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EXTENDED OBJECTS AS ULTRADISTRIBUTION-MARKED CONFIGURATION SPACES

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Configuration spaces of point particles with and without internal quantum numbers (marks) were studies earlier as a framework for the description of the dynamics of multiagent systems. Here, we propose to extend this framework to configuration spaces of extended objects of any shape. This is attained by identifying the shape with the support of an ultradistribution of compact support. Then, by using the multipole expansion property of these ultradistributions, we conclude that the configuration spaces of extended objects are a simple extension of the marked configuration spaces.

Dedicated to the memory of Yuri Kondratiev
who always was a source of enlightenment and inspiration

1. Introduction

Configuration spaces are a nice framework to describe complex multi-agent systems in statistical mechanics, biology, economy, ecology, etc. This was a field where Yuri Kondratiev made remarkable contributions. A mathematical theory of configuration spaces of non-overlapping point particles, idealized as delta function point measures, was developed in [1, 2]. Analysis and geometry of these configuration spaces as well as an harmonic analysis theory [3] have also been extensively developed. And, for particles with internal quantum numbers, marked configuration spaces were developed in [4, 5]. Application of the theory to several stochastic dynamical systems was worked out [6–13].

In some systems of statistical physics, an idealization of the molecules as point particles may be a reasonable approximation. However, in many other systems, the extended nature of the complex system components has to be taken into account. Goldin [14, 15] has suggested the use of configuration spaces involving derivatives of deltas to describe multipoles, quadropoles, etc. and also that the study of systems of loops and ribbons would be of interest. On the other hand Ismagilov [16] considered configuration spaces of compact sets. However, except for Ismagilov's compact set theory, no detailed theory of configuration spaces of extended systems has been developed. The purpose of the present work is to develop a framework that may deal in a unified manner with configuration spaces of both point-like and extended entities. The framework is so constructed to profit from the point-like analysis, in the already developed theory, while also taking into account the extended nature of the components in the system. The basic mathematical entities to be used are ultradistributions of compact support, which may always be coded as a series of multipoles based at a single point.

In Section 2 some results on the theory of tempered ultradistributions and ultradistributions of compact support are reviewed, which are important for the formulation of extended objects as well defined mathematical entities. In Section 3, some examples are worked out and a general result is obtained. Finally in Section 4, the analysis and geometry of these ultradistribution configuration spaces is developed. This section makes extensive use of the

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theory of marked configuration spaces, developed by other authors [4, 5]. In the Appendix a graphical illustration (adapted from [17, 18]) is included, as a mnemonic for the several spaces of test functions and distributions that are useful in the applications.

2. Silva's Tempered Ultradistributions. Compact Support

The space of *ultradistributions* $\mathcal{Z}'$ is the topological dual of $\mathcal{Z}$, a space of test functions for which the Fourier transform is in $\mathcal{D}$, the space of infinitely differentiable functions of compact support. Equivalently, a function φ belongs to $\mathcal{Z}$ if it can be extended onto the complex plane as an entire function satisfying a set of inequalities

$$\left|z^k \varphi(z)\right| \leq C_k e^{a|\mathrm{Im}(z)|}$$

with $a > 0$ and $k = 0, 1, 2, \ldots$. Convergence in $\mathcal{Z}$ corresponds to the convergence of the Fourier transforms in $\mathcal{D}$. $\mathcal{Z}$ is a dense subspace of $\mathcal{S}$ and the fact that the Fourier transform of each element of $\mathcal{Z}$ has compact support, endows ultradistributions with a rich analytical structure, which makes these generalized functions more convenient than distributions in many applications. An important dense subspace of $\mathcal{Z}'$ is Silva space of *tempered ultradistributions* $\mathcal{U}'$ which may be characterized as Fourier transforms of distributions of exponential type $\mathcal{K}' \subset \mathcal{S}'$, that is, distributions which locally are $\mu(x) = D^k \left(e^{a|x|} f\right)$, f bounded and continuous [17, 19, 20].

Let $\mathcal{U}'(\mathbb{C})$ be the topological dual of $\mathcal{U}(\mathbb{C})$, the space of entire functions rapidly decreasing on horizontal strips of the complex plane, topologized by the family of norms

$$\|\phi\|_{\mathcal{U},k} = \sup_{|\mathrm{Im} z|<k} (1+|z|)^k |\phi(z)|,$$

$\mathcal{U}(\mathbb{C})$ is a Fréchet space. The Fourier transform $\mathcal{F}$ is an isomorphism of the space $\mathcal{U}(\mathbb{C})$ onto $\mathcal{K}(\mathbb{R})$, the space of smooth functions with exponential decay, topologized by

$$\|\phi\|_{\mathcal{K},k} = \max_{0\leq m\leq k} \sup_{x\in\mathbb{R}} e^{k|x|} \left|\phi^{(m)}(x)\right|.$$

Defining, by duality, the Fourier transform on $\mathcal{K}'(\mathbb{R})$

$$(\mathcal{F}(f), \phi) = (f, \mathcal{F}(\phi))$$

$f \in \mathcal{K}'(\mathbb{R}), \phi \in \mathcal{U}(\mathbb{C})$, it follows that

$$\mathcal{F}: \mathcal{K}'(\mathbb{R}) \to \mathcal{U}'(\mathbb{C})$$

is also an isomorphism.

