PHYSICS OF PLASMAS 24, 000000 (2017)

Analytical study of growth estimates, control of fluctuations, and conservative structures in a two-field model of the scrape-off layer

R. Vilela Mendes[a)] and João P. S. Bizarro[b)]
Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal

(Received 1 August 2016; accepted 9 December 2016; published online xx xx xxxx)

Anomalous transport, turbulence, and generation of large-scale, collective structures (so-called blobs) in the scrape-off layer (SOL) of tokamaks are some of the main issues that control the machine performance and the life expectancy of plasma-facing components, and here one tries to achieve some understanding of these questions through a theoretical, analytical study of a reduced two-dimensional two-field (density plus vorticity) model of the SOL. The model is built around a conservative system describing transport perpendicular to the magnetic field in a slab geometry, to which terms are added to account for diffusion and parallel losses (both for particles and current) and to mimic plasma flow from the core (in the form of a source). Nonlinear estimates for the growth rates are derived, which show the growth in the density gradient to be bounded above by the vorticity gradient, and vice-versa, therefore suggesting a nonlinear instability in the model. The possibility of controlling fluctuations by means of a biasing potential is confirmed (negative polarisations being shown to be more effective in doing so, thus providing an explanation for what is seen in experiments), as well as the advantage in reducing the inhomegeneity of the magnetic field in the SOL to decrease the plasma turbulence there. In addition, focusing on the conservative part of the equations, exact solutions in the form of travelling waves are obtained which might be the conservative ancestors of the blobs that are observed in experiments and in numerical simulations. [http://dx.doi.org/10.1063/1.4973222]

I. INTRODUCTION

Transport in the plasma boundary is one of the main pending issues pertaining to the design and operation of future fusion machines, the physics of particle and energy flows in the boundary impacting both on machine performance and on the life expectancy of plasma-facing components.[1–4] The modelling of this plasma boundary, or scrape-off layer (SOL), being extremely complex,[5–11] numerical codes have been developed with recourse to reduced two-field models and conservation laws, greatly simplifying the analysis of the SOL transport while retaining the fundamental properties of the underlying physics, which is that of plasma turbulence.[12–18] The broad picture coming out from these efforts is that, although small-scale turbulence dominates, there may also be large-scale coherent structures, associated with intermittent events known as blobs, that are observed both in experiments and in numerical simulations and greatly contribute to the nature of particle and energy transport.[2,4,6,9–11,14–16,19–22] These blobs, besides modifying the local transport properties through nonlinear interaction with small-scale turbulence, generate machine-scale flows that may seriously damage the wall due to thermal charge asymmetries and plasma ejection.

In this article, one will try to achieve some understanding on these questions through the analytical study of a reduced two-dimensional (2D) fluid model, describing the plasma dynamics (electrons plus current) perpendicular to the magnetic field, built from a conservative model[23] (itself a cold-ion, reduced version of more complete four-field models[24,25]) to which phenomenological terms are added accounting for particle and current diffusion, parallel losses, and a core-plasma source.[13–18] Emphasis will be on some mathematical properties of the model, more precisely, on obtaining nonlinear growth estimates for the physical quantities and on discussing the eventual control of fluctuations, not only by polarization[14,17,18,26,27] but also by acting on the magnetic inhomogeneity. In addition, and focusing on the conservative part of the equations, one aims at deriving exact solutions that might be the conservative ancestors of the collective structures related with blobs. In fact, one will revisit in this paper, but in a purely analytical approach, a model that is fairly standard in SOL turbulence,[13–18] so that one will not only recover some results already known from linear stability analysis of interchange turbulence in the SOL (such as the destabilising role of a negative density gradient and the stabilising effects of the vorticity gradient and of a reduced magnetic-field inhomogeneity[14,15]), but will also derive other results that are indeed new. These include the setting up of upper bounds on the nonlinear growth rates for the gradients of the density and electrostatic potential, the unveiling of the benefits in using negative instead of positive biasing when looking for improved stability (in agreement with experimental results[26,27]), and the obtention of exact solutions of the traveling-wave type to the conservative kernel around which the model is built.

