Ultradistribution Spaces: Superprocesses and Nonlinear Differential Problems

R. Vilela Mendes

Abstract From branching particle systems one obtains, in the scaling limit, measure-value processes called superprocesses. In addition to providing models for evolving populations, superprocesses provide probabilistic representations of the solutions of nonlinear partial differential equations (PDE's). However, the class of PDE's that can be handled by measure-valued superprocesses is rather limited. This suggests an extension of the configuration space of superprocesses to ultradistribution-valued processes which have a wider range of applications in the solution of PDE's. The relevance of the superprocess representation of PDE's to deal with nonlinear singular problems is also discussed.

Keywords Ultradistributions · Superprocesses · Nonlinear PDE's

1 Distributions and Ultradistributions

One of the motivations to develop distribution theory arose from the need to deal with non-smooth entities in differential equations. In particular the generalization of the notion of derivative led to the spaces of distributions ($\mathscr{D}'$) and tempered distributions ($\mathscr{S}'$). However, the theory of distributions is not just $\mathscr{D}'$ and $\mathscr{S}'$. There are many other interesting spaces of "*generalized functions*". In the Figs. 1 and 2 (adapted from [1]) are displayed some other test function spaces, their dense embeddings and Fourier maps (Fig. 1) as well as their corresponding duals (distribution spaces) (Fig. 2).

As $\mathscr{D}'$ and $\mathscr{S}'$ are a tool of choice to deal with linear differential and partial differential equations, some of the other spaces might be more appropriate to deal with other types of mathematical problems. The main properties of the spaces listed in the figures are summarized here:

Test Function Spaces

R. Vilela Mendes (✉)
Centro de Matemática e Aplicações Fundamentais, University of Lisbon, Lisbon, Portugal
e-mail: rvmendes@fc.ul.pt; rvilela.mendes@gmail.com

P. Gonçalves and A.J. Soares (eds.), *From Particle Systems to Partial Differential Equations*, Springer Proceedings in Mathematics & Statistics 209,
DOI 10.1007/978-3-319-66839-0_12

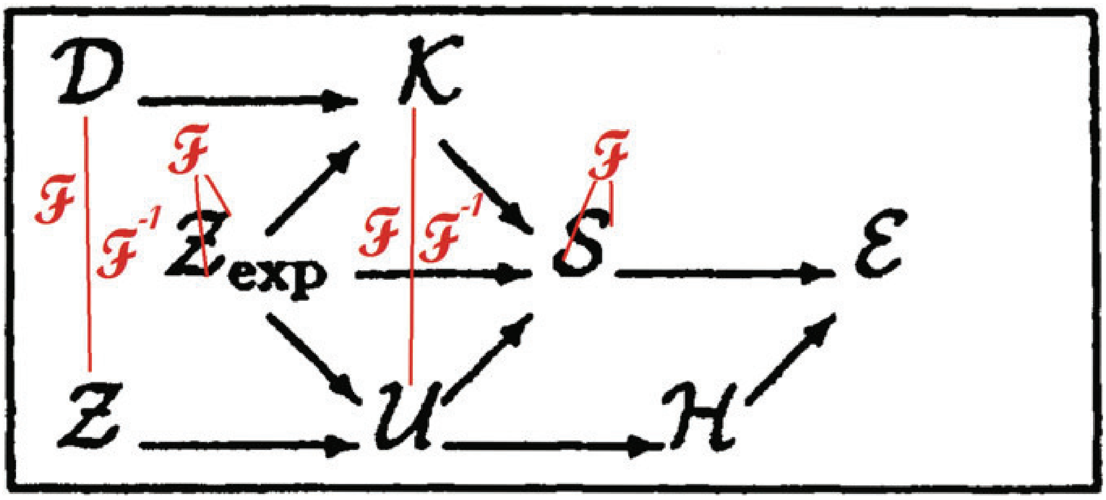

Fig. 1 Test function spaces (adapted from [1])

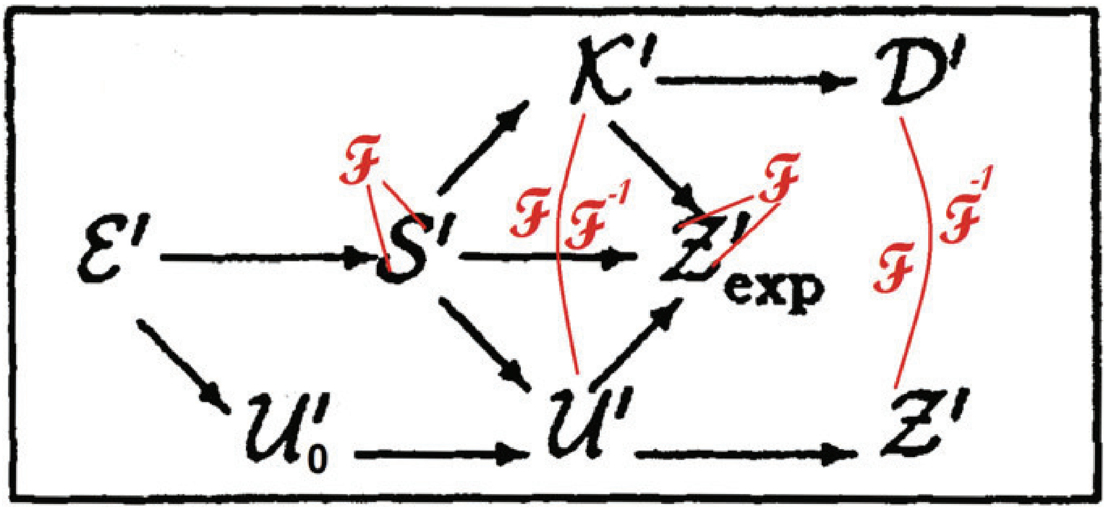

Fig. 2 Distribution spaces (adapted from [1])