Using the Cauchy formula, it follows that ultradistributions on $\mathcal{U}'(\mathbb{C}) \supset \mathcal{S}'(\mathbb{R})$ have an analytic representation by (a pair of) functions that are analytic outside an horizontal strip around the real axis. Let Λ_b denote the open horizontal strip of size b on each side of the real axis, $\mathcal{M}\left(\mathbb{C}\backslash\overline{\Lambda_b}\right)$ the space of functions that are analytic on $\mathbb{C}\backslash\overline{\Lambda_b}$ and polynomially bounded on $\mathbb{C}\backslash\Lambda_b$ and $\mathcal{P}$ the space of polynomials. Then, taking into account all horizontal strips away from the real axis, one has the isomorphism

$$\mathcal{U}'(\mathbb{C}) \simeq \mathcal{M}(\mathbb{C}\backslash\mathbb{R})/\mathcal{P}$$

implemented by the mapping

$$(\mu(F),\phi) = -\oint_{\Gamma_b} F(z)\phi(z)dz$$

$\mu(F) \in \mathcal{U}'(\mathbb{C}),\ F \in \mathcal{M}\left(\mathbb{C}\backslash\overline{\Lambda_b}\right)$ and $\phi \in \mathcal{U}(\mathbb{C})$, the integral being taken counterclockwise around the boundaries of the Λ_b strip. Distributions in $\mathcal{S}'$ correspond to the case where the strip collapses to the real axis and more general tempered ultradistributions to the case $b > 0$.

Analytic functions in $\mathcal{M}\left(\mathbb{C}\backslash\overline{\Lambda_b}\right)$ represent the same ultradistribution if they differ by a polynomial. This equivalence relation $\div$ turns out to be very useful to obtain a unique characterization of ultradistributions of compact support.

Operations on tempered ultradistributions are performed using their analytical images $F(z)$. For example $\mu(F)$ is integrable in $\mathbb{R}$ if there is an $y_0 \in \mathbb{R}$ and a $F(z)$ in $\mathcal{M}\left(\mathbb{C}\backslash\overline{\Lambda_b}\right)$ such that $F\left(x+iy_0\right) - F\left(x-iy_0\right)$ is integrable in $\mathbb{R}$ in the sense of distributions. Then

$$\langle\mu(F)|\phi\rangle = -\oint_{\Gamma_{y_0}} F(z)\phi(z)dz \tag{1}$$

$\mu(F) \in \mathcal{U}'(\mathbb{C}),\ \phi \in \mathcal{U}(\mathbb{C})$ and the integral runs around the boundaries of the strip $\mathrm{Im}(z) \leq y_0$.

When $b = 0$, the functions $F(z)$ represent tempered distributions $f(x)$ in $\mathcal{S}'$, $F(z)$ and $f(x)$ being related by the generalized Stieltjes transform

$$F(z) = \frac{p(z)}{2\pi i}\int_{\mathbb{R}} \frac{f(x)}{p(x)\,(t-z)}dx + P(z)$$

$p(z)$ being a polynomial without real zeros and $P(z)$ an arbitrary polynomial. The inverse Stieltjes transform in $\mathcal{S}'$ is

$$f(x) = F(x+i0^+) - F(x-i0^-)$$

An ultradistribution $\mu(F) \in \mathcal{U}'(\mathbb{C})$ is said to vanish on an open set $\Omega \subset \mathbb{R}$ if $F \in \mathcal{M}\left(\mathbb{C}\backslash\overline{\Lambda_b}\right)$ has a analytic continuation in $\{z \in \mathbb{C}\colon \mathrm{Re} z \in \Omega\}$. The support of $\mu(F)$ is the complement in $\mathbb{R}$ of the largest open set where $\mu(F)$ vanishes. As a result, an ultradistributions of compact support has a unique representation by a function, analytic outside some ball, that vanishes at infinity. Then, using the Laurent series and the analytic representation of the delta derivatives, it follows that ultradistributions of compact support $(\mathcal{U}_0')$, are represented by series of multipoles [20]

$$\mu(F) = \sum_{n=0}^{\infty} a_n \delta^{(n)}(x-x_0).$$

All these notions are generalized to $\mathbb{R}^n$ [20, 21] by considering products of semiplans and the corresponding polynomial bounds. For the equivalence relation $\div$ one uses pseudopolynomials, that is, functions of the form

$$\sum_{j,k} \rho\big(z_1,\ldots,\overset{\wedge}{z_j},\ldots,z_n\big) z_j^k,$$

the $\rho' s$ being holomorphic functions of polynomial growth and $\overset{\wedge}{z_j}$ meaning that this variable is absent from the arguments of ρ.