It should be stressed that one of the motivations for the present article was to push as far as possible the analytical exploration of a model frequently used in studies of SOL turbulence,[13–18] an approach that is rarely followed in the field

[a)]Also at Centro de Matemática e Aplicações Fundamentais, Universidade de Lisboa. Electronic mail: rvilela.mendes@gmail.com
[b)]Electronic mail: bizarro@ipfn.tecnico.ulisboa.pt

given the extreme complexity of SOL physics. Moreover, faced with model equations one suspects describe unstable behavior, it is essential to have some analytic control over their solutions, otherwise it might be hard to unravel what is a physically relevant behavior from what is a result of numerical instability when running simulation codes. Whenever necessary, simplifications will be made, yet always keeping in mind the aphorism according to which "everything should be made as simple as possible, but not simpler." This is why, for instance, one chose to work in the limit of negligible ion temperature, and thus started from the cold-ion limit of a conservative two-field model for the density and vorticity,[23] to end up with a model which, albeit simpler for not containing finite-ion-temperature effects, still retains the essential features of turbulent transport in the SOL.[13–18]

So, this paper is structured as follows: the model is presented and discussed in Sec. II, some estimates on nonlinear growth rates are offered in Sec. III, the control of turbulent fluctuations is dealt with in Sec. IV, the link between coherent structures and conservative dynamics is examined in Sec. V, and, finally, the findings and conclusions are summarized in Sec. VI. This outline corresponds to the story told below using an analytical language: after recognizing (from estimates of the global growth rates) that the model may exhibit a nonlinear unstable behavior, the control of unstable modes in the fluctuation spectrum (while still in the linear regime) is addressed, before showing how solutions of the conservative part of the model can lead to large propagating structures (that very much look like ejected blobs). Also, for completeness and while avoiding to burden the main text, mathematics that is eminently calculatory and straightforward, but may be helpful when going through the details of the derivations, is provided in an Appendix through a series of notes.

II. THE MODEL

The starting point is the two-field (density plus vorticity) model

$$\frac{\partial Ln}{\partial t} = -[\phi, Ln] - \left[\phi, \frac{1}{B}\right] + \left[Ln, \frac{1}{B}\right] \tag{1a}$$

$$\frac{\partial \nabla^2\phi}{\partial t} = -\left[\phi, \nabla^2\phi\right] + \left[Ln, \frac{1}{B}\right] \tag{1b}$$

describing the dynamics perpendicular to the magnetic field, which can be extracted from a four-field model including finite-Larmor-radius effects, drift-velocity ordering, and gyroviscous terms,[24] provided the dynamics along the magnetic field (the so-called parallel dynamics) is suppressed, the poloidal magnetic fluctuations are neglected, and the cold-ion limit is taken.[23] Hereabove, $Ln(x_1, x_2, t)$ stands for the log-density, meaning the logarithm of the normalised density $n(x_1, x_2, t)$, $\phi(x_1, x_2, t)$ for the normalised electrostatic potential, and $B(x_1)$ for the normalised ambient magnetic field, $x_1 = (r - a)$ and $x_2 = a\theta$ are the radial and poloidal coordinates, respectively, with a the minor radius of the core plasma (space being rescaled by the ion sonic Larmor radius and time by the ion cyclotron frequency), the canonical Poisson bracket $[f, h]$ is given by $[f, h] = \partial_1 f \partial_2 h - \partial_2 f \partial_1 h$, and the del operator reads $\nabla = (\partial_1, \partial_2)$, with $\partial_i = \partial/\partial x_i$. The conservative model in (1) has a simple Hamiltonian structure, which would be lost if finite ion-temperature effects (in particular, ion gyroviscosity) were retained in Equation (1b) for $\nabla^2\phi$ while keeping compressibility (conveyed by the $1/B$ terms) in the continuity equation (1a), the manufacturing of a Hamiltonian form in such a case having been addressed by means of a so-called gyromap.[23,25] To expect a reduced model to keep the Hamiltonian structure of the distant parent (Maxwell–Vlasov) model seems too stringent a requirement, one that might collide with the physical interpretation and the simplicity of reduced models, inasmuch as Hamiltonian systems are, in fact, just a small class among the conservative systems (in the Liouville sense).[28]