$\mathscr{D} = \cup_K \{\mathscr{D}_K : \varphi \in C^\infty, \text{supp}(\varphi) \subset K\}$; $\|\varphi\|_{(p,K)} = \max_{0\le r\le p}\left\{\sup\left|\varphi^{(r)}\right|\right\}$

$\mathscr{K} = \cap_{p=0}^\infty \mathscr{K}_p$; $\mathscr{K}_p =$ completion of $\mathscr{D}$ for the norm $\|\varphi\| = \max_{0\le q\le p}\left\{\sup\left|e^{p|x|}\varphi^{(q)}\right|\right\}$

$\mathscr{S} = \cap \mathscr{S}_{p,r} = \left\{\varphi \in C^\infty : \|\varphi\|_{p,r} = \sup\left|x^p\varphi^{(r)}\right|\right\}$

$\mathscr{E} = \varphi \in C^\infty$ with uniform convergence on compacts

$\mathscr{Z} = \varphi : \mathscr{F}\{\varphi\} \in \mathscr{D}$, $\varphi(z)$ entire: $\left|z^k\varphi(z)\right| \le C_k e^{a|Im(z)|}$

$\mathscr{U} = \cap_{p=0}^\infty \mathscr{U}_p$; $\mathscr{U}_p = \left\{\varphi : \mathscr{F}\{\varphi\} \in \mathscr{K}_p\right\}$; $\|\varphi\|_p = \sup_{z\in\Lambda_p}\left\{(1+|z|^p)\,|\varphi(z)|\right\}$

$\mathscr{H}$ = Entire functions with topology of uniform convergence on compacts of $\mathbb{C}$

$\mathscr{Z}_{\exp} = \cap_{j=1}^\infty \mathscr{Z}_{\exp,j}$; $\mathscr{Z}_{\exp,j} = \left\{\varphi : \|\varphi\|_{\exp,j} = \max_{k\le j}\left\{e^{j|Re(z)|}\left|\varphi^{(k)}(z)\right|\right\}\right\}$

Distribution Spaces

$\mathscr{D}'$ = Schwartz distributions; locally $\mu(x) = D^k(f(x))$

$\mathscr{K}'$ = Distributions of exponential type, $\mu(x) = D^k\left(e^{a|x|}f\right)$

$\mathscr{S}'$ = Tempered distributions

$\mathscr{E}'$ = Subspace of $\mathscr{D}'$ of distributions of compact support

$\mathscr{Z}'$ = Ultradistributions, $\mathscr{D}' \overset{\mathscr{F}}{\to} \mathscr{Z}'$; $\mathscr{Z}' \overset{\mathscr{F}^{-1}}{\to} \mathscr{D}'$
$\mathscr{U}'$ = Tempered ultradistributions
$\mathscr{U}_0'$ = Dual of $\mathscr{H}$, ultradistributions of compact support
$\mathscr{Z}_{\exp}'$ = Topological dual of $\mathscr{Z}_{\exp}$, contains $\mathscr{U}'$ and $\mathscr{K}'$ as proper subspaces

Of particular relevance is the relation of the upper and lower lines in the figures through the Fourier transform. In particular the fact that the Fourier transform of $\mathscr{Z}$ has compact support endows the ultradistribution space $\mathscr{Z}'$ with a rich analytical structure. Ultradistributions not only have derivatives of all orders, like the distributions, but also have Taylor series expansions. This fact, among other things, makes them more convenient than distributions in some application problems.

These generalized function spaces emphasize the role of the Fourier transform. Other techniques have been used to define test function spaces smaller than $\mathscr{D}$ and therefore distribution spaces larger than $\mathscr{D}'$. Among them are the test function spaces defined through weight functions or weight sequences. Given an integrable function of compact support φ, the condition for infinite differentiability ($\varphi \in \mathscr{D}$) is expressed through the Fourier transform by

$$\int \mathscr{F}\{\varphi\}(x)\, e^{n \log(1+|x|)} dx < \infty \qquad \forall n.$$

Replacing $\log(1+|x|)$ by a larger function $\omega(|x|)$

$$\int \mathscr{F}\{\varphi\}(x)\, e^{\lambda\omega(|x|)} dx < \infty$$

one obtains a smaller test function space. With different definitions of projective limit on λ and inductive limit on the compacts, one obtains ultradistribution spaces of class ω of Beurling [2] or Roumieu [3] type.

Another trend defines ultradifferentiable functions [3–6] of class (M_p) (or $\{M_p\}$) by $\forall h > 0\ \exists C > 0$ (or $\exists h > 0$ and $C > 0$) such that

$$\sup \left|\varphi^{(n)}(x)\right| \leq C h^n M_n \qquad \forall n \in \mathbb{N}$$

on compacts, M_p being a sequence of positive numbers.

In this paper I will be mainly concerned with the space $\mathscr{U}'$ of Silva tempered ultradistributions [7, 8] and the space $\mathscr{U}_0'$ of ultradistributions of compact support.

1.1 Silva Tempered Ultradistributions

$\mathscr{U} \subset \mathscr{S}$, is the space of functions in $\mathscr{S}$ that may be extended into the complex plane as entire functions of rapid decrease on strips. Namely $\mathscr{U}(\mathbb{C})$ consists of all entire functions φ for which

$$\|\varphi\|_p = \sup_{|Imz|<p} \left\{ \left(1+|z|^p\right) |\varphi(z)| \right\} < \infty \qquad \forall p \in \mathbb{N} \tag{1}$$

$\mathscr{U}$ topologized by the norms $\|\varphi\|_p$ is a Fréchet space and for each $\varphi(z) \in \mathscr{U}$, $\varphi(z)|_{\mathbb{R}} \in \mathscr{S}(\mathbb{R})$.