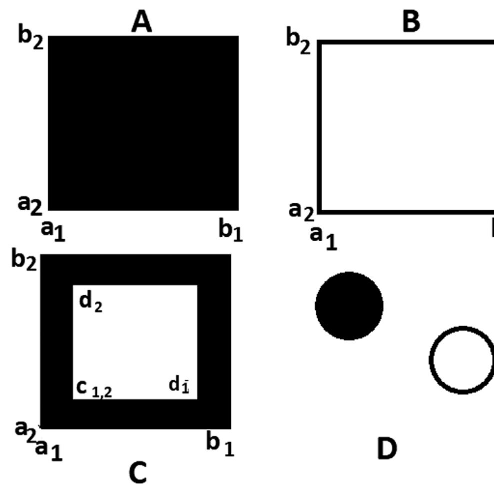

Fig. 1. Extended objects: Square, loop, strip, ball and shell to be represented as ultradistributions of compact support.

An ultradistribution in $\mathbb{R}^n$ has compact support if there is a disk D such that any F representing $\mu(F) \in \mathcal{U}'(\mathbb{C})$ has an analytic extension to $(\mathbb{C}/D)^n$. Then the integral in (1) is around a closed contour containing the support of the ultradistribution. And for ultradistributions of compact support there is also a representation as a series of multipoles,

$$\nu(x) = \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} p_{r_1,\ldots,r_n} \delta^{(r_1,\ldots,r_n)}(x-a).$$

with the $p_{r_1,\ldots,r_n}$ being constants and the $\delta^{(r_1,\ldots,r_n)}$'s derivatives of the delta distribution.

A few analytic representations (as ultradistributions) that will be useful for the examples:

$$\delta(x-x_0) \to -\frac{1}{2\pi i}\frac{1}{z-x_0},$$

$$\delta^{(n)}(x-x_0) \to (-1)^{n+1}\frac{n!}{2\pi i}\frac{1}{(z-x_0)^{n+1}}.$$

For the Heaviside function H, $DH(x-x_0) = \delta(x-x_0)$, hence

$$H(x-x_0) \to -\frac{1}{2\pi i}\log(x_0 - z)$$

3. Extended Objects as Ultradistributions

To obtain an intuitive feeling for what an ultradistribution of compact support is and how they may code extended objects, several examples, as depicted in Fig. 1, will be studied. Of particular relevance is how they may be represented as multipole series.

Consider, in $\mathbb{R}$, a function f_1 that is equal to $K > 0$ in the interval $[a, b]$, $b > a$, and zero outside the interval,

$$f_1(x) = K\{H(x-a) - H(x-b)\}, \tag{2}$$

H being the Heaviside function. As an ultradistribution, f_1 is the equivalence class (up to polynomials) of the analytic function

$$F_1(z) = \frac{K}{2\pi i} \log \frac{b-z}{a-z}. \tag{3}$$

Because $f_1(x)$ has compact support, it has a unique representation by an analytic function vanishing at infinity. For any ultradistribution of compact support, such a representation is obtained by writing the Laurent series of any function in the equivalence class $[F_1(z)]$ and discarding the positive powers,

$$F_1(z) = \sum_{n\geq 1} \frac{c_{-n}}{(z-\alpha)^n},$$

α being the expansion point. The c_{-n} coefficients may be obtained recursively from the behavior of the function at infinity

$$c_{-n} = \lim_{z\to\infty} z^n \left(F_1(z) - \sum_{k=0}^{n-1} \frac{c_{-k}}{(z-\alpha)^k} \right).$$

With

$$\alpha = \frac{a+b}{2}$$

one obtains for the function in (3)

$$F_1(z) = -\frac{2K}{2\pi i} \sum_{k\geq 0} \left(\frac{b-a}{2} \right)^{2k+1} \frac{1}{\left(z - \frac{a+b}{2}\right)^{2k+1}}. \tag{4}$$

The Laurent series is convergent in an annulus around the expansion point between r and R with

$$r = \lim_{n\to\infty} \sup |c_{-n}|^{\frac{1}{n}},$$

$$\frac{1}{R} = \lim_{n\to\infty} \sup |c_n|^{\frac{1}{n}}$$

leading to

$$r = \frac{b-a}{2}.$$

This means that, by choosing $\frac{a+b}{2}$ as the expansion point, the series in (4) is convergent everywhere outside the support of f_1.

