An extended Dirac equation in noncommutative space-time

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Abstract

Stabilizing, by deformation, the algebra of relativistic quantum mechanics a non-commutative space-time geometry is obtained. The exterior algebra of this geometry leads to an extended massless Dirac equation which has both a massless and a large mass solution. The nature of the solutions is discussed, as well as the effects of coupling the two solutions.

1 The stable Heisenberg-Poincaré algebra and noncommutative space-time

When models are constructed for the natural world, it is reasonable to expect that only those properties of the models that are robust have a chance to be observed. Models or theories being approximations to the natural world, it is unlikely that properties that are too sensitive to small changes (that is, that depend in a critical manner on particular values of the parameters) will be well described in the model. If a fine tuning of the parameters is needed to reproduce some natural phenomenon, then the model is basically unsound and its other predictions expected to be unreliable. For this reason it would seem that a good methodological point of view, in the construction of physical theories, would be to focus on the robust properties of the models or, equivalently, to consider only models which are stable, in the sense that they do not change, in a qualitative manner, when some parameter changes.

This point of view had a large impact in the field of non-linear dynamics, where it led to the rigorous notion of *structural stability* [1] [2]. As pointed out by Flato [3] and Faddeev [4] the same pattern seems to occur in the fundamental theories of Nature. In fact, the two physical revolutions of the last century, namely the passage from non-relativistic to relativistic and from classical to quantum mechanics are deformations of two unstable Lie algebras to two stable

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ones. A mathematical structure is said to be *stable* (or *rigid*) for a class of *deformations,* if any deformation in this class leads to an equivalent (isomorphic) structure. When going from Galilean to relativistic mechanics, the Galilean algebra, an isolated point, is deformed to the stable Lorentz algebra and on the transition from classical to quantum mechanics the unstable Poisson algebra is deformed to the stable Moyal algebra.

This situation motivated the question of whether the full algebra of relativistic quantum mechanics, the Heisenberg-Poincaré algebra, would itself be stable. The answer was that it is not and that one possible deformation to a stable one is defined by the following commutators [5]:

$$
\begin{array}{rcl}
[M_{\mu\nu}, M_{\rho\sigma}] & = & i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}) \\
[M_{\mu\nu}, P_{\lambda}] & = & i(P_{\mu}\eta_{\nu\lambda} - P_{\nu}\eta_{\mu\lambda}) \\
[M_{\mu\nu}, x_{\lambda}] & = & i(x_{\mu}\eta_{\nu\lambda} - x_{\nu}\eta_{\mu\lambda}) \\
[P_{\mu}, P_{\nu}] & = & -i\frac{\epsilon_4}{R^2}M_{\mu\nu} \\
[x_{\mu}, x_{\nu}] & = & -i\epsilon_5\ell^2 M_{\mu\nu} \\
[P_{\mu}, x_{\nu}] & = & i\eta_{\mu\nu}\Im \\
[P_{\mu}, \Im] & = & -i\frac{\epsilon_4}{R^2}x_{\mu} \\
[x_{\mu}, \Im] & = & i\epsilon_5\ell^2 P_{\mu}
\end{array}
\tag{1}
$$

In this algebra, that will be denoted $\Re_{\ell,R} = \{M_{\mu\nu}, P_{\mu}, x_{\mu}, \Im\}$, $M_{\mu\nu}$ are the generators of the Lorentz group, P_{μ} and x_{μ} the momenta and coordinates, $\Im$ a non-trivial operator that replaces the center of the Heisenberg algebra and ϵ_4, ϵ_5 are $\pm$ signs. Velocities and actions are in units of c and $\hbar$ (that is $c = \hbar = 1$). The stable algebra $\Re_{\ell,R}$ is isomorphic to the algebra of the 6−dimensional pseudo-orthogonal group with metric

$$
\eta_{aa} = (1, -1, -1, -1, \epsilon_4, \epsilon_5), \ \ \epsilon_4, \epsilon_5 = \pm 1
\tag{2}
$$