$\mathscr{U}'$, the dual of $\mathscr{U}$ is Silva space of tempered ultradistributions [7, 8]. It may also be characterized as the space of all Fourier transforms of distributions of exponential type, that is

$$\mathscr{U}' = \mathscr{F}\left\{\mathscr{K}'\right\} \tag{2}$$

$\mathscr{K}'$ being the space of finite order derivatives of some exponentially bounded continuous function, i.e. for each $f \in \mathscr{K}'$ there is $b \geq 0$, $m \in N$ and a bounded, continuous function g such that

$$f(t) = (e^{b|t|}g(t))^{(m)}$$

However, the representation of tempered ultradistributions by analytical functions is the most convenient one for practical calculations. Define B_η as the complement in $\mathbb{C}$ of the strip $\Lambda_\eta = \{z : \text{Im}(z) \leq \eta\}$

$$B_\eta = \{z : \text{Im}(z) > \eta\} \tag{3}$$

and H_η as the set of functions which are holomorphic and of polynomial growth in B_η

$$\varphi(z) \in H_\eta \Longrightarrow \exists M, \alpha : |\varphi(z)| < M|z|^\alpha, \forall z \in B_\eta. \tag{4}$$

Let H_ω be the union of all such spaces

$$H_\omega = \bigcup_{\eta \geq 0} H_\eta \tag{5}$$

and in H_ω define the equivalence relation Ξ by

$$\varphi \overset{\Xi}{\simeq} \psi \text{ if } \varphi - \psi \text{ is a polynomial.}$$

Then, the space of tempered ultradistribution is

$$\mathscr{U}' = H_\omega / \Xi \tag{6}$$

and $[\phi(z)]$ will denote the equivalence class. The vectorial operations as well as derivation and multiplication by polynomials, defined on H_ω, are compatible with the equivalence relation and $\mathscr{U}'$ becomes a vector space with these operations.

The Schwartz space $\mathscr{S}'$ of tempered distributions may be identified with a subspace of $\mathscr{U}'$ by the Stieltges transform, that is, a linear mapping of $\mathscr{S}'$ on a subspace $\mathscr{U}'^*$ of $\mathscr{U}'$. Namely, given $\nu(x) \in \mathscr{S}'$

$$\varphi(z) = \frac{p(z)}{2\pi i} \int \frac{\nu(x)}{p(x)(x-z)} dx + P(z) \tag{7}$$

$[\varphi(z)] \in \mathscr{U}'$. Here $p(z)$ is a polynomial such that $\nu/p \sim O\left(x^{-1}\right)$ in the Silva-Cesàro sense and $P(z)$ is an arbitrary polynomial [4, 9, 10].

Operations on tempered ultradistributions $f \in \mathscr{U}'$ are performed using their analytical images $\varphi(z)$. For example f is integrable in $\mathbb{R}$ if there is an $y_0 \in \mathbb{R}$ and a $\varphi(z)$ in $[\varphi(z)] \in \mathscr{U}'$ such that $\varphi(x+iy_0) - \varphi(x-iy_0)$ is integrable in $\mathbb{R}$ in the sense of distributions. Then

$$\langle \varphi | g \rangle = \oint_{\Gamma_{y_0}} \varphi(z)\, g(z)\, dz \tag{8}$$

$\varphi \in \mathscr{U}'$, $g \in \mathscr{U}$ and the integral runs around the boundaries of the strip $\mathrm{Im}(z) \le y_0$.

An ultradistribution vanishes in an open set $A \in \mathbb{R}$ if $\varphi(x+iy) - \varphi(x-iy) \to 0$ for $x \in A$ when $y \to 0$ or, equivalently, if there is an analytical extension of φ to the vertical strip Re $z \in A$, being of at most polynomial growth there. The support of ν is the complement in $\mathbb{R}$ of the largest open set where ν vanishes.

All these notions are easily generalized to $\mathbb{R}^n$ [8, 11] by considering products of semiplans as in (3) and the corresponding polynomial bounds. For the equivalence relation Ξ one uses pseudopolynomials, that is, functions of the form

$$\sum_{j,k} \rho\left(z_1, \cdots, \overset{\wedge}{z_j}, \cdots, z_n\right) z_j^k,$$

$\overset{\wedge}{z_j}$ meaning that this variable is absent from the arguments of ρ.

An ultradistribution ν in $\mathbb{R}^n$ has compact support if there is a disk D such that any φ in $[\varphi(z)] \in \mathscr{U}'$ has an analytic extension to $(\mathbb{C}/D)^n$, being of at most polynomial growth there. Then the integral in (8) is around a closed contour containing the support of the ultradistribution. In particular for a tempered ultradistribution of compact support there is a unique representative function $\varphi^0(z)$ vanishing at ∞. Then from its Laurent expansion it follows

$$[\mu^0] = \left[\sum_{i=1}^{\infty} c_n \frac{1}{(z-a)^n}\right] \to -\sum_{i=1}^{\infty} (-1)^n \frac{2\pi i}{n!} c_n \delta^{(n)}(z-a) \tag{9}$$

showing that any ultradistribution of compact support has a representation as a series of multipoles [8]. The space of *tempered ultradistributions of compact support* will be denoted $\mathscr{U}_0'$. $\mathscr{U}_0'$ may be identified with $\mathscr{H}'$, the space of analytic functionals, dual of the space $\mathscr{H}$ of entire functions with the topology of uniform convergence on compacts of $\mathbb{C}$.