Taking into account the fact that, as an ultradistribution,

$$\delta^{(n)}(x-x_0) = (-1)^{n+1} \frac{n!}{2\pi i} \frac{1}{(z-x_0)^{n+1}} \tag{5}$$

one finally obtains the following multipole expansion for the function f_1

$$f_1(x) = K(b-a) \sum_{k\geq 0} \left(\frac{b-a}{2} \right)^{2k} \frac{1}{(2k+1)(2k)!} \delta^{(2k)} \left(x - \frac{a+b}{2} \right).$$

In conclusion: an object of intensity K and extension $[a, b]$ is represented by a point in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$, the first entry being a point x_0 in the support and the others the coefficients of $\delta^{(n)}(x - x_0,\ n = 0, 1, 2, 3, \ldots,$ in the multipole expansion. The same applies to any compact support object in $\mathbb{R}$. The only restriction in the coefficients in $\mathbb{R}^{\mathbb{N}}$ is that, in the corresponding Laurent series,

$$\lim_{n \to \infty} \sup |c_{-n}|^{\frac{1}{n}} < \infty. \tag{6}$$

For a solid cube in $\mathbb{R}^n$ the result generalizes to

$$f_N(\overrightarrow{x}) = K \prod_{i=1}^{n} (b_i - a_i) \sum_{k_i \geq 0} \left(\frac{b_i - a_i}{2} \right)^{2k_i} \frac{\delta^{(2k_i)} \left(x_i - \frac{a_i + b_i}{2} \right)}{(2k_i + 1)(2k_i)!}$$

the defining coefficients of the cube being now in $\left(\mathbb{R} \times \mathbb{R}^{\mathbb{N}}\right)^n$.

For a rectangular loop in $\mathbb{R}^2$ (Fig. 1B)

$$\begin{aligned} g_2(x_1, x_2) &= (\delta(x_1 - a_1) + \delta(x_1 - b_1)) (H(x_2 - a_2) - H(x_2 - b_2)) \\ &\quad + (\delta(x_2 - a_2) + \delta(x_2 - b_2)) (H(x_1 - a_1) - H(x_1 - b_1)). \end{aligned} \tag{7}$$

The multipole expansion is obtained by the replacements

$$\begin{aligned} \delta(x_1 - a_1) + \delta(x_1 - b_1) &= -\frac{1}{2\pi i} \left(\frac{1}{z_1 - a_1} + \frac{1}{z_1 - b_1} \right), \\ H(x_2 - a_2) - H(x_2 - b_2) &= \frac{1}{2\pi i} \log \frac{b_2 - z_2}{a_2 - z_2} \end{aligned} \tag{8}$$

multiplying these analytic functions and expanding in Laurent series around some point, for example

$$\left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2} \right).$$

Then from the coefficients of the nonpolynomial part of the Laurent series, one obtains the coefficients of the multipole expansion.

For a rectangular strip (Fig. 1C) the function to be considered is

$$\begin{aligned} h_2(x_1, x_2) &= (H(x_1 - a_1) - H(x_1 - b_1)) (H(x_2 - a_2) - H(x_2 - b_2)) \\ &\quad - (H(x_1 - c_1) - H(x_1 - d_1)) (H(x_2 - c_2) - H(x_2 - d_2)) \end{aligned}$$

and with the replacements as in (8), the procedure is similar.

In conclusion one sees that extended shapes in $\mathbb{R}^n$ of compact support may always be represented by a point in $\mathbb{R}^n$ and a sequence of coefficients in $\left(\mathbb{R}^{\mathbb{N}}\right)^n$ (or simply in $\mathbb{R}^{\mathbb{N}}$ by reordering).

The coefficients of the multipole series depend on the choice of the expansion point in the Laurent series, which should be chosen inside the support for a wider convergence radius of the series, as well as to simplify the multipole series. For example consider the following Laurent series in the neighborhood of infinity

$$\sum_{n\geq 1} \frac{a^n}{z^n}$$

which would imply a multipole series with an infinite number of terms and expansion point at $z=0$. However the series is equivalent to $\frac{a}{z-a}$, that is, a delta at $x=a$.

For the configuration spaces the expansion point will always be taken at the origin and the extended object will be characterized by a point in the base space and the coefficients of the multipole series. For a compact extended object of arbitrary shape one considers the Laurent expansion near infinity of the analytical function vanishing at infinity and with support matching the object,

$$f\left(z_1, z_2, \ldots, z_n\right) = \sum_{\substack{m,k\geq 1 \\ i,j\in\{1,\ldots,n\}}} c(i,j,m,k) \frac{1}{z_i^k z_j^m},$$

and then use (5) to obtain the corresponding multipole expansion. By relabeling, any compact extended object will be characterized by a point in the base space and a point in $\mathbb{R}^{\mathbb{N}}$.

