastic solution may also be constructed by a backwards-in-time stochastic process from time t to time zero. This is obtained by rewriting Eq. (9) as

$$u(t, x) = E_{\alpha,1} \left(-t^\alpha \right) \int_{-\infty}^{\infty} dy \mathcal{F}^{-1} \left(\frac{E_{\alpha,1} \left(-\left(1 + \frac{1}{2} \psi_\beta^\theta (k) \right) t^\alpha \right)}{E_{\alpha,1} \left(-t^\alpha \right)} \right)$$

$$\times (x - y) u \left(0^+, y \right) + \int_0^t d\tau \tau^{\alpha-1} E_{\alpha,\alpha} \left(-\tau^\alpha \right)$$

$$\int_{-\infty}^{\infty} dy \mathcal{F}^{-1} \left(\frac{E_{\alpha,\alpha} \left(-\left(1 + \frac{1}{2} \psi_\beta^\theta (k) \right) \tau^\alpha \right)}{E_{\alpha,\alpha} \left(-\tau^\alpha \right)} \right) (x - y) u^2 (t - \tau, y), \qquad (15)$$

and noticing also that

$$E_{\alpha,1} \left(-t^\alpha \right) + \int_0^t d\tau \tau^{\alpha-1} E_{\alpha,\alpha} \left(-\tau^\alpha \right) = 1.$$

Then, we obtain the following stochastic construction of the solution:

Starting at time t a particle propagates backwards in time according to the process $\Pi^{\beta}_{\alpha,1}$ if it reaches time zero or according to $\Pi^{\beta}_{\alpha,\alpha}$ if it branches at time $t - \tau$. The branching probability is controlled by the process B_{α} (that is, the branching probability density is $\tau^{\alpha-1}E_{\alpha,\alpha}(-\tau^{\alpha})$). When it branches, two new particles are born which propagate independently and the process is repeated until all surviving particles reach time zero.

Appendix: The Green's functions
and the characterization of the processes

The processes $\Pi^{\beta}_{\alpha,1}$ *and* $\Pi^{\beta}_{\alpha,\alpha}$:

$$\mathcal{F}\left\{G^{\beta}_{\alpha,1}(t,x)\right\}(t,k) = \frac{E_{\alpha,1}\left(-\left(1 + \frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)}{E_{\alpha,1}(-t^{\alpha})}, \qquad (16)$$

$$\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}(t,x)\right\}(t,k) = \frac{E_{\alpha,\alpha}\left(-\left(1 + \frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)}{E_{\alpha,\alpha}(-t^{\alpha})}. \qquad (17)$$

For a propagation kernel $G(t,x)$ to be the Green function of a stochastic process, the following conditions should be satisfied:

(i) $G(0, x - y) = \delta(x - y)$ or $\mathcal{F}\{G\}(0, k) = 1 \; \forall k$;

(ii) $\int dx G(t,x) = 1 \; \forall t$ or $\mathcal{F}\{G\}(t,0) = 1$;

(iii) $G(t,x)$ should be real and ≥ 0.

For the processes $\Pi^{\beta}_{\alpha,1}$ and $\Pi^{\beta}_{\alpha,\alpha}$:

(i) $\mathcal{F}\left\{G^{\beta}_{\alpha,1}\right\}(0,k) = \frac{E_{\alpha,1}(0)}{E_{\alpha,1}(0)} = 1$ and $\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}\right\}(0,k) = \frac{E_{\alpha,\alpha}(0)}{E_{\alpha,\alpha}(0)} = 1$;

(ii) $\mathcal{F}\left\{G^{\beta}_{\alpha,1}\right\}(t,0) = \frac{E_{\alpha,1}(-t^{\alpha})}{E_{\alpha,1}(-t^{\alpha})} = 1$ and $\mathcal{F}\left\{G^{\beta}_{\alpha,\alpha}\right\}(t,0) = \frac{E_{\alpha,\alpha}(-t^{\alpha})}{E_{\alpha,\alpha}(-t^{\alpha})} = 1$;

(iii) If $\mathcal{F}\{G\}(t,-k) = (\mathcal{F}\{G\}(t,k))^{*}$, then $G(t,x)$ is real.

Because $\psi^{\theta}_{\beta}(-k) = \left(\psi^{\theta}_{\beta}(k)\right)^{*}$, it follows

$$E_{\alpha,1}\left(-\left(1 + \frac{1}{2}\psi^{\theta}_{\beta}(-k)\right)t^{\alpha}\right) = \left(E_{\alpha,1}\left(-\left(1 + \frac{1}{2}\psi^{\theta}_{\beta}(k)\right)t^{\alpha}\right)\right)^{*}$$

and

$$E_{\alpha,\alpha}\left(-\left(1+\frac{1}{2}\psi_\beta^\theta\left(-k\right)\right)t^\alpha\right) = \left(E_{\alpha,1}\left(-\left(1+\frac{1}{2}\psi_\beta^\theta\left(k\right)\right)t^\alpha\right)\right)^*$$

implying that both $G_{\alpha,1}^\beta\left(t,x\right)$ and $G_{\alpha,\alpha}^\beta\left(t,x\right)$ are real.

Finally, for the positivity, one notices that for $0 < \alpha \leq 1$ and $\rho \geq \alpha$, $E_{\alpha,\rho}\left(-x\right)$ is a completely monotone function, see [16]. Therefore,

$$E_{\alpha,\rho}\left(-x\right) = \int_0^\infty e^{-rx}dF\left(r\right),$$

with F nondecreasing and bounded.

For $G_{\alpha,\rho}^\beta\left(t,x\right)$ ($\rho = 1$ and $\rho = \alpha$) one has

$$
\begin{aligned}
G_{\alpha,\rho}^\beta\left(t,x\right) &= \frac{1}{2\pi E_{\alpha,\rho}\left(-t^\alpha\right)}\int_0^\infty dF\left(r\right)\int_{-\infty}^\infty dk\, e^{-ikx}e^{-rt^\alpha\left(1+\frac{1}{2}\psi_\beta^\theta\left(-k\right)\right)} \\
&= \frac{1}{2\pi E_{\alpha,\rho}\left(-t^\alpha\right)}\int_0^\infty dF\left(r\right)e^{-rt^\alpha}\int_{-\infty}^\infty dk\, e^{-ikx}e^{-\frac{rt^\alpha}{2}\psi_\beta^\theta\left(-k\right)}.
\end{aligned}
$$

We recognize the last integral (in k) as the Green's function of a Levy process. Therefore one has an integral in r of positive quantities implying that $G_{\alpha,1}^\beta\left(t,x\right)$ and $G_{\alpha,\alpha}^\beta\left(t,x\right)$ are positive.

The process B_α:
The decaying probability in time $d\tau$ of this process is

$$\tau^{\alpha-1}E_{\alpha,\alpha}\left(-\tau^\alpha\right).$$

From

$$\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\tau^\alpha\right)d\tau = 1 - E_{\alpha,1}\left(-t^\alpha\right)$$

it follows that $E_{\alpha,1}\left(-t^\alpha\right)$ is the survival probability up to time t. The process B_α is a fractional generalization of the exponential process.

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