For all practical purposes an analytic representative $\varphi(z)$ in $\mathbb{C}\backslash\Lambda_\eta$ of a tempered ultradistribution corresponds (up to a common polynomial) to a pair of functions $\left[\varphi_+ (z), \varphi_- (z)\right]$ which are holomorphic and of polynomial growth respectively above and below some strip $\Lambda_\eta = \{z : \mathrm{Im}(z) \leq \eta\}$.

For comparison, it is perhaps useful to recall the corresponding analytic representation of distributions and tempered distributions. The Cauchy representation of a distribution of compact support $f \in \mathscr{D}' (\Omega)$

$$C(f)(z) = \frac{1}{2\pi i}\left\langle f(t), \frac{1}{t-z}\right\rangle \tag{10}$$

is analytic in $\mathbb{C}\backslash\mathbb{R}$ and f is recovered by

$$f(x) = \lim_{y\to 0^+} \{C(f)(x+iy) - C(f)(x-iy)\} \tag{11}$$

For a general distribution on an open interval I, the Cauchy representation cannot be defined as above. Nevertheless, for $\Lambda_I = \{I + i\mathbb{R}\}$, there is a function F analytic in $\Lambda_I\backslash I$ such that the jump operator as in (11) recovers the distribution. One has the isomorphism

$$\mathscr{D}'(I) \cong \mathscr{H}_{\mathscr{D}'}(\Lambda_I\backslash I) / \mathscr{H}_{\mathscr{D}'}(\Lambda_I)$$

$\mathscr{H}_{\mathscr{D}'}(\Lambda_I\backslash I)$ being the space of functions analytic in $\Lambda_I\backslash I$ and for which $\exists N$ such that

$$\sup |Imz|^N F(z) < \infty$$

on vertical strips contained on $\Lambda_I\backslash I$. The kernel of the homomorphism is the space of such functions which are analytic in the whole of Λ_I.

For $\mathscr{S}'(\mathbb{R})$ one has

$$\mathscr{S}'(\mathbb{R}) \cong \mathscr{H}_{\mathscr{S}'}(\mathbb{C}\backslash\mathbb{R}) / (\mathscr{H}_{\mathscr{S}'}(\mathbb{C}\backslash\mathbb{R}) \cap \mathscr{H}_{\mathscr{S}'}(\mathbb{C}))$$

$\mathscr{H}_{\mathscr{S}'}(\mathbb{C}\backslash\mathbb{R})$ being the functions analytic in $\mathbb{C}\backslash\mathbb{R}$ such that $\forall R > 0\ \exists m, N \in \mathbb{N}$

$$\sup_{z\in\Lambda_R\backslash\mathbb{R}} \frac{|Imz|^N |F(z)|}{(1+|Rez|)^m} < \infty$$

and $\mathscr{H}_{\mathscr{S}'}(\mathbb{C})$ all such functions which are analytic in the whole of $\mathbb{C}$.

2 Superprocesses on Ultradistributions

A superprocess describes the evolution of a population, without a fixed number of units, that evolves according to the laws of chance. It involves both propagation and branching of paths. They have been extensively used to model population dynamics and, more recently, as a tool for the construction of solutions of nonlinear partial differential equations [12–14].

Given a countable dense subset Q of $[0, \infty)$ and a countable dense subset F of a separable metric space E, the countable set

$$M_1 = \left\{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} : x_1 \cdots x_n \in F; \alpha_1 \cdots \alpha_n \in Q; n \geq 1 \right\} \tag{12}$$

is dense (in the topology of weak convergence) on the space $M(E)$ of finite Borel measures on E [14]. This is at the basis of the interpretation of the limits of evolving particle systems as measure-valued superprocesses. The representation of an evolving measure as a collection of measures with point support is useful for the construction of solutions of nonlinear partial differential equations as rescaling limits of measure-valued superprocesses.

However, as far as representations of solutions of nonlinear PDE's, superprocesses constructed in the space $M(E)$ of finite measures have serious limitations. The set of nonlinear terms that can be handled is limited (essentially to powers $u^{\alpha}(x)$ with $\alpha \leq 2$) and derivative interactions cannot be included as well. The first obvious generalization would be to construct superprocesses on distributions of point support, because any such distribution is a finite sum of deltas and their derivatives [15]. However, because in a general branching process the number of branches is not bounded, one really needs a framework that can handle arbitrary sums of deltas and their derivatives. This requirement leads naturally to the space of ultradistributions of compact support $\mathscr{U}_0'$, by virtue of the multipole expansion property (9) mentioned before.

The limitations of superprocesses on measures and the generalization to superprocesses on ultradistributions are described in detail in Refs. [16–18]. Here the main results will be summarized.

Let the underlying space of the superprocess be $\mathbb{R}^n$and denote by $\left(X_t, P_{0,\nu}\right)$ a branching stochastic process with values in $\mathscr{U}_0'$ and transition probability $P_{0,\nu}$ starting from time 0, $x \in \mathbb{R}^n$ and $\nu \in \mathscr{U}_0'$. The process is assumed to satisfy the *branching property,* that is, given $\nu = \nu_1 + \nu_2$

$$P_{0,\nu} = P_{0,\nu_1} * P_{0,\nu_2}. \tag{13}$$

After the branching $\left(X_t^1, P_{0,\nu_1}\right)$ and $\left(X_t^2, P_{0,\nu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $\left(X_t, P_{0,\nu}\right)$. In terms of the *transition operator* V_t operating on functions on $\mathscr{U}$ this would be

$$\langle V_t f, \nu_1 + \nu_2 \rangle = \langle V_t f, \nu_1 \rangle + \langle V_t f, \nu_2 \rangle \tag{14}$$

with V_t defined by $e^{-\langle V_t f, \nu \rangle} = P_{0,\nu} e^{-\langle f, X_t \rangle}$ or

$$\langle V_t f, \nu \rangle = -\log P_{0,\nu} e^{-\langle f, X_t \rangle} \tag{15}$$

$f \in \mathscr{U}, \nu \in \mathscr{U}_0'$.

