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ORIGINAL ARTICLE

The dynamical nature of a backlash system with and without fluid friction

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Abstract We study the dynamics of a simple system with backlash and impacts. Both the presence or the absence of fluid friction is considered. The fluid friction is modeled by a fractional derivative, but it is also shown how an inhomogeneous time scale, although not arising from a fractional differential equation, may lead to some features similar to fractional solutions.

Keywords Backlash · Fractional dynamics · Inhomogeneous time scale

1. Introduction

Mechanical devices with backlash and impacts play an important role in technology (see for example [1–3]). Due to their nonlinear or piecewise linear characteristics, even simple systems may display complex dynamical behavior. Gearboxes, for example, may display either regular vibrations or chaotic motions depending on the system parameters [4–9]. Recently it has been pointed out [10] that this type of systems might also be analyzed in the perspective of fractional calculus. The fractional nature of the dynamics in these systems would explain the improved effectiveness of fractional-order controllers [11,12].

Fractional calculus, a well developed mathematical field, has recently found a growing range of applications in physics and engineering [13–22].

Here we study a simple system with backlash and impacts to identify the origin and nature of its dynamics. In the system we consider both the presence or the absence of fluid friction, to exhibit the way in which either a fractional derivative or a time inhomogeneity appear in the description of the system. We also point out how an inhomogeneous time scale, although not arising from a fractional differential equation, displays some features similar to fractional solutions.

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2. A dynamical backlash system with fluid friction

We consider the system with backlash and impacts depicted in the Fig. 1. On the bodies 1 and 2 act a (driving force) force F_1 and a (load) force F_2. Between impacts body 1 moves in a Newtonian viscous fluid. The mass

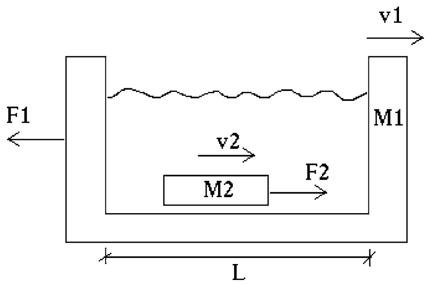

Fig. 1 A dynamical backlash system with impacts and fluid friction

M_2 includes the mass of the fluid as well. At the k-th impact

$$v_1(k+) - v_2(k+) = -\varepsilon (v_1(k-) - v_2(k-))$$
$$M_1 v_1(k+) + M_2 v_2(k+) = M_1 v_1(k-) + M_2 v_2(k-) \tag{1}$$

$v_{1,2}(k-)$ and $v_{1,2}(k+)$ being the velocities before and after the impact and $\varepsilon \in [0, 1]$ the inelasticity coefficient.

From (1) it follows

$$\begin{aligned} v_1(k+) &= \frac{(1-\varepsilon\mu) v_1(k-) + (1+\varepsilon)\mu v_2(k-)}{1+\mu} \\ v_2(k+) &= \frac{(1+\varepsilon) v_1(k-) + (\mu-\varepsilon) v_2(k-)}{1+\mu} \end{aligned} \tag{2}$$

with $\mu = \frac{M_2}{M_1}$.

In-between impacts body 1 is subject to friction arising from the shear stress of the fluid [18]

$$\begin{aligned} \sigma(t, z) &= -\sqrt{\nu\rho} \frac{1}{\Gamma(1/2)} \frac{\partial}{\partial t} \int_0^t \frac{v(\tau, z)}{(t-\tau)^{1/2}} d\tau \\ &= -\sqrt{\nu\rho} D_{(t)}^{1/2} v_2(t, z) \end{aligned} \tag{3}$$

ν being the viscosity, ρ the fluid density, and $v(t, z)$ the fluid density at the vertical coordinate z. Applying the boundary condition $v(t, 0) = v_2(t)$ one obtains the equations of motion for the interval between impacts

$$\begin{aligned} M_1 \frac{dv_1(t)}{dt} &= F_1 + 2A\sigma(t, 0) \\ M_2 \frac{dv_2(t)}{dt} &= F_2 - 2A\sigma(t, 0) \end{aligned} \tag{4}$$

