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Conditional exponents, entropies and a measure of dynamical self-organization

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Abstract

In dynamical systems composed of interacting parts, conditional exponents, conditional exponent entropies and cylindrical entropies are shown to be well-defined ergodic invariants which characterize the dynamical self-organization and statistical independence of the constituent parts. An example of interacting Bernoulli units is used to illustrate the nature of these invariants. © 1998 Elsevier Science B.V.

1. Conditional exponents

The notion of conditional Lyapunov exponents (originally called sub-Lyapunov exponents) was introduced by Pecora and Carroll in their study of synchronization of chaotic systems [1,2]. It turns out, as I will show below, that, like the full Lyapunov exponent, the conditional exponents are well defined ergodic invariants. Therefore they are reliable quantities to quantify the relation of a global dynamical system to its constituent parts and to characterize dynamical self-organization.

Given a dynamical system defined by a map $f : M \longrightarrow M$, with $M \subset \mathbb{R}^m$ the *conditional exponents associated to the splitting* $\mathbb{R}^k \times \mathbb{R}^{m-k}$ are the eigenvalues of the limit

$$\lim_{n\to\infty} (D_k f^{n*}(x) D_k f^n(x))^{1/2n}, \tag{1}$$

where $D_k f^n$ is the $k \times k$ diagonal block of the full Jacobian.

Lemma 1. The existence of the conditional exponents as well-defined ergodic invariants is guaranteed under the same conditions that establish the existence of the Lyapunov exponents.

Proof. Let μ be a probability measure in $M \subset \mathbb{R}^m$ and f a measure-preserving $M \longrightarrow M$ mapping such that μ is ergodic. Oseledec's multiplicative ergodic theorem [3], generalized for non-invertible f [4], states that if the map $T : M \longrightarrow M_m$ from M to the space of $m \times m$ matrices is measurable and

$$\int \mu(\mathrm{d}x) \log^+ \|T(x)\| < \infty \tag{2}$$

(with $\log^+ g = \max(0, \log g)$) and if

$$T_x^n = T(f^{n-1}x) \ldots T(fx)T(x), \tag{3}$$

then

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$$\lim_{n\to\infty} (T_x^{n*} T_x^n)^{1/2n} = \Lambda_x \tag{4}$$

exists μ almost everywhere.

If T_x is the full Jacobian $Df(x)$ and if $Df(x)$ satisfies the integrability condition (2) then the Lyapunov exponents exist μ almost everywhere. But if the Jacobian satisfies (2), then the $m \times m$ matrix formed by the diagonal $k \times k$ and $m-k \times m-k$ blocks also satisfies the same condition and the conditional exponents too are defined a.e. Furthermore, under the same conditions as for Oseledec's theorem, the set of regular points is Borel of full measure and

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_k f^n(x) u\| = \xi_i^{(k)} \tag{5}$$

with $0 \neq u \in E_x^i / E_x^{i+1}$, E_x^i being the subspace of $\mathbb{R}^k$ spanned by eigenstates corresponding to eigenvalues $\leqslant \exp(\xi_i^{(k)})$.

2. Conditional entropies and dynamical self-organization

For measures μ that are absolutely continuous with respect to the Lebesgue measure of M or, more generally, for measures that are smooth along unstable directions (SBR measures) Pesin's [5] identity

$$h(\mu) = \sum_{\lambda_i > 0} \lambda_i \tag{6}$$

holds, relating the Kolmogorov–Sinai entropy $h(\mu)$ to the sum of the Lyapunov exponents. By analogy we may define the *conditional exponent entropies* associated to the splitting $\mathbb{R}^k \times \mathbb{R}^{m-k}$ as the sum of the positive conditional exponents counted with their multiplicity,

$$h_k(\mu) = \sum_{\xi_i^{(k)} > 0} \xi_i^{(k)}, \tag{7}$$

$$h_{m-k}(\mu) = \sum_{\xi_i^{(m-k)} > 0} \xi_i^{(m-k)}. \tag{8}$$

The Kolmogorov–Sinai entropy of a dynamical system measures the rate of information production per unit time. That is, it gives the amount of randomness in the system that is not explained by the defining equations (or the minimal model [6]). Hence, the conditional exponent entropies may be interpreted as a measure of the randomness that would be present if the two parts $S^{(k)}$ and $S^{(m-k)}$ were uncoupled. The difference $h_k(\mu) + h_{m-k}(\mu) - h(\mu)$ represents the effect of the coupling.

Given a dynamical system S composed of N parts $\{S_k\}$ with a total of m degrees of freedom and invariant measure μ, one defines a *measure of dynamical self-organization* $I(S, \Sigma, \mu)$ as

$$I(S, \Sigma, \mu) = \sum_{k=1}^{N} \{h_k(\mu) + h_{m-k}(\mu) - h(\mu)\}. \tag{9}$$

Of course, for each system S, this quantity will depend on the partition Σ into N parts that one considers. $h_{m-k}(\mu)$ always denotes the conditional exponent entropy of the complement of the subsystem S_k. Being constructed out of ergodic invariants, $I(S, \Sigma, \mu)$ is also a well-defined ergodic invariant for the measure μ. $I(S, \Sigma, \mu)$ is formally similar to mutual information. However, not being strictly mutual information, in the information theory sense, $I(S, \Sigma, \mu)$ may take negative values.