In $M = [0, \infty) \times \mathbb{R}^n$ consider an open regular set $Q \subset M$ and the associated exit process $\xi = (\xi_t, \Pi_{0,x})$ with parameter k defining the lifetime. The process starts from $x \in \mathbb{R}^n$ carrying along an ultradistribution in $\mathcal{U}_0'$ indexed by the path coordinate. At each branching point (ruled by $\Pi_{0,x}$) of the ξ_t-process there is a transition ruled by a P probability in $\mathcal{U}_0'$ leading to one or more elements in $\mathcal{U}_0'$. These $\mathcal{U}_0'$ elements are then carried along by the new paths of the ξ_t-process. By construction, in each path, the process never leaves $\mathcal{U}_0'$. The whole process stops at the boundary ∂Q, finally defining an exit process $(X_Q, P_{0,\nu})$ on $\mathcal{U}_0'$. If the initial ν is δ_x and $f \in \mathcal{U}$ a function on ∂Q one writes

$$u(x) = \langle V_Q f, \delta_x \rangle = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \tag{16}$$

$\langle f, X_Q \rangle$ being computed on the (space-time) boundary with the exit ultradistribution generated by the process.

The connection with nonlinear PDE's is established by defining the whole process to be a $(\xi, \psi)-$*superprocess* if $u(x)$ satisfies the equation

$$u + G_Q \psi(u) = K_Q f \tag{17}$$

where G_Q is the Green operator,

$$G_Q f(0, x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s)\, ds \tag{18}$$

and K_Q the Poisson operator

$$K_Q f(x) = \Pi_{0,x} 1_{\tau < \infty} f(\xi_\tau) \tag{19}$$

$\psi(u)$ means $\psi(0, x; u(0, x))$ and τ is the first exit time from Q, Eq. (17) being recognized as the integral version of a nonlinear partial differential equation with the Green operator determined by the linear part of the equation and $\psi(u)$ by the nonlinear terms. If the equation does not possess a natural Poisson clock for the branching one has to introduce an artificial lifetime for the particles in the process (e^{-k}), which in the end must vanish ($k \to \infty$) through a rescaling method.

The superprocess is constructed as follows: Let $\varphi(s, x; z)$ be the branching function at time s and point x. Then denoting $P_{0,x} e^{-\langle f, X_Q \rangle}$ as $e^{-w(0,x)}$ one has

$$e^{-w(0,x)} = \Pi_{0,x} \left[e^{-f(\tau, \xi_\tau)} + k \int_0^\tau ds \left[\varphi\left(s, \xi_s; e^{-w(\tau - s, \xi_s)}\right) - e^{-w(\tau - s, \xi_s)} \right] \right] \tag{20}$$

where τ is the first exit time from Q and $f(\tau, \xi_\tau) = \langle f, X_Q \rangle$ is computed with the exit boundary ultradistribution. Existence of $\langle f, X_Q \rangle$ and hence of $e^{-w(0,x)}$ is insured if $f \in \mathcal{U}$ and the branching function is such that the exit $X_Q \in \mathcal{U}_0'$.

Equation (17) is then obtained by a limiting process. Let in (20) replace $w(0, x)$ by $\beta w_\beta(0, x)$ and f by βf. In a branching particle interpretation of the superprocess β may be interpreted as the mass of the particles and when the $\mathcal{U}_0'$-valued process

$X_Q \to \beta X_Q$ then $P_\mu \to P_{\frac{\mu}{\beta}}$.

$$e^{-\beta w(0,x)} = \Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta \int_0^\tau ds \left[\varphi_\beta\left(s, \xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right] \tag{21}$$

With

$$u_\beta^{(1)} = \left(1 - e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta^{(1)} = \left(1 - e^{-\beta f}\right)/\beta$$

or

$$u_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta w_\beta} - e^{-\beta w_\beta}\right) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta f} - e^{-\beta f}\right)$$

and

$$\psi_\beta^{(i)}\left(0, x; u_\beta^{(i)}\right) = \frac{k_\beta}{\beta}\left(\varphi\left(0, x; 1 - \beta u_\beta^{(i)}\right) - 1 + \beta u_\beta^{(i)}\right)$$

$u_\beta^{(i)} \to w_\beta$ and $f_\beta^{(i)} \to f$ when $\beta \to 0$, Eq. (17) being obtained in this limit.

Let $z = e^{-\beta w(\tau-s,\xi_s)} = P_{0,x} e^{-\langle \beta f, X\rangle}$. For the branching function $\varphi(s, x; z)$, in contrast with the measure-valued case, in addition to branchings of deltas into other deltas one also has:

(1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f, \delta_x\rangle} = e^{\beta f(x)} \to e^{\langle \beta f, -\delta_x\rangle} = e^{-\beta f(x)}$$

which corresponds to

$$z \to \frac{1}{z}$$

and

(2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle \beta f, \delta_x\rangle} = e^{\beta f(x)} \to e^{\left\langle \beta f, \pm\delta'_x\right\rangle} = e^{\mp\beta f'(x)}$$

which corresponds to

$$z \to e^{\mp\partial_x \log z}.$$

Case (1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and the second to superprocesses in $\mathscr{U}'_0$.