The nonvanishing right hand side of the $[P_{\mu}, P_{\nu}]$ commutator simply means that flat space is an isolated point in the set of arbitrarily curved spaces. This is why Faddeev [4] points out that Einstein's theory of gravity may also be considered as a deformation in a stable direction. Einstein's theory is based on curved pseudo Riemann manifolds. Therefore, in the set of Riemann spaces, Minkowski space is a kind of degeneracy whereas a generic Riemann manifold is stable in the sense that in its neighborhood all spaces are curved. However, as long as one is concerned with the kinematical group of the tangent space to the space-time manifold, and not with the group of motions in the manifold itself, it is perfectly consistent to take $R \to \infty$ and this deformation would be removed. In particular, because the curvature is not a constant, R cannot have the status of a fundamental constant.

In contrast, for the other stabilizing deformation, associated to the ℓ constant, there is no obvious reason to remove it and ℓ might play the role of a new fundamental constant. Taking the $\Re_{\ell,\infty}$ algebra as the kinematical algebra of tangent space, the main features are the non-commutativity of the x_{μ} coordinates and the fact that $\Im$, previously a trivial center of the Heisenberg algebra,

becomes now a non-trivial operator. These are however the minimal changes that seem to be required if stability of the algebra of observables (in the tangent space) is a good guiding principle. Two constants define this deformation. One is ℓ, a *fundamental length*, the other the sign ϵ_5.

The algebra $\Re_{\ell,\infty}$ is seen to be the algebra of the pseudo-Euclidean groups $E(1,4)$ or $E(2,3)$, depending on whether ϵ_5 is -1 or $+1$. It has a simple representation by differential operators in a five-dimensional space with coordinates $(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4)$

$$
\begin{aligned}
P_\mu &= i\frac{\partial}{\partial \xi^\mu} + iD_{P_\mu} \\
M_{\mu\nu} &= i(\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}) + \Sigma_{\mu\nu} \\
x_\mu &= \xi_\mu + i\ell(\xi_\mu \frac{\partial}{\partial \xi^4} - \epsilon_5 \xi^4 \frac{\partial}{\partial \xi^\mu}) + \ell\Sigma_{\mu 4} \\
\Im &= 1 + i\ell\frac{\partial}{\partial \xi^4} + i\ell D_{\xi^4}
\end{aligned}
\tag{3}
$$

The set $(\Sigma_{\mu\nu}, \Sigma_{\mu 4})$ is an internal spin operator for the groups $O(4,1)$ (if $\epsilon_5 = -1$) or $O(3,2)$ (if $\epsilon_5 = +1$) and D_{P_μ} and D_{ξ^4} are derivations operating in the space where $(\Sigma_{\mu\nu}, \Sigma_{\mu 4})$ acts. For practical calculations, in particular for the construction of quantum fields, it may be convenient to use this representation. Notice however that only the Poincaré part of $E(1,4)$ or $E(2,3)$ corresponds to symmetry operations and only this part has to be implemented by unitary operators. Also, although an extra dimension is used in the representation space, the space-time coordinates are still only four, noncommutative ones. Physical consequences of the non-commutative space-time structure implied by the $\Re_{\ell,\infty}$ algebra have already been explored in a series of publications [6]-[11]. Depending on the sign of ϵ_5 the time ($\epsilon_5 = +1$) or one space variable ($\epsilon_5 = -1$) will have discrete spectrum. In any case ℓ, a new fundamental constant, sets a natural scale for time and length. If ℓ is of the order of Planck's length, observation of most of the effects worked out in the cited references will be beyond present experimental capabilities. However, if ℓ is much larger than Planck´s length (for example of order $10^{-27} - 10^{-26}$ seconds) the effects might already be observable in the laboratory or in astrophysical observations. Some of the most noteworthy effects arise from the modification of the phase space volume and from interference effects.

However, most of the consequences worked out in the references [6]-[10] are rather conservative, in the sense that they simply explore the nonvanishing of the right-hand-side of the commutators of previously commuting variables. Deeper consequences are to be expected from the radical change from a commutative to a non-commutative space-time geometry, in particular from the new differential algebra associated to the geometry. One such consequence will be described in this paper. The new geometry was studied in Ref.[12] to which I will refer for details and notation.