For objects of uniform symmetry the labelling may be simplified. For example for objects with spherical symmetry in $\mathbb{R}^p$, for example

$$B_p = H(R-\rho)$$

a radius R ball or

$$S_p = \delta(R-\rho)$$

a radius R shell (Figs. 1D), one may consider the analytical extension in ρ, and then

$$B_p \to -\frac{1}{2\pi i} \log \frac{z}{z-R} = -\frac{1}{2\pi i} \sum_{n\geq 1} \frac{1}{n} \left(\frac{R}{z}\right)^n,$$

which with (5) leads to a $\mathbb{R}^{\mathbb{N}}$ label

$$\left(R, -\frac{R^2}{2}, \frac{R^3}{3!}, -\frac{R^4}{4!}, \ldots\right) = \left\{\frac{(-1)^{n-1}}{n!} R^n; n = 1, 2, \ldots\right\}.$$

For $S_p \to -\frac{1}{2\pi i}\frac{1}{z-R} = -\frac{1}{2\pi i}\sum_{n=1}\frac{R^{n-1}}{z^n}$

$$\left(1, -R, \frac{R^2}{2}, -\frac{R^3}{3!}, \frac{R^4}{4!}, \ldots\right) = \left\{\frac{(-1)^{n-1}}{(n-1)!} R^{n-1}; n = 1, 2, \ldots\right\},$$

different points in $\mathbb{R}^{\mathbb{N}}$ corresponding to spherical objects with different radial density distributions.

In general, for objects of arbitrary shape a description is obtained in a configuration space $X \times \mathbb{R}^{\mathbb{N}}$, X being the base space. Choosing the expansion point at a central position of the object, each entry in $\mathbb{R}^{\mathbb{N}}$ corresponds to a different multipole structure of the object shape. The first entry roughly corresponds to the the mass of the object, the second and the third to the dipolar and quadrupolar structures, etc. Intuitively this is what one expects when describing the shape of an object. What the space of compact support ultradistributions ($\mathcal{U}_0'$) provides is a rigorous formulation of the intuitive notion. In practice, a reasonable description might be obtained by the truncation to a few multipoles, obtaining a $X \times \mathbb{R}^p$ configuration space, p being the number of retained multipoles. Summarizing,

Proposition 1. *Any extended object in $\mathbb{R}^n$ which may be identified with the support of an ultradistribution in $\mathcal{U}_0'$, may be coded as a point in $\mathbb{R}^n$ and a (finite or infinite) sequence of real numbers.*

Similar results apply if instead of $\mathbb{R}^n$ one has a Riemannian manifold.

4. Ultradistribution-Marked Configuration Spaces

Compact extended objects being characterized by a point in a base space X and a point in $\mathbb{R}^{\mathbb{N}}$, ultradistribution-marked configuration spaces (*uconfig spaces*, for short) may be constructed in a way similar to the marked configuration spaces of Refs. [4, 5], with X a Riemannian manifold and a space of marks in $\mathbb{R}^{\mathbb{N}}$.

Not all sequences of real numbers code a ultradistribution of compact support, only those that insure the convergence of the Laurent series (6), which one denotes as $\mathbb{M} \subset \mathbb{R}^{\mathbb{N}}$.

Definition 1. *A uconfig space is*

$$\Omega_X^{\mathbb{M}} = \left\{ \omega = (\gamma, s) \mid \gamma \in \Gamma_X,\ s \in \mathbb{M}^{\gamma} \right\}$$

X being a $\mathbb{C}^\infty$ Riemannian manifold,

$$\Gamma_X = \left\{ \gamma \subset X \mid \#(\gamma \cap K) < \infty \text{ for each compact } K \subset X \right\}$$

and $\mathbb{M}^\gamma$ is the set of maps $\gamma \ni x \to s_x \in \mathbb{M}$, s_x being a (finite or infinite) sequence of real numbers coding for an ultradistribution in $\mathcal{U}_0'(\mathbb{R}^n)$.

The topology in $\mathbb{M} \subset \mathbb{R}^{\mathbb{N}}$ is the subspace topology. In $\mathbb{R}^{\mathbb{N}}$ the topology is defined by the countable collection of seminorms $(p_1, p_2, \ldots)$, where $p_n = |x_n|$, for $x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$ becomes a locally convex space metrizable by the translational invariant metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

$\mathbb{R}^{\mathbb{N}}$ is a Fréchet space but not a Banach space.

For each open subset of X with compact closure $\Lambda \in \mathcal{O}_c(X)$ define

$$\Omega_\Lambda^{\mathbb{M}} = \left\{ \omega \in \Omega_X^{\mathbb{M}} \,\middle|\, \Pr_X(\omega) \subset \Lambda \right\}$$

with the restriction map

$$p_\Lambda : \Omega_X^{\mathbb{M}} \to \Omega_\Lambda^{\mathbb{M}}$$

Also

$$\Omega_\Lambda^{\mathbb{M}}(n) = \left\{ \omega \in \Omega_\Lambda^{\mathbb{M}} \;\middle|\; \#\omega = n \right\}$$

For practical applications with objects of finite size some restrictions may be put on n for each Λ. For the moment no such limitations will be considered here. Notice that factoring by the permutation group does not apply here because one is allowing for different nature of the objects at each point $x \in X$.

