Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Physica D 214 (2006) 182–186
PHYSICA D
www.elsevier.com/locate/physd

Deterministic Bak–Sneppen model: Lyapunov spectrum and avalanches as return times

R. Vilela Mendes*

Centro de Matemática e Aplicações Fundamentais, Complexo Interdisciplinar, Universidade de Lisboa, Av. Gama Pinto, 2-1649-003 Lisboa, Portugal
Centro de Fusão Nuclear, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

Received 21 October 2004; received in revised form 9 December 2005; accepted 11 January 2006

Communicated by C.K.R.T. Jones

Abstract

A deterministic version of the Bak–Sneppen model is studied. The role of the Lyapunov spectrum in the onset of scale-free behavior is established, as well as the measure-theoretic nature of the Bak–Sneppen self-organized state. Avalanches are interpreted as return times to a small measure set and the problem of accurate determination of the scaling exponents near the critical barrier is addressed.

Keywords: Bak–Sneppen model; Lyapunov spectrum; Self-organized criticality

1. Introduction

In Ref. [1] several structure-generating mechanisms, in multi-agent dynamical systems, have been analyzed. A *structure* in a dynamical system is characterized by the emergence of phenomena at scales different from those typical of the dynamics of the agents, when they evolve in isolation. A *space-structure* corresponds to phenomena operating at a scale much larger than the typical size of one agent and a *time-structure* is a phenomenon with a time scale much longer than the cycle time of the agents.

An important role, in the generation of some types of time-structures, is played by modifications to the Lyapunov spectrum, occurring either as a result of the agents' interaction or of a change of parameters. A new structure is created each time one Lyapunov exponent crosses zero from above.[1] In particular, in one of the mechanisms described in [1], the joint effect of chaos in the individual agent evolution law and extremal dynamics implies that all Lyapunov exponents tend to 0^+ in the infinite agents' limit. This leads to a characteristic behavior free of time scales. It has been suggested in Ref. [1] that it is this mechanism that is behind many of the dynamical manifestations of what has been called self-organized criticality (SOC). A similar behavior has been established in [2] for the infinite size limit of the Zhang model. Here, this research is pursued by analyzing a situation [3] where breakdown of self-organized criticality is exhibited. It is shown that the breakdown of SOC may in fact be understood from the behavior of the Lyapunov spectrum.

Associated with the idea of self-organized criticality (SOC) several phenomena have been studied. Here, only the absence of natural time scales is emphasized, without much concern about space scales, nature of the driving, separation of time scales and other relevant issues, useful for a precise characterization of SOC. Also I will use as a basic prototype of SOC the Bak–Sneppen model [4]. In this, as in other SOC models, starting from a randomly initialized configuration, the system is found to self-organize close to a characteristic state, called the SOC state. In the Bak–Sneppen model this is the state with all the agents' coordinates above a critical barrier ($b \simeq 0.667$ for

* Corresponding address: Complexo Interdisciplinar, University of Lisbon, Av. Gama Pinto 2, 1649-003 Lisboa, Portugal. Tel.: +351 914739442; fax: +351 217954288.
E-mail address: vilela@cii.fc.ul.pt.

[1] There are also other mechanisms for the creation of structures in complex systems which do not depend on the existence of positive Lyapunov exponents. For examples see the coupled map minority model and the market-like game in Ref. [1].

0167-2789/$ - see front matter
doi:10.1016/j.physd.2006.01.004

the one-dimensional model). Subsequently the system displays excursions away from the SOC state returning to it after an interval of time. These excursions are called *avalanches*. The probability distribution of the avalanche sizes (return times to the self-organized state) displays a non-exponential fall-off, suggesting the absence of a natural time scale for these systems.

Absence of a time scale, as a property of dynamics, is naturally related to the Lyapunov spectrum. Time scales disappear whenever the Lyapunov exponents vanish. This may lead to non-exponential fall-offs for the time correlations. However, scaling of the avalanches, that is, a power law for the return times to the self-organized return set, is a subtler effect. This is easy to understand when the multi-agent system is formulated as a measure-preserving dynamical system. If the return set A, which serves as reference for the counting of return times (k), has non-zero measure ($\mu(A) \neq 0$), the measure $\mu(A)$ itself serves as a natural time scale. Then, the large time behavior of the return times distribution would be dominated by an exponential factor $\exp(-k\nu(\mu))$ [5,6]. Therefore, for cases that fit in the ergodic dynamical systems setting, power laws may occur only if the return set has vanishing measure. In the $\mu(A) \to 0$ limit the return times distribution is dominated by the prefactor that multiplies the exponential and this one might or might not be a power law.