A being the effective lateral contact area of body 1.

$$\begin{aligned} M_1 \frac{dv_1(t)}{dt} &= F_1 + 2A\sqrt{\nu\rho} D_{(t)}^{1/2} v_2(t) \\ M_2 \frac{dv_2(t)}{dt} &= F_2 - 2A\sqrt{\nu\rho} D_{(t)}^{1/2} v_2(t) \end{aligned} \tag{5}$$

The fractional nature of these equations of motion originates naturally from the fluid friction. However, a numerical study by Barbosa and Machado [10] of the Nyquist diagram of a similar system without the fluid friction and excited by a sinusoidal input force, suggests that also in this case the dynamics has fractional-order characteristics. This is somewhat surprising, because without the fluid friction, the system has a very simple exponential solution. Here we analyze first this case to understand where the fractional-like behavior might come from. It turns out that, rather than originating from the shear stress of the fluid (as above), it arises from a non-homogeneous time scale.

2.1. The zero-viscosity limit

Defining sum and difference coordinates

$$\begin{aligned} \nu_- &= v_1 - v_2 \\ \nu_+ &= \nu_1 + v_2 \end{aligned} \tag{6}$$

let, for simplicity, consider $F_1 = F_2 = 0$. Then, from (1) one obtains

$$v_-(k+) = (-1)^k \varepsilon^k v_-(0) \tag{7}$$

$$v_+(k+) = v_+(0) + \frac{1-\mu}{1+\mu} v_-(0)\,(1 - (-1)^k \varepsilon^k) \tag{8}$$

Hence

$$\begin{aligned} v_1(k+) &= \frac{(1+(-1)^k\varepsilon^k) v_1(0) + (\mu - (-1)^k\varepsilon^k) v_2(0)}{1+\mu} \\ v_2(k+) &= \frac{(1-\mu(-1)^k\varepsilon^k) v_1(0) + \mu(1+(-1)^k\varepsilon^k) v_2(0)}{1+\mu} \end{aligned} \tag{9}$$

One sees that the velocities have a simple exponential dependence on the impact number k and no symptom of fractional behavior. A similar situation occurs if $F_1, F_2 \neq 0$. Notice however that the interval between impacts is not uniform. Denoting by t_k the time of the $k-$th impact

$$t_{k+1} = t_k + \frac{L}{|v_1(k+) - v_2(k+)|} \tag{10}$$

Then

$$t_k = \frac{L(\varepsilon^{-k} - 1)}{|v_1(0) - v_2(0)|(\varepsilon^{-1} - 1)} \tag{11}$$

Therefore the evolution of $v_-(t)$ in physical time is

$$v_-(t_k) = (-1)^k v_-(0) \frac{1}{1 + \frac{t_k(\varepsilon^{-1}-1)v_-(0)}{L}} \tag{12}$$

Considering the discrete times t_k embedded on continuous time, $|v_-(t_k)|$ is a discrete version of the solution of the differential equation

$$\frac{d}{dt} f(t) = -\frac{\alpha}{1 + \alpha t} f(t) \tag{13}$$

with $\alpha = \frac{(\varepsilon^{-1}-1)v_-(0)}{L}$. Hence, in *physical time* the differential behavior is more complex than in *impact number*, but it is not fractional. Nevertheless there is a feature than bears some similarities to a behavior typical of fractional solutions. It is the non-homogeneity of the time scale that arises from the dissipation at the impacts and entails violation of the invariance under time translations. To understand this fact consider the fractional integral

$$\left(I_{0+}^{\beta}\phi\right)(t) = \frac{1}{\Gamma(\beta)} \int_0^t \phi(\tau)(t-\tau)^{\beta-1} d\tau \tag{14}$$

rewritten as

$$\left(I_{0+}^{\beta}\phi\right)(t) = \int_0^t \phi(\tau) d\zeta_t(\tau) \tag{15}$$

with

$$\zeta_t(\tau) = \frac{1}{\Gamma(\beta+1)}(t^{\beta} - (t-\tau)^{\beta}) \tag{16}$$

That is, the fractional integral may be interpreted as an integral over a modified time scale $\zeta_t(\tau)$.