The topology in $\Omega_X^{\mathbb{M}} \subset \Omega_X^{\mathbb{R}^{\mathbb{N}}}$ is obtained from the metrics on X and $\mathbb{R}^{\mathbb{N}}$, the topology in $\Omega_X^{\mathbb{R}^{\mathbb{N}}}$ being the weakest topology making all the p_Λ mappings continuous. The associated σ-algebra is denoted $\mathcal{B}\left(\Omega_X^{\mathbb{M}}\right)$.

In the Riemannian space X one has the usual measure m (the Riemannian volume) and for $\mathbb{M} \subset \mathbb{R}^{\mathbb{N}}$ consider a probability measure μ_n in each $\mathbb{R}$ entry, the measure in $\mathbb{M}$ being the product measure

$$\mu(dy) = \Pi_n \mu_n(dy_n)$$

$y = (y_1, y_2, \ldots) \in \mathbb{M}$. Notice that we may chose different probability measures for each n in case we want to emphasize special multipole features of the objects. Consider in X a intensity measure σ absolutely continuous with respect to the volume measure. Then a measure $\widetilde{\sigma}(A)$ is defined in $\mathcal{B}(X \times \mathbb{M})$ by

$$\widetilde{\sigma}(A) = \int_A \mu(dy)\sigma\,(dx)$$

$A \in \mathcal{B}(X \times \mathbb{M})$. For $\Omega_X^{\mathbb{M}}$ one starts by considering the product measure $\widetilde{\sigma}^{\otimes n}$ in $(X \times \mathbb{M})^n$ which, for each $\Lambda \in \mathcal{O}_c(X)$ yields a finite measure $\widetilde{\sigma}_\Lambda^{\otimes n}$. Then a family of measures is defined on $\Omega_\Lambda^{\mathbb{M}}$ by

$$\lambda_\sigma^\Lambda = \sum_{n=0}^{\infty} \beta(n) \widetilde{\sigma}_\Lambda^{\otimes n}$$

with $\beta(n)$ chosen such that

$$N(\Lambda) = \sum_{n=0}^{\infty} \beta(n) \widetilde{\sigma}(\Lambda)^n$$

is finite, $\widetilde{\sigma}(\Lambda)$ being the measure of $(\Lambda \times \mathbb{M})$. A probability measure in $\Omega_\Lambda^{\mathbb{M}}$ is

$$\pi_\sigma^\Lambda = \frac{1}{N(\Lambda)} \lambda_\sigma^\Lambda$$

These Λ-families of measures $\left\{\pi_\sigma^\Lambda \;\middle|\; \Lambda \in \mathcal{O}_c(X)\right\}$, being consistent, define probability measures π_σ on $\Omega_X^{\mathbb{M}}$. For

$$\beta(n) = \frac{1}{n!}, \quad N(\Lambda) = e^{\sigma(\Lambda)},$$

π_σ is a Poisson measure, for

$$\beta(n) = \frac{1}{\Gamma(\alpha n + 1)}, \quad N(\Lambda) = E_\alpha(\sigma(\Lambda)) \quad (\alpha > 0,\ \alpha \neq 1)$$

it would be a fractional Poisson measure [22], etc.

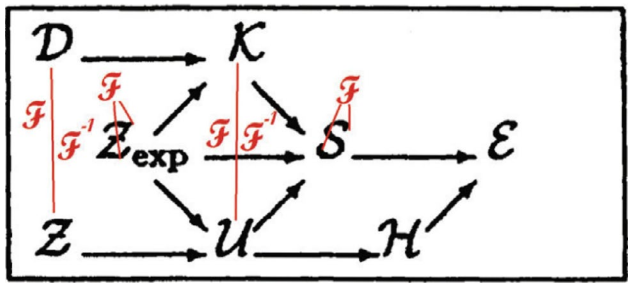

Fig. 2. Test function spaces.

For the transformation properties of the π_σ measures in $\Omega_X^{\mathbb{M}}$ one considers a transformation group $\mathcal{G}$ that is the semidirect product of the group $\mathrm{Diff}_0(X)$, diffeomorphisms of X with compact support, with the smooth currents

$$X \ni x \to \theta(x) \in \mathbb{M}$$

θ being equal to one for all but a finite number of entries in $\mathbb{M}$ and also equal to one outside a compact set (dependent on θ and on each particular entry). The group acts on $X \times \mathbb{M}$ as follows.

$$X \times \mathbb{M} \ni (x, s) \to \big(\psi(x), \theta(\psi(x))s\big) \in X \times \mathbb{M}$$

$\psi \in \mathrm{Diff}_0(X)$.