When μ is an ergodic measure, by Kac's lemma, the mean return time to a set A is $1/\mu(A)$. Therefore when $\mu(A) = 0$, the numerical evaluation of the return time (avalanche) law has to be carried out for a set slightly larger than A. This implies that it may be difficult to disentangle the prefactor dependence from the exponential one, leading to some uncertainty about the exact values of the numerically measured scaling exponents. In particular because, as seen in Section 3 the exponent factor $\nu(\mu)$ may be a non-trivial function of the measure.

The paper is organized as follows. In Section 2 a deterministic version of the Bak–Sneppen model is studied to put into evidence the role of the vanishing Lyapunov exponents in the removal of a natural time scale for this system. The zero measure nature of the self-organized return set is also established. Finally, in Section 3, one discusses the problems involved in a precise determination of the scaling exponents.

2. The deterministic Bak–Sneppen model. The role of the Lyapunov exponents and the measure of the return set

The original Bak–Sneppen model [4,7] may be converted into a deterministic dynamical system by defining

$$x_i(t+1) = \Gamma_i(\underset{\sim}{x})x_i(t) + (1-\Gamma_i(\underset{\sim}{x}))f(x_i(t)) \tag{1}$$

where $\underset{\sim}{x} = \{x_i\}$ is the vector of agent coordinates and f a deterministic pseudo-random generator, for example

$$f(x_i) = kx \quad \text{mod}.1$$

$k = 2, 3, \ldots$. $\Gamma_i(\underset{\sim}{x})$ is a function which is nearly zero if i corresponds to the agent with the minimum x_i value or to one of its $2n_v$ neighbors and is nearly one otherwise. In Ref. [1] the following function was proposed[2]

$$\Gamma_i^{(1)}(\underset{\sim}{x}) = \prod_{j=i-n_V}^{j=i+n_V}\left(1 - \prod_{l\neq j}(1+\mathrm{e}^{-\alpha(x_l-x_j)})^{-1}\right) \tag{2}$$

which, for large α, satisfies the above conditions.

Here, instead, the function

$$\Gamma_i^{(2)}(\underset{\sim}{x}) = \prod_{j=i-n_V}^{j=i+n_V}\left(1 - \frac{\mathrm{e}^{-x_j/T}}{\sum_{l=1}^{N}\mathrm{e}^{-x_l/T}}\right) \tag{3}$$

will be used, which has a similar behavior for $T \to 0^+$. Using a similar function in a stochastic model Head [3] has shown the breakdown of scale-free behavior for finite non-zero T. By using $\Gamma_i^{(2)}(\underset{\sim}{x})$ in the deterministic model one may compare the Lyapunov spectrum analysis with the numerical results of Head. In the $T \to 0$ limit both models are equivalent to the original Bak–Sneppen model.

The deterministic system (1)–(3) may have many invariant measures. Here I assume that when subjected to random perturbations (like in the numerical calculations) it has a unique (physical) measure. It is to the zero-noise limit of this measure that I refer when speaking about the "invariant measure of the deterministic Bak–Sneppen model".

The Lyapunov spectrum of the model defined by (1) and (3) cannot be computed exactly for $T \neq 0$. However it is easy to verify that the Lyapunov exponents should vanish in the $N \to \infty, T \to 0$. For $T = 0$ the Jacobian of the dynamics is a diagonal matrix with ones in the diagonal except for $2n_V + 1$ elements equal to k. Let N be the number of agents. When the successive Jacobian matrices are multiplied, one obtains, in the limit of large n, a diagonal matrix with diagonal elements $\frac{2n_V+1}{N}n$. Then,

$$\lambda = \lim_{n\to\infty} \ln\left(k^{\frac{(2n_V+1)n}{N}}\right)^{\frac{1}{n}} = \frac{2n_V+1}{N}\ln k \tag{4}$$

which tends to 0^+ in the infinite number of agents' limit.