To this conservative model, and as described in the literature,[13–18] one adds diffusion (to govern the damping at small scales via the coefficients D and ν, respectively, for particles and vorticity), sinks (to mimic the parallel losses to the wall or limiter as set, from sheath physics,[1,12] by the floating potential Λ and some $\sigma_{||}$ that measures the characteristic time for parallel transport), and a source $S(x_1)$ (to account for the plasma flow from the core). Hence, and putting

$$\tilde{g} = g = \partial_1\left(\frac{1}{B}\right), \tag{2}$$

(1) is completed to become [see (a) in the Appendix]

$$\frac{\partial Ln}{\partial t} = -[\phi, Ln] - \tilde{g}\partial_2(Ln - \phi) + D\left(\nabla^2 Ln + |\nabla Ln|^2\right) - \sigma_{||}e^{(\Lambda-\phi)} + S \tag{3a}$$

$$\frac{\partial \nabla^2\phi}{\partial t} = -\left[\phi, \nabla^2\phi\right] - g\partial_2 Ln + \nu\nabla^4\phi + \sigma_{||}\left[1 - e^{(\Lambda-\phi)}\right], \tag{3b}$$

which account for particle balance (for electrons) and charge conservation, respectively. It is worth mentioning that (3) is very much the same as a 2D model used for flux-driven uniform-temperature (so-called FDUT) simulations,[15,16] and also very similar to TOKAM-2D,[12–14,17,18] the difference with the latter lying in the presence hereabove of the term associated with the magnetic field inhomogeneity $\tilde{g}$ in the continuity equation, so all results in this paper apply to TOKAM-2D by making $\tilde{g} = 0$. The compressibility term in (3a), namely, $\tilde{g}\partial_2(Ln - \phi)$, being of the same order as the coupling term $g\partial_2 Ln$ in the vorticity equation, one has decided to keep it in the model, as other authors have done.[6,8,9,15,16] For completeness, note that interchange-like models similar to the one above have also been used to study the nonlinear dynamics of transport-barrier relaxations and of tearing magnetic islands.[29–31]

III. NONLINEAR GROWTH ESTIMATES

To estimate the dependence of the temporal growth rate of physical quantities, (3) will be simplified: first, one will assume that the difference $\phi(x_1, x_2, t) - \Lambda$ between the electric and plasma floating potentials may be neglected and, second, that the parallel current losses are compensated by the source $S(x_1)$ (a quasi-equilibrium hypothesis). Therefore, one expects the following equations to display the same qualitative behavior as those in (3):

$$\frac{\partial Ln}{\partial t} = -[\phi, Ln] - \tilde{g}\partial_2(Ln - \phi) + D\left(\nabla^2 Ln + |\nabla Ln|^2\right) \quad \text{(4a)}$$

$$\frac{\partial \nabla^2 \phi}{\partial t} = -\left[\phi, \nabla^2\phi\right] - g\partial_2 Ln + \nu\nabla^4\phi. \quad \text{(4b)}$$

Remark that, the source and sink terms having been dropped, (4) describes a closed system, yet this does not prevent the local growth of physical quantities or of their derivatives, being precisely this growth, which arises from the nonlinear nature of the problem, that one wants to unveil. Putting it differently, the potentially dangerous concentration of mass and energy in the blobs does not necessarily imply a global increase of these quantities in the SOL. Then, introducing the unit vector $\widehat{\mathbf{b}}$ along the direction of the magnetic field (the so-called parallel direction, so $\widehat{\mathbf{b}} \cdot \nabla = 0$), which is such that $[f, h] = \widehat{\mathbf{b}} \cdot \nabla f \times \nabla h$, (4) can be rewritten as

$$\frac{\partial Ln}{\partial t} = \widehat{\mathbf{b}} \cdot \nabla Ln \times \nabla\phi - \tilde{g}\partial_2(Ln - \phi) + D\left(\nabla^2 Ln + |\nabla Ln|^2\right) \quad \text{(5a)}$$

$$\frac{\partial \nabla^2 \phi}{\partial t} = \widehat{\mathbf{b}} \cdot \nabla\nabla^2\phi \times \nabla\phi - g\partial_2 Ln + \nu\nabla^4\phi. \quad \text{(5b)}$$