2 The differential algebra and an extended Dirac equation

In the framework of the non-commutative geometry implied by the deformed algebra, the differential algebra may be constructed either by duality from the derivations of the algebra or from the triple $(H, \pi(U_\Re), D)$, where $U_\Re$ is the enveloping algebra of $\Re_{\ell,\infty}$, to which a unit and, for later convenience, the inverse of $\Im$, are added.

$$U_\Re = \{x_\mu, M_{\mu\nu}, p_\mu, \Im, \Im^{-1}, 1\} \tag{4}$$

$\pi(U_\Re)$ is a representation of the $U_\Re$ algebra in the Hilbert space H and D is the Dirac operator, the commutator with the Dirac operator being used to generate the one-forms. In a general non-commutative framework [13] [14] it is not always possible to use the derivations of the algebra to construct by duality the differential forms. In particular, many algebras have no derivations at all. However when the algebra has enough derivations it is useful to consider them [15] [16] because the correspondence of the non-commutative geometry notions to the classical ones becomes very clear. One considers here the set V of derivations with basis $\{\partial_\mu, \partial_4\}$ defined as follows[1]

$$
\begin{aligned}
\partial_\mu(x_\nu) &= \eta_{\mu\nu}\Im \\
\partial_4(x_\mu) &= -\epsilon_5 \ell p_\mu \Im \\
\partial_\sigma(M_{\mu\nu}) &= \eta_{\sigma\mu} p_\nu - \eta_{\sigma\nu} p_\mu \\
\partial_\mu(p_\nu) &= \partial_\mu(\Im) = \partial_\mu(1) = 0 \\
\partial_4(M_{\mu\nu}) &= \partial_4(p_\mu) = \partial_4(\Im) = \partial_4(1) = 0
\end{aligned}
\tag{5}
$$

In the commutative ($\ell = 0$) case a basis for 1-forms is obtained, by duality, from the set $\{\partial_\mu\}$. In the $\ell \neq 0$ case the set of derivations $\{\partial_\mu, \partial_4\}$ is the minimal set that contains the usual ∂_μ's, is maximal abelian and is action closed on the coordinate operators, in the sense that the action of ∂_μ on x_ν leads to the operator $\Im$ associated to ∂_4 and conversely.

The operators that are associated to the physical coordinates are just the four x_μ, $\mu \in (0,1,2,3)$. However, an additional degree of freedom appears in the set of derivations. This is not a conjectured extra dimension but simply a mathematical consequence of the algebraic structure of $\Re_{\ell,\infty}$ which, in turn, was a consequence of the stabilizing deformation of relativistic quantum mechanics. No extra dimension appears in the set of physical coordinates, because it does not correspond to any operator in $\Re_{\ell,\infty}$. However the derivations in V introduce, by duality, an additional degree of freedom in the exterior algebra. For example, all quantum fields that are Lie algebra-valued connections will pick up additional components. These additional components, in quantum fields that are connections, are a consequence of the length parameter ℓ which does not depend on its magnitude, but only on ℓ being $\neq 0$.

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[1] Notice that the definition of ∂_4 here is slightly different from the one in Ref.[12].

The Dirac operator [12] is

$$D = i\gamma^a \partial_a \tag{6}$$

with $\partial_a = (\partial_\mu, \partial_4)$, the γ's being a basis for the Clifford algebras $C(3,2)$ or $C(4,1)$

$$\begin{array}{ll} \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^5\right) & \epsilon_5 = +1 \\ \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = i\gamma^5\right) & \epsilon_5 = -1 \end{array} \tag{7}$$