Another ergodic invariant that may be associated to the splitting of a dynamical system into its constituent parts is the notion of *cylindrical entropies*.

Consider, as before, a μ-preserving and μ-ergodic mapping $f: M \to M$ and a splitting $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$. A measure in $\mathbb{R}^m$ induces a measure in $\mathbb{R}^k$ by

$$\nu(x) = \int_{\mathbb{R}^{m-k}} d\mu(y, x), \tag{10}$$

$x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{m-k}$.

Given a ν-measurable partition $P(\mathbb{R}^k)$ in $\mathbb{R}^k$,

$$\mathbb{R}^k = \bigcup_i P_i, \tag{11}$$

$P_i \in P(\mathbb{R}^k)$, it induces a partition in $\mathbb{R}^m$ by the associated cylinder sets

$$\mathbb{R}^m = \bigcup_i P_i^c, \tag{12}$$

$P_i^c = P_i \times \mathbb{R}^{m-k} \in P^c(\mathbb{R}^m)$.

Let $P^c(M) = P^c(\mathbb{R}^m) \cap M$ be the corresponding partition of M. Denote by $P^c(x)$ the element of

$P^c(M)$ that contains x. If all powers of f are ergodic, for any nontrivial partition

$$\lim_{n\to\infty} \mu(P_n^c(x)) = 0, \tag{13}$$

where

$$P_n^c(x) = \bigcap_{j=0}^{n} f^{-j}(P^c(f^j(x))), \tag{14}$$

then the Shannon–MacMillan–Breiman theorem states that if

$$\sum_i \mu(P_i^c) \log \mu(P_i^c) < \infty, \tag{15}$$

the limit

$$h^c(f, P^c, x) = -\lim_{n\to\infty} \frac{1}{n} \log \mu(P_n^c(x)) \tag{16}$$

exists μ a.e. and converges in L^1. This limit is the entropy at x associated to the cylindrical partition P^c. The *cylindrical entropy relative to the splitting* $\mathbb{R}^k \times \mathbb{R}^{m-k}$ may be defined as the integral of the supremum of this limit over all finite cylindrical partitions

$$h^c(f) = -\int_M d\mu(x) \sup_{P^c} \lim_{n\to\infty} \frac{1}{n} \log \mu(P_n^c(x)). \tag{17}$$

The full Kolmogorov–Sinai entropy is a similar limit where now the supremum would be taken over all finite partitions. Therefore, for a smooth measure, if the parts of a composite dynamical system are all uncoupled, the full entropy is simply the sum of the cylindrical entropies. In the uncoupled case each cylindrical entropy is determined by the corresponding conditional exponents. However, for coupled mixing systems, the cylindrical partitions may, by themselves, already generate the full entropy of the coupled system. Therefore the relation of the cylindrical entropies to the total entropy is simply a *measure of the statistical independence* of the constituent parts. The conditional exponent entropies defined in (7), (8) seem to be a better quantitative characterization of dynamical self-organization.

3. An example

Consider a fully coupled system defined by

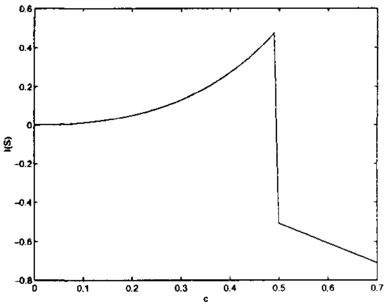

Fig. 1. Coupling dependence of the self-organization invariant $I(S, \Sigma, \mu)$ in the coupled Bernoulli system.

$$x_i(t+1) = (1-c) f(x_i(t)) + \sum_{j\neq i} \frac{c}{N-1} f(x_j(t)) \tag{18}$$

with $f(x) = 2x \pmod 1$.

The Lyapunov exponents are $\lambda_1 = \log 2$ and $\lambda_i = \log[2(1 - N/(N-1)c)]$ with multiplicity $N-1$. Therefore, for an absolutely continuous measure

$$\begin{aligned} h(\mu) &= \log 2 + (N-1) \log\left(2 - \frac{2Nc}{N-1}\right) \quad \text{for } c \leqslant \frac{N-1}{2N} \\ &= \log 2 \quad \text{for } c \geqslant \frac{N-1}{2N}. \end{aligned} \tag{19}$$