Existence of the superprocess is existence of a unique solution for the Eq. (21) and its rescaling limit. It will depend on the appropriate choice of the branching function $\varphi(s, y; z)$. Suppose that such a ultradistribution branching is specified. Associated to the ultradistribution superprocess Γ with branching function φ there is an *enveloping measure superprocess* $\widetilde{\Gamma}$ with branching function $\widetilde{\varphi}$ that has the same branching topology as Γ but without any derivative change in the original delta measure nor on its sign. General existence conditions for measure-valued superprocesses have been found in the past [19–21]. Namely $\widetilde{\varphi}$ should have the form

$$\widetilde{\varphi}(s, y : z) = -b(s, y)\, z - c(s, y)\, z^2 + \int_0^\infty \left(e^{-\lambda z} + \lambda z - 1\right) n(s, y; d\lambda) . \quad (22)$$

Suppose that the branching $\widetilde{\varphi}$ for the process $\widetilde{\Gamma}$ is of the form (22). This insures almost sure existence of $e^{-\langle f, \widetilde{X} \rangle}$, $\widetilde{X}$ being the exit measure generated by the $\widetilde{\Gamma}$ process. Then, for the corresponding ultradistribution Γ superprocess one has

$$\langle f, X \rangle \leq M \int_{\partial Q} \sum_{n=0} \left| f^{(n)} \right| .$$

and the following result is obtained [18]:

Proposition 1 *A $\mathscr{U}_0'$ ultradistribution-valued exit superprocess Γ exists if the branching function $\widetilde{\varphi}$ of the associated enveloping exit measure process $\widetilde{\Gamma}$ is as in Eq. (22) and the boundary function f is such that the integral over the exit boundary of $\Sigma_n \left| f^{(n)} \right|$ is finite.*

3 Nonlinear Differential Problems, Distributions and Superprocesses

3.1 Nonlinear Theories of Generalized Functions

The general treatment of derivatives in distribution theory provides a powerful symbolic calculus for linear differential problems. However, when modeling natural phenomena, the most interesting problems are very often nonlinear. Of course, if the solutions of a nonlinear problem are known to be smooth, there is no problem because many nice algebras can be found in the domain of smooth functions. However in cases where the solutions are singular or the sources are concentrated (point, line or sheet charges, for example) the application of distribution theory becomes problematic. The problem was identified long ago by Schwartz in his impossibility result [22] which implies that $\mathscr{D}'$ cannot be linearly embedded into a differential algebra with the unit function as unit, with a derivation D satisfying Leibnitz rule, $D|_{C^1(\mathbb{R})}$ being the usual derivative and containing $C(\mathbb{R})$ as a subalgebra.

The need to deal with systems of PDE's without classical solutions, shock waves and singular sources led to many attempts to embed the space of distributions into an algebra, of course violating in a minimal way, one or more of the conditions of Schwartz result. They may be broadly classified into sequential and complex analysis methods and the type of products that are defined are either *intrinsic*, in the sense that the product of two distributions is still a distribution, or products which may lead to generalized functions different from distributions. For reviews see [23] and

[1]. Colombeau's algebra of generalized functions, which is of the second type, has found a widespread use in the applications. It has been applied to singular shock problems [24, 25], to symmetric hyperbolic systems with discontinuous coefficients [26], to equations with distributions as initial conditions [27], to generalized stochastic processes [28, 29], to general relativity [30, 31], etc.

Colombeau's approach [30, 32–34], in line with the sequential approach to distribution theory [35], considers a distribution as an equivalence class of weakly converging sequences of smooth functions and choosing the appropriate quotient constructs a differential algebra $\mathscr{A}$ with a product $\circ$ that satisfies the desired conditions except that instead of having $C(\mathbb{R})$ as a subalgebra it is $\circ|_{C^\infty \times C^\infty}$ that corresponds to the usual pointwise product of smooth functions.

Given some $\Phi \in \mathscr{D}(\mathbb{R}^n)$ with integral one, a family of functions

$$\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\Phi\left(\frac{x}{\varepsilon}\right) \tag{23}$$

has the property $\Phi_\varepsilon \to \delta$ in $\mathscr{D}'$ as $\varepsilon \to 0$. Φ is called a *molifier*. Convolution of $f \in C^\infty(\mathbb{R}^n)$ with Φ_ε yields a family

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n}\int f(y)\Phi\left(\frac{y-x}{\varepsilon}\right)d^n y. \tag{24}$$

of smooth functions that converge to f in $\mathscr{D}'$ as ε tends to zero. Using a Taylor series expansion to compare the difference between two such sequences they are said to be equivalent if they differ by a *negligible function*. To define a differential algebra $\mathscr{G}$ as the quotient by negligible functions one needs to restrict the set of functions to *moderate functions*. The canonical choice introduces a grading on the space of molifiers

$$\begin{aligned} \mathscr{A}_0 &:= \{\Phi \in \mathscr{D}(\mathbb{R}^n)\ :\ \int \Phi(x)\,dx = 1\} \\ \mathscr{A}_q &:= \{\Phi \in \mathscr{A}_0\ :\ \int \Phi(x)x^\alpha\,dx = 0\,,\ 1 \le |\alpha| \le q\}\ \ (q \in \mathbb{N}) \end{aligned} \tag{25}$$

taking the basic function space to be

$$\mathscr{E}^e := \{f\ :\ \mathscr{A}_0 \times \mathbb{R}^n \to \mathbb{R}^n,\ \ f \text{ smooth }\} \tag{26}$$

and defines as *moderate functions*

$$\begin{aligned} \mathscr{E}^e_M(\mathbb{R}^n) := \ &\{f \in \mathscr{E}^e : \forall K \subset\subset \mathbb{R}^n\ \forall\alpha \in \mathbb{N}_0^n\ \exists p \in \mathbb{N}_0\ \forall\Phi \in \mathscr{A}_p : \\ &\sup_{x\in K}|D^\alpha f(\Phi_\varepsilon, x)| = O(\varepsilon^{-p}) \text{ as } \varepsilon \to 0\} \end{aligned} \tag{27}$$

and *negligible functions*

$$\mathscr{N}^e(\mathbb{R}^n) := \{f \in \mathscr{E}^e(\Omega) : \forall K \subset\subset \mathbb{R}^n \; \forall \alpha \in \mathbb{N}_0^n \; \forall p \in \mathbb{N}_0 \; \exists q \; \forall \Phi \in \mathscr{A}_q : \sup_{x \in K} |D^\alpha f(\Phi_\varepsilon, x)| = O(\varepsilon^p) \text{ as } \varepsilon \to 0\} \quad (28)$$