For small (non-vanishing) T, the leading corrections to the Jacobian will be of order

$$\beta_i = \frac{1}{T}\exp(-(x_i - x_{\min})/T) \tag{5}$$

$x_{\min}$ being, at each time, the smaller coordinate. Keeping only terms of this order one has

[2] Notice that in Ref. [1] there is a typing mistake in the definition of this function.

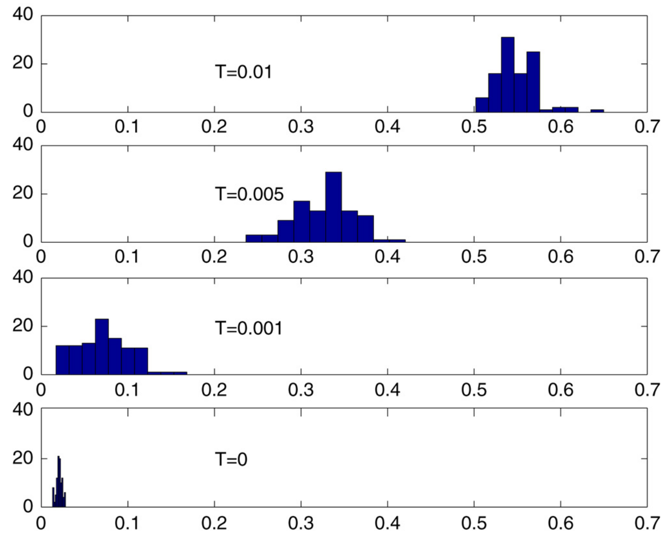

Fig. 1. Approximate Lyapunov spectrum for several T values in the deterministic Bak–Sneppen model.

$$
\begin{aligned}
\frac{\partial x_{\min}(t+1)}{\partial x_{\min}(t)} &\simeq f'(x_{\min}) - (f(x_{\min}) - x_{\min}) \sum_{j\neq\min} \beta_j \\
\frac{\partial x_{\min}(t+1)}{\partial x_j(t)} &\simeq (f(x_{\min}) - x_{\min})\beta_j \\
\frac{\partial x_V(t+1)}{\partial x_V(t)} &\simeq f'(x_V)(f(x_V) - x_V)\beta_V \\
\frac{\partial x_V(t+1)}{\partial x_{\min}(t)} &\simeq -(f(x_V) - x_V) \sum_{j\neq\min} \beta_j \\
\frac{\partial x_V(t+1)}{\partial x_j(t)} &\simeq (f(x_V) - x_V)\beta_j \\
\frac{\partial x_j(t+1)}{\partial x_j(t)} &\simeq 1 - (f(x_j) - x_j)\beta_j \\
\frac{\partial x_j(t+1)}{\partial x_r(t)} &\simeq \begin{cases} -(f(x_j) - x_j)\beta_r & r = j+1, j-1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{6}
$$

$x_{\min}$ denotes the minimum coordinate, x_V the coordinate of one of its neighbors and x_j a generic element. It is clear from (6) that the 0^+ limit at $T = 0$ will be disturbed. However, because of the dependence on the factors $(f(x_j) - x_j)\beta_j$ it is not obvious in which way the Lyapunov spectrum is going to change.

To quantitatively illustrate the effect, a numerical calculation of the evolution of the Lyapunov spectrum is made for successively decreasing T values, with N agents. The results, shown in Fig. 1, were obtained for $N = 100$, $2n_V = 2$ and $k = 2$. The analytical expression of the tangent map is obtained from (1) to (3) and the Lyapunov exponent obtained by multiplication of the tangent map along a numerically computed trajectory. The spread in the $T = 0$ limit, around the exact value for N agents ($\frac{3}{N} \ln 2$ in this case), results from the finite number of time steps. The deterministic system defined by (1)–(3) closely corresponds to the stochastic system studied in Ref. [3]. Therefore the behavior of the Lyapunov spectrum explains why the distribution of jump sizes and activation times obeys power laws only in the $T \to 0$ limit. In Ref. [3] the loss of criticality for finite T is interpreted in terms of the correlated or uncorrelated nature of the active site jumps. The Lyapunov spectrum interpretation seems to be clearer. Notice also that it is only in the $N \to \infty$ limit that all the Lyapunov exponents reach 0^+. It is only in this limit that all time scales disappear.