The conditional exponents associated to the splitting $\mathbb{R}^1 \times \mathbb{R}^{N-1}$ are

$$\xi^{(1)} = \log(2 - 2c) \tag{20}$$

$$\begin{aligned} \xi_1^{(N-1)} &= \log\left(2 - \frac{2c}{N-1}\right), \\ \xi_i^{(N-1)} &= \log\left(2 - \frac{2Nc}{N-1}\right) \end{aligned} \tag{21}$$

with multiplicity $N-2$. Therefore, for a partition Σ of a system with N parts one obtains

$$\begin{aligned} I(S, \Sigma, \mu) = N\Big\{ &\log\left(1 - \frac{c}{N-1}\right) \\ &+ \max(\log(2-2c), 0) \\ &- \max\left[\log\left(2 - \frac{2Nc}{N-1}\right), 0\right]\Big\}, \end{aligned} \tag{22}$$

which in the limit of large N becomes

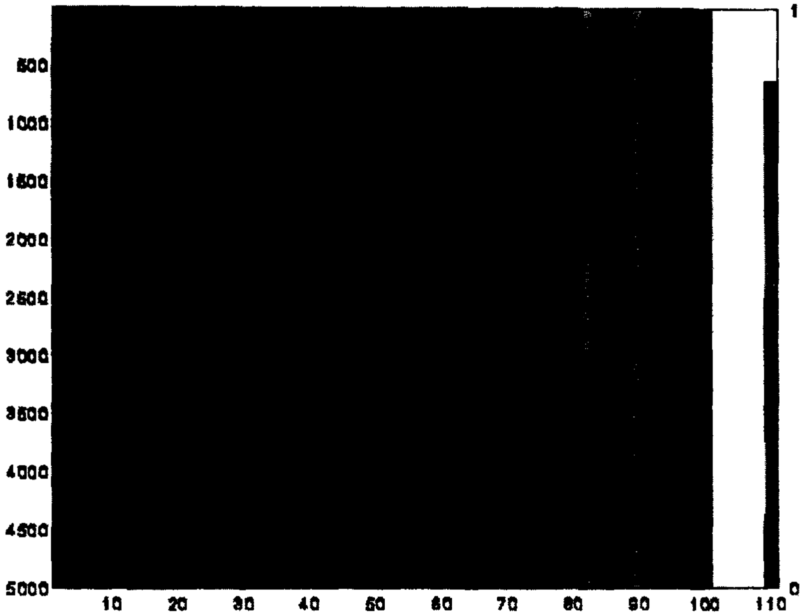

Fig. 2. The first 5000 time steps for $c = 0.495$ (maximum $I(S, \Sigma, \mu)$). The last column on the right is the color map.

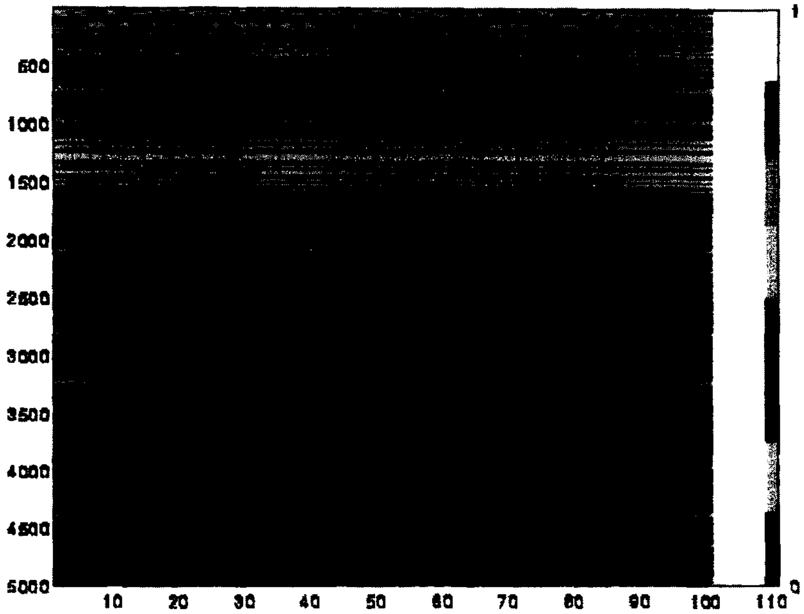

Fig. 3. The first 5000 time steps for $c = 0.51$.

$$\begin{aligned} I(S, \Sigma, \mu) &= \frac{c^2}{1-c}, \quad c \leqslant \frac{N-1}{2N}, \\ &= -c, \qquad c \geqslant \tfrac{1}{2}. \end{aligned} \tag{23}$$

Fig. 1 shows the variation with c of $I(S, \Sigma, \mu)$ for $N = 100$.

At $c = 0$, and starting from a random initial condition, the motion of the system is completely disorganized. When c starts to grow the system shows the coexistence of disorganized behavior with patches of synchronized clusters. At the point where $I(S, \Sigma, \mu)$ is a maximum, $c = 0.495$, starting from a random initial condition, the system settles rapidly in a state with many different synchronized clusters. Fig. 2 shows the first 5000 time steps. It is indeed at this point that the system shows what intuitively we would call a large organizational structure. Above $c = 0.5$, after a short transition period, the system becomes fully synchronized (Fig. 3 for $c = 0.51$).

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