One now assumes the solution of (5) to be defined in a domain D with either periodic boundary conditions or vanishing values for all functions at its boundary, meaning, in practice, that boundary terms can be dropped in all partial integrations. This simplifying assumption may not seem fully appropriate to the boundary between the SOL and the core plasma (at $x_1 = 0$), yet any eventual nonvanishing boundary terms would not be dynamical, but fixed instead, and hence one does not expect them to qualitatively change the inequalities obtained below. Furthermore, the growth rates thus estimated are global, in that they are spatially integrated quantities, and nonlinear, in that they proceed from the model equations without any previous linearisation of the latter (unlike in the approach followed below when addressing the control of fluctuations). So, multiplying (5b) by ϕ and integrating over D yields [see (b) in Appendix]

$$\frac{1}{2}\frac{d}{dt}\int_D |\nabla\phi|^2 = \int_D g\phi\partial_2 Ln - \nu\int_D \left(\nabla^2\phi\right)^2, \quad (6)$$

whence the inequality [see (c) in the Appendix]

$$\frac{d}{dt}\int_D |\nabla\phi|^2 \leq \int_D |\nabla Ln|^2 + \int_D (g\phi)^2, \quad (7)$$

which means the growth of $|\nabla\phi|$ is partially controlled by the gradient $|\nabla Ln|$ of the log-density.

Proceeding to work out (5a) in order to establish a bound on the growth of $|\nabla Ln|$, one takes the gradient of this equation, makes the inner product with ∇Ln, and integrates by parts on the domain D to obtain [see (d) in the Appendix]

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_D |\nabla Ln|^2 = &-\widehat{\mathbf{b}} \cdot \int_D (\nabla Ln \times \nabla\phi)\nabla^2 Ln \\ &+\int_D \tilde{g}\partial_2(Ln - \phi)\nabla^2 Ln \\ &-\int_D D\left(\nabla^2 Ln + |\nabla Ln|^2\right)\nabla^2 Ln \end{aligned} \quad (8)$$

and, from (8), the following inequality can be derived [see (e) in the Appendix]

$$\begin{aligned}\frac{d}{dt}\int_D |\nabla Ln|^2 \leq &\int_D |\nabla\phi|^2 + \int_D |\nabla(Ln - \phi)|^2 + \int_D D|\nabla Ln|^4 \\ &+\int_D \left(\tilde{g}^2 + D + |\nabla Ln|^2\right)\left(\nabla^2 Ln\right)^2, \end{aligned} \quad (9)$$

or still [see (f) in the Appendix]

$$\begin{aligned}\frac{d}{dt}\int_D |\nabla Ln|^2 \leq 3&\int_D |\nabla\phi|^2 + \int_D \left(2 + D|\nabla Ln|^2\right)|\nabla Ln|^2 \\ &+\int_D \left(\tilde{g}^2 + D + |\nabla Ln|^2\right)\left(\nabla^2 Ln\right)^2. \end{aligned} \quad (10)$$

The growth of $|\nabla Ln|$ is thus bounded by $|\nabla\phi|$, whose growth is itself bounded by $|\nabla Ln|$, suggesting a nonlinear instability of the model, at least for some domains of the configuration space. Admittedly, these estimates for the mutual control of the log-density and the potential are estimates over global integrated quantities yet, being integrals over the modulus of gradients, or derivatives, an eventual growth of one such integrated quantity carries information on short range variations, hence on local instabilities.

IV. CONTROL OF FLUCTUATIONS

Having seen in Sec. III the model may become nonlinearly unstable, the next step is to perform a linear stability analysis. Indeed, the ability to control plasma behavior in the SOL is highly desirable for tokamak operation, a possible route to achieve this being through polarisation, either of the wall (biased limiter or divertor elements) or via biased probes, including the injection of current by way of heated emissive electrodes, both situations having been explored experimentally,[26,27] as well as numerically.[14,17,18] Here, the model in (3) will be used to understand how the control of fluctuations might be achieved with the aid not only of biasing, but also of magnetic-field modulation, another effect worthy to be explored. So, let the source $S(x_1)$ and $V_{\text{bias}}(x_1, t)$ be such that