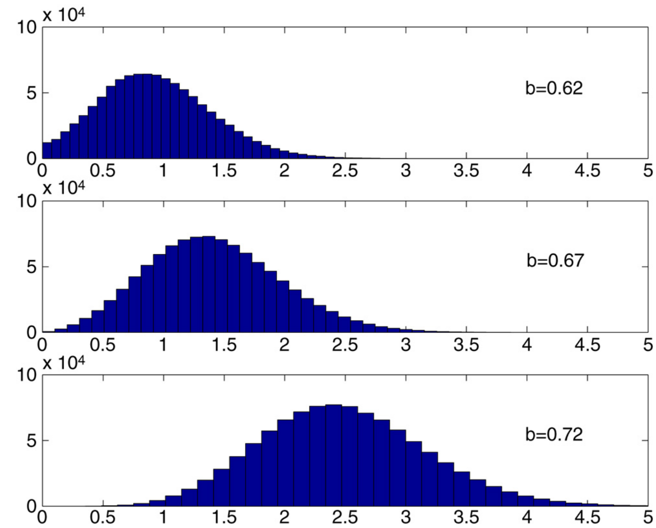

Fig. 2. Probability density for the distance process.

One of the most remarkable features of the Bak–Sneppen model is the existence of a sharp probability distribution for the agents' coordinates above a barrier $b \simeq 0.667$. However, this sharp distribution is merely a one-agent marginal, which is the one-dimensional projection of the support of the invariant measure in the N-dimensional space. In the N-dimensional space the self-organized return set to which the system returns after each avalanche has zero measure. This is consistent with Kac's lemma. Kac's lemma states that for an ergodic measure μ the average return time to a set A of measure $\mu(A)$ is $1/\mu(A)$. Therefore for a return time probability distribution scaling like $p(k) \sim 1/k^\tau$, $\tau \leq 2$ implies $\mu(A) \to 0$.

To exhibit the zero-measure nature of the Bak–Sneppen return set consider the *distance process* d defined by

$$
d = \sum_i \max(b - x_i, 0). \tag{7}
$$

Fig. 2 shows the probability distribution of the distance process for several values of the barrier b. One sees that it is for b around 0.67 that the neighborhood of the $d = 0$ point becomes of zero measure.

The Bak–Sneppen self-organized return set is an N-dimensional hypercube of volume $(1 - 0.667)^N$. This set has repelling directions corresponding to the agents that are active and neutral directions for all others. Not being an invariant set it falls outside the usual definition of "weak repeller". It has been called a "ghost weak repeller" in Ref. [1].

3. Measuring the scaling exponents

As mentioned before, the zero measure of the "repeller" makes the direct measurement of the distribution law of avalanches a delicate matter. On the one hand the barrier value defining the avalanches must be placed close to the critical barrier to avoid the exponential finite-measure effects. However because the average size of the avalanches is $1/\mu(A)$, the closer we are to the critical barrier the worse the statistics becomes. A bin size has to be chosen to fit the avalanche's data to a power law and the accuracy of the exponent is found to be sensitive to the bin size.

Here one analyzes the nature of the probability distribution of the avalanches using a construction of the characteristic function,

$$C(x) = \langle \mathrm{e}^{\mathrm{i}kx} \rangle \tag{8}$$

from the data. The probability density may then be obtained from the computation of the inverse Fourier transform

$$p(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x. \tag{9}$$

The merit of using a characteristic function approach lies in:

(i) The fact that all the data is used for each k, instead of only the events in the neighborhood of k, allows for easier extrapolations to high k.

(ii) It allows the direct comparison of several functional forms with the data.

In our case we are not only interested in an accurate determination of $p(k)$ but also on the extraction of the power law prefactor that multiplies the measure-dependent exponential. In cases where the exponent of the exponential is also a non-trivial function of the measure it is difficult to separate the effect of the two terms. This is especially true if the measure-dependent exponent decreases with the measure (see the example below).

More accurate statements, concerning the nature of the prefactors, are obtained by comparing the characteristic function constructed from the data with the characteristic functions for trial distributions

$$p(k, \mu) = ck^{-\alpha(\mu)} \mathrm{e}^{-\nu_\alpha(\mu)k} \tag{10}$$

with c and $\nu_\alpha(\mu)$ obtained from normalization and Kac's lemma, $\langle k \rangle = \frac{1}{\mu}$.