This construction is of course a simple extension to $X \times \mathbb{M}$ of the results obtained in [5] for $X \times \mathbb{R}_+$. Likewise the results on integration by parts, divergence, representations of the group $\mathcal{G}$, Dirichlet forms, etc. in [5] may be similarly extended to $X \times \mathbb{M}$.

The important point to retain is that, with the ultradistribution interpretation, extension to $X \times \mathbb{M}$ or to truncations $X \times \mathbb{M}$ allows one to deal with configuration spaces of extended objects of arbitrary shape. For dynamical applications where extended objects might represent interacting physical systems, a few interesting questions have to be addressed. For example distance between the systems may be defined from the metric in X ambient space, with care to chose the ultradistribution expansion point in a central position. Collisions between the systems would correspond to the overlap of the supports of the ultradistributions, etc.

Appendix: Test Function and Distribution Spaces [17]

$\mathcal{D} = \cup_K \{\mathcal{D}_K : \varphi \in C^\infty,\ \mathrm{supp}(\varphi) \subset K\}$; $\|\varphi\|_{(p,K)} = \max_{0\le r\le p}\left\{\sup\left|\varphi^{(r)}\right|\right\}$,

$\mathcal{K} = \cap_{p=0}^{\infty}\mathcal{K}_p$; $\mathcal{K}_p =$ completion of C^∞ for the norm $\|\varphi\| = \max_{0\le q\le p}\left\{\sup\left|e^{p|x|}\varphi^{(q)}\right|\right\}$,

$\mathcal{S} = \cap\mathcal{S}_{p,r} = \left\{\varphi \in C^\infty : \|\varphi\|_{p,r} = \sup\left|x^p\varphi^{(r)}\right|\right\}$,

$\mathcal{E} = \varphi \in C^\infty$ with ω-convergence on compacts,

$\mathcal{Z} = \varphi : \mathcal{F}\{\varphi\} \in \mathcal{D}$, $\varphi(z)$ entire: $\left|z^k\varphi(z)\right| \le C_k e^{a|\mathrm{Im}(z)|}$,

$\mathcal{U} = \cap_{p=0}^{\infty}\mathcal{U}_p$; $\mathcal{U}_p = \{\varphi : \mathcal{F}\{\varphi\} \in \mathcal{K}_p\}$; $\|\varphi\|_p = \sup_{z\in\Lambda_p}\{(1+|z|^p)\,|\varphi(z)|\}$,

$\mathcal{H} =$ Entire functions with topology of uniform convergence on compacts of $\mathbb{C}$,

$\mathcal{Z}_{\exp} = \cap_{j=1}^{\infty}\mathcal{Z}_{\exp,j}$; $\mathcal{Z}_{\exp,j} = \left\{\varphi : \|\varphi\|_{\exp,j} = \max_{k\le j}\left\{e^{j|\mathrm{Re}(z)|}\left|\varphi^{(k)}(z)\right|\right\}\right\}$.

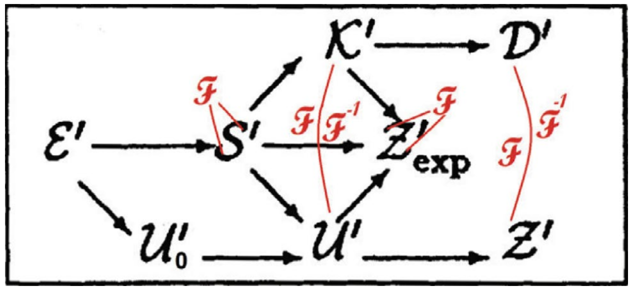

Fig. 3. Distribution spaces.

$\mathcal{D}'$ = Schwartz distributions; locally $\mu(x) = D^k(f(x))$,

$\mathcal{K}'$ = Distributions of exponential type, $\mu(x) = D^k\left(e^{a|x|}f\right)$,

$\mathcal{S}'$ = Tempered distributions,

$\mathcal{E}'$ = Subspace of $\mathcal{D}'$ of distributions of compact support,

$\mathcal{Z}'$ = Ultradistributions, $\mathcal{D}' \overset{\mathcal{F}}{\to} \mathcal{Z}'$; $\mathcal{Z}' \overset{\mathcal{F}^{-1}}{\to} \mathcal{D}'$,

$\mathcal{U}'$ = Tempered ultradistributions,

$\mathcal{U}'_0$ = Dual of $\mathcal{H}$, ultradistributions of compact support,

$\mathcal{Z}'_{\text{exp}}$ = Topological dual of $\mathcal{Z}_{\text{exp}}$, contains $\mathcal{U}'$ and $\mathcal{K}'$ as proper subspaces,

$\mathcal{F}$ denotes Fourier morphisms. For details see Refs. [17, 18].