Then the algebra is

$$\mathscr{G}^e(\mathbb{R}^n) := \mathscr{E}^e_M(\mathbb{R}^n) \,/\, \mathscr{N}^e(\mathbb{R}^n)\,. \quad (29)$$

with the distributions being embedded into $\mathscr{G}^e$ by convolution with the molifiers

$$\iota(T) = [T * \Phi] \quad (30)$$

$\mathscr{G}^e$ is a commutative differential algebra where $\mathscr{D}'$ is embedded as a linear space but not as an algebra. The results of multiplication in this algebra may frequently be interpreted in terms of distributions by using the concept of *association*. A generalized function f is said to be associated to a distribution $T \in \mathscr{D}'$ if for one (hence any) representative $\{f_\varepsilon\}$ we have

$$\forall \phi \in \mathscr{D}, \quad \lim_{\varepsilon \to 0} \int f_\varepsilon(x)\phi(x)d^n x = \langle T, \phi \rangle \quad (31)$$

Not all elements of $\mathscr{G}^e$ are associated to distributions. Association is an equivalence relation which respects addition and differentiation. It also respects multiplication by smooth functions but by the Schwartz impossibility result cannot respect multiplication in general.

For hyperbolic systems with rapidly growing nonlinear terms or with time-dependent nonregular coefficient the need arises of going beyond the nonlinear theory of generalized functions to a nonlinear theory of ultradistributions. After several other attempts, in a recent paper, Debrouwere, Vernaeve and Vindas have achieved an embedding of ultradistributions into differential algebras [36] of the same type as Colombeau algebra.

When applying these differential algebras to physical problems and in addition to the fact that many potentially interesting products in the Colombeau algebra cannot be associated to distributions, there is a problem of interpretation of the results because of the departure from the notion of pointwise multiplication that is behind the derivation of the physical equations. Instead the multiplication is a kind of tensor product multiplication. For example, if a weak solution of an equation behaves locally like a delta and if the equation has a term u^2 one has locally a δ^2, but in Colombeau-type algebras deltas are multiplied as

$$(\delta, \varphi)\,(\delta, \varphi)$$

which, even in a sequential approach, is very far from a pointwise multiplication and leads to a nonlinear functional

$$\left(\delta^2, \varphi\right) = \varphi^2(0)$$

therefore not a distribution.

3.2 Superprocesses and Nonlinear Differential Equations

The extension of superprocess from measures to ultradistributions, discussed in Sect. 2 allows to deal with a large class of nonlinear differential equations. When solutions are constructed by superprocesses, nonlinear terms do not raise any special problem. Let u be the solution of some equation. Existence of $n-$powers of u in the equation means that there is a splitting of the stochastic path into n paths, a derivative means a transition from $\delta^{(n)}$ to $\delta^{(n+1)}$, etc. It all boils down to the choice of the appropriate branching function and rescaling limit. No nonlinear operations on distributions are required. In the end the boundary process, the ultradistribution X_Q, need not have smooth properties. The only limitation to insure existence of $\left\langle f, X_Q\right\rangle$ is a sufficiently smooth boundary condition.

How superprocesses on ultradistributions provide solutions to nonlinear differential equations, which cannot be obtained by superprocesses on measures and avoid any explicit use of the nonlinear theory of generalized functions, is illustrated by the following results, proved in [18].

Proposition 2 *The superprocess with branching function*

$$\varphi(0, x; z) = p_1 e^{\partial_x \log z} + p_2 e^{-\partial_x \log z} + p_3 z^2$$

provides a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2u^2 - \frac{1}{2}\left(\partial_x u\right)^2$$

whenever the boundary function $u|_{\partial Q}$ *satisfies the condition of Proposition 1.*

Proposition 3 *The superprocess associated to the branching function*

$$\varphi(0, x; z) = p_1 z^2 + p_2 \frac{1}{z}$$

provides a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^3$$

whenever the boundary function $u|_{\partial Q}$ *satisfies the condition of Proposition 1.*

Here the construction of solutions of nonlinear differential equations by stochastic processes (*stochastic solutions*) has been discussed using superprocesses. There is another method, which has been called the McKean method, which may also deal with nonlinear terms and derivative interactions [37, 38]. Refer to [16] for a comparison of the two methods. In general the superprocess method seems more appropriate for Dirichlet boundary conditions and McKean's for Cauchy conditions. A limitation of this last method, in some cases, is that only finite time solutions may be constructed.