How to construct quantum, scalar, spinor and gauge fields, as operators in $U_\Re$, has been described in [12]. In particular the role of the additional dimension in the exterior algebra, on gauge interactions, has been emphasized (see also [10]). Here another potentially interesting consequence for spinor fields will be described. Because

$$\left[p_\mu, e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+}\right] = k_\mu e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} \tag{8}$$

a spinor field is written

$$\Psi = \int d^4 k \, \delta(k^2 - m^2) \left\{ b_k u_k e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} + d_k^* v_k e^{\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} \right\} \tag{9}$$

$$\Psi \in U_\Re : D\Psi - m\Psi = 0 \tag{10}$$

From the field a wave function is constructed operating on a vacuum state

$$\psi = \Psi \left| 0 \right\rangle \tag{11}$$

Notice that both b_k, d_k^* and the elements of $U_\Re$ operate on $\left| 0 \right\rangle$, in particular $p_\mu \left| 0 \right\rangle = 0$. Now, for a massless field, the (extended) Dirac equation becomes

$$D\psi = i\gamma^a \partial_a \psi = \left(i\gamma^\mu \partial_\mu + i\gamma^4 \partial_4\right)\psi = 0 \tag{12}$$

Write

$$\psi = e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} u(k)$$

From

$$\begin{array}{ll} \partial_\mu e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} & = -ik_\mu e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} \\ \partial_4 e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} & = -i\epsilon_5 \ell \left(-k^\mu p_\mu + \frac{1}{2}k^2\right) e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} \end{array} \tag{13}$$

one obtains, using (13), (8) and (11)

$$\begin{array}{ll} \left(\gamma^\mu k_\mu - \gamma^5 \ell \frac{1}{2}k^2\right) u(k) = 0 & \epsilon_5 = +1 \\ \left(\gamma^\mu k_\mu + i\gamma^5 \ell \frac{1}{2}k^2\right) u(k) = 0 & \epsilon_5 = -1 \end{array} \tag{14}$$

Let $\epsilon_5 = -1$. Iterating (14)

$$\left(k^2 - \frac{\ell^2}{4}\left(k^2\right)^2\right) u(k) = 0 \tag{15}$$

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This equation has two solutions, the massless solution ($k^2 = 0$) and another one, of large mass (ℓ being small)

$$k^2 = \frac{4}{\ell^2} \tag{16}$$

For $\epsilon_5 = +1$ one would obtain $k^2 = 0$ and

$$k^2 = -\frac{4}{\ell^2} \tag{17}$$

a tachyonic large $\left|k^2\right|$ solution.

The solutions of the extended Dirac equation for $k^2 = 0$ are the usual ones. To find the nature of the solutions for $\left|k^2\right| = \frac{4}{\ell^2}$, $\epsilon_5 = -1$ and $+1$, use a Majorana imaginary representation for the gamma matrices

$$
\gamma^0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} ; \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix} ; \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}
$$

$$
\gamma^3 = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix} ; \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{18}
$$

2.1 $\epsilon_5 = -1, k^2 = \frac{4}{\ell^2}$

In the rest frame $k = (m_0 = \pm\frac{2}{\ell}, 0, 0, 0)$. The second equation in (14) leads to

$$\left(\pm\gamma^0 + i\gamma^5\right) u = 0$$

$$
\begin{array}{ll}
u = \begin{pmatrix} a \\ ia \end{pmatrix} & \text{Positive energy } \left(m_0 = \frac{2}{\ell}\right) \\[2ex]
u = \begin{pmatrix} a \\ -ia \end{pmatrix} & \text{Negative energy } \left(m_0 = -\frac{2}{\ell}\right)
\end{array}
$$

where a is an arbitrary two-vector. The solutions of non-zero momentum are obtained by the application of a proper Lorentz transformation $\exp\left(i\frac{1}{2}\alpha_{\mu\nu}\left\{\gamma^\mu, \gamma^\nu\right\}_+\right)$.

One has $u^* \neq u$, hence these solutions are Dirac spinors.

2.2 $\epsilon_5 = +1, k^2 = -\frac{4}{\ell^2}$

Here one makes $k = (0, 0, 0, \frac{2}{\ell})$, obtaining

$$\left(\gamma^3 - \gamma^5\right) u' = 0$$

Making $u' = \begin{pmatrix} a \\ b \end{pmatrix}$, a and b being two-vectors, yields

$$
\begin{array}{rcl}
\left(\sigma_2 + i\sigma_3\right) a & = & 0 \\
\left(\sigma_2 - i\sigma_3\right) b & = & 0
\end{array}
$$

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meaning that a and b are independent two vectors

$$a = \begin{pmatrix} a_1 \\ a_1 \end{pmatrix}; b = \begin{pmatrix} b_1 \\ -b_1 \end{pmatrix}$$

with a_1 and b_1 real numbers $u'^* = u$ and this tachyonic, large $|k^2|$, solution is a Majorana spinor.