For general distributions of the form

$$p_g(k, \mu) \sim g(k) \mathrm{e}^{-k\nu(\mu)}$$

the characteristic function must satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}\nu} C(x, \mu) = \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}x} C(x, \mu) + \frac{1}{\mu} C(x, \mu)$$

with solution

$$C(x, \mu) = \exp\left(\int_{\nu_0}^{\nu} \frac{\mathrm{d}\tau}{\mu(\tau)} - \int_{\nu_0}^{\nu - \mathrm{i}x} \frac{\mathrm{d}\tau}{\mu(\tau)} \right).$$

Therefore a large family of solutions is obtained simply by specifying the function $\nu(\mu)$ (and its inverse $\mu(\nu)$).

A particular case is

$$p_\alpha(k, \mu) \sim k^{-\alpha} \mathrm{e}^{-k\nu_\alpha(\mu)}.$$

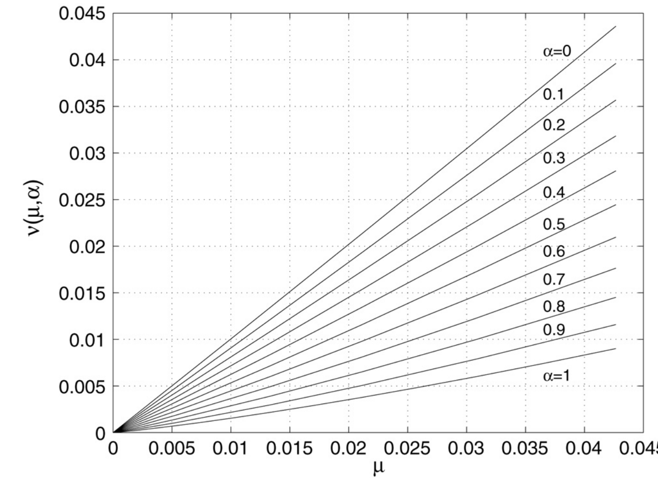

Fig. 3. Exponential argument $\nu_\alpha(\mu)$ for the probability densities $p_\alpha(k, \mu) \sim k^{-\alpha(\mu)} \exp(-k\nu_\alpha(\mu))$.

For $\alpha = 0$ this is the geometric distribution with

$$\nu_\alpha(\mu) = -\log(1 - \mu)$$

and for $\alpha = 1$

$$C_1(x, \mu) = \frac{\log(1 - \exp(\mathrm{i}x - \nu))}{\log(1 - \exp(-\nu))}$$

with

$$\mu = (1 - \mathrm{e}^{\nu}) \log(1 - \mathrm{e}^{-\nu}).$$

The functions $\nu_\alpha(\mu)$ for other values of α are plotted in Fig. 3.

The characteristic functions obtained by numerical simulations of the B–S model (in the $T = 0$ limit) were fitted to the trial distribution (10) for several values of the measure. The results are displayed in Fig. 4 (for $2n_v = 2$). Both the exponential factor ν and the scaling exponent α depend on the measure of the avalanche return set.

This puts into evidence the unavoidable uncertainties in the direct evaluation of the scaling exponent, in particular because even very close to the critical barrier the scaling exponent still shows a noticeable dependence on the measure of the return set.

The discussion above refers to the problem of direct determination of the scaling exponents. One sees that without further assumptions the accurate determination of the scaling exponents is delicate, in particular because of the μ dependence of the function $\nu(\mu)$.

With the additional assumption, as done by several authors, of a scaling form for $p(k)$ near the critical barrier, an estimate of the $\mu \to 0$ value may be obtained. This however relies on the validity of the scaling assumption. Assuming that close to $\mu = 0$[3]

$$p(k, \mu) = k^{-\alpha} f(k^s \mu) \tag{11}$$

[3] The scaling functions used in the past to obtain the scaling exponents have in general a dependence on the size of the system. Here that dependence is implicit on the measure because, as seen before, it is only in the $N \to \infty$ limit that the Lyapunov exponents vanish and the return set becomes a zero-measure set.