References

1. Hoskins, R.F., Sousa Pinto, J.: Theories of Generalized Functions: Distributions, Ultradistributions and Other Generalized Functions, 2nd edn. Woodhead Publ, Philadelphia (2011)
2. Björck, G.: Linear partial differential operators and generalized distributions. Ark. Mat. **6**, 351–407 (1966)
3. Roumieu, C.: Sur quelques extensions de la notion de distribution. Ann. Sci. École Norm. Sup. Sér. **3**(77), 41–121 (1960)
4. Pilipović, S.: Beurling-Gevrey tempered ultradistributions as boundary values. Portugaliae Math. **48**, 483–504 (1991)
5. Komatsu, H.: Ultradistributions I. Structure theorems and a characterization. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **20**, 25–105 (1973)
6. Debrouwere, A., Vindas, J.: On the non-triviality of certain spaces of analytic functions. Hyperfunctions and ultrahyperfunctions of fast growth. arXiv:1608.07859
7. Sebastião e Silva, J.: Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel. Math. Annalen **136**, 58–96 (1958)
8. Sebastião e Silva, J.: Les séries de multip ôles des physiciens et la théorie des ultradistributions. Math. Annalen **174**, 109–142 (1967)
9. Sebastião e Silva, J.: Integrals and order of distributions. Inst. Gulbekian de Ciencia, Lisboa **35** (1964)
10. Estrada, R.: The Cesàro behavior of distributions. Proc. R. Soc. Lond. A **454**, 2425–2443 (1998)
11. Hasumi, M.: Note on the n-dimensional tempered ultradistributions. Tôhoku Math. J. **13**, 94–104 (1960)
12. Dynkin, E.B.: Diffusions, Superdiffusions and Partial Differential Equations. AMS Colloquium Pubs, Providence (2002)
13. Dynkin, E.B.: Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations. AMS, Providence (2004)
14. Li, Z.: Measure-Valued Branching Markov Processes. Springer, Berlin (2011)
15. Stein, E.M., Shakarchi, R.: Princeton Lectures in Analysis IV, Functional Analysis: Introduction to Further Topics in Analysis. Princeton Universiyt Press, Princeton (2011)
16. Vilela Mendes, R.: Stochastic solutions of nonlinear PDE's: McKean versus superprocesses. In: Leoncini, X., Leonetti, M. (eds.) Chaos, Complexity and Transport, pp. 143–154. World Scientific (2012)
17. Vilela Mendes, R.: Stochastic solutions of nonlinear pde's and an extension of superprocesses. In: Stochastic and Infinite dimensional Analysis, Trends in Math., pp. 243–261. Birkhäuser (2016)
18. Vilela Mendes, R.: Superprocesses on ultradistributions. Stochastics (2016). doi:10.1080/17442508.2016.1269768
19. Dynkin, E.B.: Branching exit Markov systems snd superprocesses. Ann. Probab. **29**, 1833–1858 (2001)

20. Fitzsimmons, P.J.: Construction and regularity of measure-valued Markov branching processes. Isr. J. Math. **64**, 337–361 (1988)
21. Fitzsimmons, P.J.: On the Martingale problem for measure-valued Markov branching processes. In: Seminar on Stochastic Processes 1991, Progress in Probability, vol. 29, pp. 39–51. Springer, Boston (1992)
22. Schwartz, L.: Sur l'impossibilité de la multiplication des distributions. C.R. Acad. Sci. Paris **239**, 847–848 (1954)
23. Li, C.K.: A review on the products of distributions. In: Kenan Tas et al. (ed.) Mathematical Methods in Engineering. Springer (2007)
24. Keyfitz, B.L., Kranzer, H.C.: A strictly hyperbolic system of conservation laws admitting singular shocks. In: Nonlinear Evolution Equations that Change Type, pp. 107–125. Springer (1990)
25. Le Floch, Ph.: Shock waves for nonlinear hyperbolic systems in nonconservative form. Inst. for Math. and Applications - Univ. Minnesotta, preprint , vol. 593 (1989)
26. Lafon, F., Oberguggenberger, M.: Generalized solutions to symmetric hyperbolic systems with discontinuous coefficients: the multidimensional case. J. Math. Anal. Appl. **160**, 93–106 (1991)
27. Langlais, M.: Generalized functions solutions of monotone and semilinear parabolic equations. Monatshefte Math. **110**, 117–136 (1990)
28. Mirkov, R., Pilipovic, S., Selesi, D.: Generalized stochastic processes in algebras of generalized functions. J. Math. Anal. Appl. **353**, 260–270 (2009)
29. Oberguggenberger, M.: Generalized functions and stochastic processes. Prog. Probab. **36**, 215–229 (1995)
30. Grosser, M., Kunzinger, M., Oberguggenberger, M., Steinbauer, R.: Geometric Theory of Generalized Functions Mathematics and its Applications, vol. 537. Kluwer (2001)
31. Steinbauer, R., Vickers, J.A.: The use of generalized functions and distributions in general relativity. Class. Q. Grav. **23**, R91–R114 (2006)
32. Colombeau, J.F.: New Generalized Functions and Multiplication of Distributions. North-Holland-Elsevier (1984)
33. Colombeau, J.F.: Nonlinear generalized functions: their origin, some developments and recent advances. São Paulo J. Math. Sci. **7**, 201–239 (2013)
34. Aragona, J., Juriaans, S.O., Colombeau, J.F.: Locally convex topological algebras of generalized functions: compactness and nuclearity in a nonlinear context. Trans. Am. Math. Soc. **367**, 5399–5414 (2015)
35. Mikusinski, J.: Sur la méthode de gé neralisation de M. Laurent Schwartz sur la convergence faible. Fundam. Math. **35**, 235–239 (1948)
36. Debrouwere, A., Vernaeve, H., Vindas, J.: Optimal embeddings of ultradistributions into differential algebras. arXiv:1601.03512
37. Vilela Mendes, R.: Stochastic solutions of some nonlinear partial differential equations. Stochastics **81**, 279–297 (2009)
38. Vilela Mendes, R.: Poisson-Vlasov in a strong magnetic field: A stochastic solution approach. J. Math. Phys. **51**, 043101 (2010)