Likewise, Equation (13) may be written in integral form as

$$f(t) = f(0) + \int_0^t f(\tau) d\varphi(\tau) \tag{17}$$

with

$$\varphi(\tau) = -\log(1 + \alpha\tau) \tag{18}$$

with $\varphi(\tau)$ behaving also as a modified time scale. Therefore the non-homogeneity of the physical time scale (as compared to the impact number scale), might in fact lead this system to have responses, to external driving forces, bearing some resemblance to fractional systems. Notice however that there is a fundamental difference in the modified time scales $\zeta_t(\tau)$ and $\varphi(\tau)$. This is the dependence on t of $\zeta_t(\tau)$.

2.2. The viscous fluid case

Here we find the solution of the dynamics between impacts of the full system with external forces and fluid friction. Let $t = 0$ be the time immediately after an impact. Then from (3) and (5) one obtains

$$v_{\mp}(t) = v_{\mp}(0) + t\left(\frac{F_1}{M_1} \pm \frac{F_2}{M_2}\right) - \frac{2A\sqrt{\nu\rho}}{\Gamma(1/2)}\left(\frac{1}{M_2} \pm \frac{1}{M_1}\right)\int_0^t \frac{v_{\mp}(\tau)}{(t-\tau)^{1/2}} d\tau \tag{19}$$

Using the Laplace transform

$$\tilde{v}_{\mp}(z) = \tilde{v}_{\mp}(0)\frac{1}{z} + \left(\frac{F_1}{M_1} \pm \frac{F_2}{M_2}\right)\frac{1}{z^2} - \theta_{\pm}\tilde{v}_{\mp}(z)\frac{1}{\sqrt{z}} \tag{20}$$

with $\theta_{\pm} = 2A\sqrt{\nu\rho}(\frac{1}{M_2} \pm \frac{1}{M_1})$, leads to

$$v_{\mp}(t) = \left(\frac{F_1}{M_1} \pm \frac{F_2}{M_2}\right) \frac{1}{\theta_{\pm}^2}\left\{\frac{2\theta_{\pm}\sqrt{t}}{\sqrt{\pi}} - 1 + e^{t\theta_{\pm}^2} erfc(\theta_{\pm}\sqrt{t})\right\} + v_{\mp}(0) e^{t\theta_{\pm}^2} erfc(\theta_{\pm}\sqrt{t}) \tag{21}$$

The fractionality of the solution is here quite clear and, as before, because the time between impacts obtained from

$$\int_0^T v_-(\tau)\,d\tau = L \tag{22}$$

is not uniform, there would also be effects arising from time non-homogeneity.

3. Conclusions

The simple backlash system studied in this paper puts into evidence both the fractional derivative nature of some friction phenomena and the fractional-like features that arise from an inhomogeneous time scale.

Inhomogeneous time or space scales or, equivalently, coexistence of many different scales are indeed at the origin of the fractional derivatives modeling of systems in viscoelasticity and other fields [19–22]. The emphasis on the inhomogeneity of the integration scale also provides a nice geometrical interpretation of fractional integration [23], similar to Bullock's interpretation of the Riemann-Stieltjes integral [24].

One should notice however that, as seen above, the fractional integral and fractional derivative represent only a particular type of inhomogeneous integration scale. It is therefore conceivable that a more general framework might be needed to model all kinds of inhomogeneous scales appearing in Nature.

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