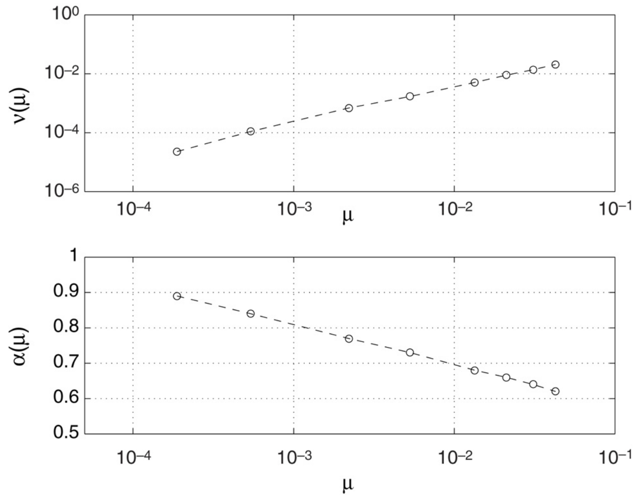

Fig. 4. The exponential argument $\nu(\mu)$ and scaling exponent $\alpha(\mu)$.

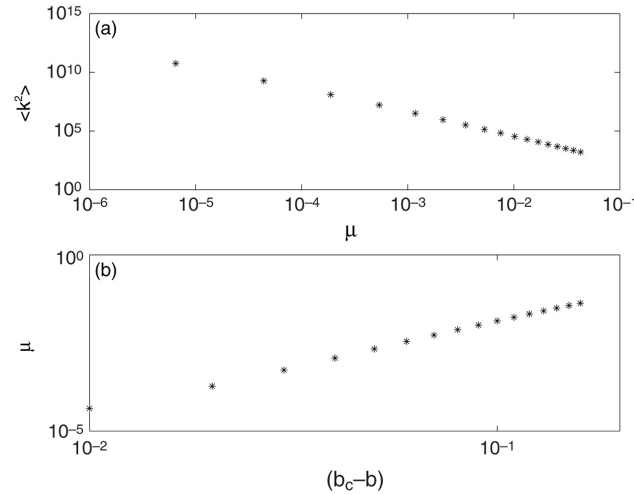

Fig. 5. Data points used to infer the exponents in $\langle k^2 \rangle \sim \mu^{\frac{\alpha-3}{s}}$ and $\mu \sim (b_c - b)^\eta$.

one obtains

$$\langle k \rangle \sim \mu^{\frac{\alpha-2}{s}}$$
$$\langle k^2 \rangle \sim \mu^{\frac{\alpha-3}{s}}. \tag{12}$$

Then, from Kac's lemma, one obtains

$$s = 2 - \alpha$$

and from the numerical data for $2n_v = 2$ (Fig. 5(a)), $\frac{\alpha-3}{s} \simeq 2.07$, leading to $\alpha \simeq 1.067$, $s \simeq 0.93$.

Another exponent relates the measure to the position of the barrier

$$\mu \sim (b_c - b)^\eta$$

with (Fig. 5(b)) $\eta \simeq 2.55$.

4. Conclusions

(i) The deterministic Bak–Sneppen model, studied in this paper, establishes a clear relation between the Lyapunov spectrum and scale-free behavior.

Extremal behavior, where only a finite number of agents is active at each time, leads to the vanishing of all Lyapunov exponents in the infinite agents' limit. On the other hand, local chaos insures that the convergence to zero is from above. These seem to be the most common ingredients in systems that display what has been called self-organized criticality.

(ii) The self-organized return set in the deterministic Bak–Sneppen model is an interesting zero-measure set. Interpreting avalanches as return times to a set A, one concludes that a scaling law with exponent ≤ 2 implies, by Kac's lemma, $\mu(A) \to 0$. Therefore one expects the zero measure feature to be present not only in the Bak–Sneppen model but also in other self-organized states with avalanche scaling laws.

References

[1] R. Vilela Mendes, Physica A 295 (2001) 537.
[2] B. Cessac, Ph. Blanchard, T. Krüger, J.L. Meunier, J. Stat. Phys. 115 (2004) 1283.
[3] D. Head, Eur. Phys. J. B 17 (2000) 289.
[4] P. Bak, K. Sneppen, Phys. Rev. Lett. 71 (1993) 4083.
[5] M. Hirata, B. Saussol, S. Vaienti, Commun. Math. Phys. 206 (1999) 33.
[6] P. Collet, Ergodic Theory Dynam. Systems 21 (2001) 401.
[7] H. Flyvbjerg, K. Sneppen, P. Bak, Phys. Rev. Lett. 71 (1993) 4087.