$$Ln^{(0)}(x_1) = Ln_0 - \mu^{-1}x_1 \quad \text{(11a)}$$

$$\phi^{(0)}(x_1, t) = \Lambda + V_{\text{bias}}(x_1, t) \quad \text{(11b)}$$

is a solution of (3) with Λ replaced by $\Lambda + V_{\text{bias}}(x_1, t)$, μ giving the scale length for the exponential decay of density in the SOL. On physical grounds, one expects $V_{\text{bias}}(x_1, t)$ to be of the form

$$V_{\text{bias}}(x_1, t) = U(x_1, t)e^{-(Z - x_1)/\lambda_D}, \quad (12)$$

Z being the radial width of the SOL in the x_1 coordinate, $\lambda_{\rm D}$ the Debye length, and $U(x_1,t)$ a slowly varying function of x_1 and t. The intensity $U(x_1,t)$ of the potential taken at the wall or at a probe would be a control parameter, whilst the decaying term in $V_{\rm bias}(x_1,t)$ takes into account the screening properties of the plasma away from $x_1 = Z$. The effect of biasing is thus emulated by modifying the plasma potential around the position of the polarized element, an approach that has been standard (although using Gaussian function forms instead of (12)),[14,17,18] but which can be replaced by more sophisticated, or "smart," control techniques.[32,33] One should further add that the situation conveyed by (11b) and (12) is that of a poloidally homogeneous biasing potential, which is experimentally more relevant for dealing with, say, a biased poloidal limiter, or ring,[14,17] than with discrete biased probes,[18,33] being clear that with a finite number of probes there would be in the biasing potential an x_2-dependent modulation that would also be present in the results here derived. If one now looks for fluctuations around the unperturbed solution (11), one may write up to first order [see (g) in the Appendix],

$$Ln(x_1,x_2,t) = Ln^{(0)}(x_1) + \delta Ln(x_1,x_2,t) \tag{13a}$$

$$\phi(x_1,x_2,t) = \phi^{(0)}(x_1,t) + \delta\phi(x_1,x_2,t), \tag{13b}$$

and subsequently plug (11) and (13) into (3) to obtain

$$\frac{\partial \delta Ln}{\partial t} \simeq -\partial_1\phi^{(0)}\partial_2\delta Ln - \mu^{-1}\partial_2\delta\phi - \tilde{g}\partial_2(\delta Ln - \delta\phi) + \sigma_{||}\delta\phi + D\left(\nabla^2\delta Ln - 2\mu^{-1}\partial_1\delta Ln\right) \tag{14a}$$

$$\frac{\partial \nabla^2\delta\phi}{\partial t} \simeq -\partial_1\phi^{(0)}\partial_2\nabla^2\delta\phi + \partial_1^3\phi^{(0)}\partial_2\delta\phi - g\partial_2\delta Ln + \sigma_{||}\delta\phi + \nu\nabla^4\delta\phi. \tag{14b}$$

Next, Fourier transforming according to

$$f(x_1,x_2,t) = \frac{1}{2\pi}\int\int \widehat{f}(k_1,k_2,t)e^{i(k_1x_1+k_2x_2)}dk_1dk_2, \tag{15}$$

(14) becomes

$$\frac{\partial\widehat{\delta Ln}}{\partial t} \simeq -\left(Dk^2 + i2D\mu^{-1}k_1 + i\tilde{g}k_2\right)\widehat{\delta Ln} - \left[i\left(\mu^{-1}-\tilde{g}\right)k_2 - \sigma_{||}\right]\widehat{\delta\phi} - \frac{1}{2\pi}\left(\widehat{\partial_1\phi^{(0)}} * \widehat{\partial_2\delta Ln}\right) \tag{16a}$$

$$\frac{\partial\widehat{\delta\phi}}{\partial t} \simeq i\frac{gk_2}{k^2}\widehat{\delta Ln} - \left[\nu k^2 + \frac{\sigma_{||}}{k^2}\right]\widehat{\delta\phi} + \frac{1}{2\pi k^2}\left(\widehat{\partial_1\phi^{(0)}} * \widehat{\partial_2\nabla^2\delta\phi}\right) - \frac{1}{2\pi k^2}\left(\widehat{\partial_1^3\phi^{(0)}} * \widehat{\partial_2\delta\phi}\right), \tag{16b}$$