Conflict of Interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

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REFERENCES

1. S. Albeverio, Yu. G. Kondratiev, and M. Röckner, "Analysis and geometry on configuration spaces," *J. Funct. Anal.*, **154**, 444–500 (1998).
2. S. Albeverio, Yu. G. Kondratiev, and M. Röckner, "Analysis and geometry on configuration spaces: The Gibbsian case," *J. Func. Anal.*, **157**, 242–291 (1998).
3. Y. G. Kondratiev and T. Kuna, "Harmonic analysis on configuration space I. General theory," *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **5**, 201–233 (2002).
4. S. Albeverio, Y. Kondratiev, E. Lytvynov, and G. Us, "Analysis and geometry on marked configuration spaces," in: *Infinite Dimensional Harmonic Analysis*, H. Heyer et al. (eds.), Gräbner, Altendorf, (2000), pp. 1–39.
5. Y. Kondratiev, E. Lytvynov, and G. Us, "Analysis and geometry on R_+-marked configuration space," *Methods Funct. Anal. Topology*, **5**, 29–64 (1999).
6. D. L. Finkelshtein, Yu. G. Kondratiev, and M. J. Oliveira, "Glauber dynamics in the continuum via generating functionals evolution," *Complex Anal. Oper. Theory*, **6**, 923–945 (2012).
7. D. L. Finkelshtein, Yu. G. Kondratiev, and M. J. Oliveira, "Kawasaki dynamics in the continuum via generating functionals evolution," *Methods Funct. Anal. Topology*, **18**, 55–67 (2012).
8. D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, "Semigroup approach to birth-and-death stochastic dynamics in continuum," *J. Funct. Anal.*, **262**, 1274–1308 (2012).

9. Y. Kondratiev, O. Kutoviy, and S. Pirogov, "Correlation functions and invariant measures in continuous contact model," *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **11**, 231–258 (2008).
10. Y. Kondratiev, O. Kutoviy, and R. Minlos, "On non-equilibrium stochastic dynamics for interacting particle systems in continuum," *J. Funct. Anal.*, **255**, 200–227 (2008).
11. Y. G. Kondratiev, T Kuna, and M. J. Oliveira, "Holomorphic Bogoliubov functionals for interacting particle systems in continuum," *J. Funct. Anal.*, **238**, 375–404 (2006).
12. D. L. Finkelshtein, Yu. G. Kondratiev, and M. J. Oliveira, "Markov evolutions and hierarchical equations in the continuum. II: multicomponent systems," *Rep. Math. Phys.*, **71**, 123–148 (2013).
13. D. Finkelshtein, Yu. Kondratiev, O. Kutoviy, and M. J. Oliveira, "Dynamical Widom–Rowlinson model and its mesoscopic limit," *J. Stat. Phys.*, **158**, 57–86 (2015).
14. G. A. Goldin, "Lectures on diffeomorphism groups in quantum physics," in: *Contemporary Problems in Mathematical Physics*, J. Govaerts et al. (eds.), World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2004), pp. 3–93.
15. G. A. Goldin, "Quantum configuration spaces of extended objects, diffeomorphism group representations and exotic statistics," in: *Geometric Methods in Physics XXX*, Workshop, Bialowieza, Poland, P. Kielanowski, S. T. Ali, T. Voronov, A. Odzijewicz, and M. Schlichenmaier (eds.), Springer, Basel (2013), pp. 239–252.
16. R. S. Ismagilov, "Representations of infinite-dimensional groups," *Translations of Mathematical Monographs*, **152**, American Mathematical Society (1996).
17. R. F. Hoskins and J. Sousa Pinto, *Theories of Generalized Functions: Distributions, Ultradistributions and Other Generalized Functions*, 2nd. ed., Woodhead Publ., Philadelphia (2011).
18. R. Vilela Mendes, "Ultradistribution spaces: Superprocesses and nonlinear differential problems," in: *From Particle Systems to Partial Differential Equations*, Springer Proc. Math. Stat., **209**, Springer, Cham (2017), pp. 247–261.
19. J. Sebastião e Silva, "Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel," *Math. Ann.*, **136**, 58–96 (1958).
20. J. Sebastião e Silva, "Les séries de multipôles des physiciens et la théorie des ultradistributions," *Math. Ann.*, **174**, 109–142 (1967).
21. M. Hasumi, "Note on the n-dimensional tempered ultradistributions," *Tôhoku Math. J.*, **13**, 94–104 (1960).
22. M. J. Oliveira and R. Vilela Mendes, "Fractional Boson gas and fractional Poisson measure in infinite dimensions," in: *From Particle Systems to Partial Differential Equations II*, *Springer Proc. Math. Stat.*, **129** (2015), pp. 293–312.