As before, general solutions would be obtained by the action of a Lorentz transformation.

3 Coupling the two solutions

In the previous section it was seen how the extended Dirac equation, following from the exterior algebra of noncommutative space-time, has both a massless and a large $|k^2|$ solution, large if ℓ is small. If, for example, ℓ is in the $10^{-27} - 10^{-26}$ seconds range, $M = \frac{2}{\ell}$ would be of the order of 1TeV. If the two solutions mix, one expects that the massless solution would acquire a small mass as in the seesaw mechanism proposed for neutrinos. In the seesaw mechanism the large mass (of right-handed neutrinos) is hypothesized to be obtained from the lepton number violation scale at grand unification. Here the large mass arises from an independent solution of the same extended equation.

Let us call u_1 the zero mass solution and u_2 the large mass solution. Then let, us assume that they are coupled by interaction with a background scalar field that acquires a nonzero vacuum expectation value $\phi = \langle \phi \rangle + h$, with Lagrangian

$$\mathcal{L} = \overline{u_1} i\gamma^\mu \partial_\mu u_1 + \overline{u_2} \left(i\gamma^\mu \partial_\mu + \gamma^4 \frac{2}{\ell} \right) u_2 + (g\overline{u_1} \left(\langle \phi \rangle + h \right) u_2 + h.c.) \qquad (19)$$

leading to the equations of motion:

$$\begin{array}{rcl} i\gamma^\mu \partial_\mu u_1 + g \left(\langle \phi \rangle + h \right) u_2 & = & 0 \\ \left(i\gamma^\mu \partial_\mu + \gamma^4 \frac{2}{\ell} \right) u_2 + g^* \left(\langle \phi \rangle + h^* \right) u_1 & = & 0 \end{array} \qquad (20)$$

With $\frac{2}{\ell}$ large, one neglects the kinetic term in the last equation and obtains, in leading order,

$$\begin{array}{rl} u_2 \simeq -\frac{\ell}{2} g^* \langle \phi \rangle \gamma^5 u_1 & \epsilon_5 = +1 \\ u_2 \simeq i\frac{\ell}{2} g^* \langle \phi \rangle \gamma^5 u_1 & \epsilon_5 = -1 \end{array} \qquad (21)$$

Substitution in the first equation of (20) yields

3.1 $\epsilon_5 = -1$

$$\left(i\gamma^\mu \partial_\mu + i |g|^2 \langle \phi \rangle^2 \frac{\ell}{2} \gamma^5 \right) u_1 \simeq 0$$

which has a small mass solution with

$$k^2 = \left(|g|^2 \langle \phi \rangle^2 \frac{\ell}{2} \right)^2$$

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For $k = \left(\pm |g|^2 \langle \phi \rangle^2 \frac{\ell}{2}, 0, 0, 0 \right)$ the solutions are

$$u_1 = \left(\begin{array}{c} a \\ \pm ia \end{array} \right)$$

a small mass Dirac spinor. a is an arbitrary two-vector.

3.2 $\epsilon_5 = +1$

$$\left(i\gamma^\mu \partial_\mu - |g|^2 \langle \phi \rangle^2 \frac{\ell}{2} \gamma^5 \right) u_1' \simeq 0$$

which has a small $\left| k^2 \right|$ solution with

$$k^2 = - \left(|g|^2 \langle \phi \rangle^2 \frac{\ell}{2} \right)^2$$

For $k = \left(0, 0, 0, |g|^2 \langle \phi \rangle^2 \frac{\ell}{2} \right)$ the solution is

$$u_1' = \left(\begin{array}{c} a_1 \\ a_1 \\ b_1 \\ -b_1 \end{array} \right)$$

a small $\left| k^2 \right|$ tachyonic Majorana spinor.

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