with $k = \sqrt{k_1^2 + k_2^2}$. The convolution products that appear in (16) of the form

$$\frac{1}{2\pi}\left(\widehat{f} * \widehat{h}\right)(k_1,k_2,t) = \frac{1}{2\pi}\int\int \widehat{f}(q_1,q_2,t)\widehat{h}(k_1-q_1,k_2-q_2,t)dq_1dq_2 \tag{17}$$

can be rewritten, with recourse to (15), as

$$\frac{1}{2\pi}\left(\widehat{f} * \widehat{h}\right)(k_1,k_2,t) = \frac{1}{4\pi^2}\int\int\int\int f(x_1,x_2,t)\widehat{h}(k_1-q_1,k_2-q_2,t)e^{-i(q_1x_1+q_2x_2)}dx_1dx_2dq_1dq_2, \tag{18}$$

yielding upon expansion [see (h) in the Appendix]

$$\frac{1}{2\pi}\left(\widehat{f} * \widehat{h}\right)(k_1,k_2,t) = \int\int\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(x_1\partial_1 + x_2\partial_2)^n f(0,0,t)}{n!}\,\frac{(-i)^{n'}(\partial_1\partial_{k1} + \partial_2\partial_{k2})^{n'}\widehat{h}(k_1,k_2,t)}{n'!}\delta(x_1)\delta(x_2)dx_1dx_2, \tag{19}$$

with $\partial_{ki} = \partial/\partial k_i$, (19) being equivalent to [see (i) in the Appendix]

$$\frac{1}{2\pi}\left(\widehat{f} * \widehat{h}\right)(k_1,k_2,t) = \sum_{n=0}^{\infty}\frac{i^n(\partial_1\partial_{k1} + \partial_2\partial_{k2})^n}{n!}f(0,0,t)\widehat{h}(k_1,k_2), \tag{20}$$

or, in operator form, to

$$\frac{1}{2\pi}\left(\widehat{f} * \widehat{h}\right)(k_1,k_2,t) = e^{i\nabla\cdot\nabla_k}f(0,0,t)\widehat{h}(k_1,k_2,t), \tag{21}$$

after defining $\nabla_k = (\partial_{k1},\partial_{k2})$. Keeping only the leading, lowest-order term in (21) and making, for short, $f_0 = f(0,0,t)$, (16) becomes

$$\frac{\partial\widehat{\delta Ln}}{\partial t} \simeq -\left[Dk^2 + i2D\mu^{-1}k_1 + i\left(\tilde{g} + \partial_1\phi_0^{(0)}\right)k_2\right]\widehat{\delta Ln} - \left[i\left(\mu^{-1}-\tilde{g}\right)k_2 - \sigma_{||}\right]\widehat{\delta\phi} \tag{22a}$$

$$\frac{\partial \widehat{\delta\phi}}{\partial t} \simeq i\frac{gk_2}{k^2}\widehat{\delta Ln} - \left[\nu k^2 + i\left(\partial_1\phi_0^{(0)} + \frac{\partial_1^3\phi_0^{(0)}}{k^2}\right)k_2 + \frac{\sigma_{||}}{k^2}\right]\widehat{\delta\phi}. \tag{22b}$$

The damping/growth rates for the fluctuation modes follow from solving the characteristic equation for the system (22) of two linear ordinary differential equations (ODE's), whose solutions are [see (j) in Appendix]

$$\lambda_\pm \simeq -\frac{1}{2}\left[(D+\nu)k^2 + i2D\mu^{-1}k_1 + i\left(\tilde{g} + 2\partial_1\phi_0^{(0)} + \frac{\partial_1^3\phi_0^{(0)}}{k^2}\right)k_2 + \frac{\sigma_{||}}{k^2}\right]$$
$$\pm\frac{1}{2}\sqrt{\left[(\nu-D)k^2 - i2D\mu^{-1}k_1 - i\left(\tilde{g} - \frac{\partial_1^3\phi_0^{(0)}}{k^2}\right)k_2 + \frac{\sigma_{||}}{k^2}\right]^2 - i\frac{4gk_2}{k^2}\left[ik_2\left(\mu^{-1}-\tilde{g}\right) - \sigma_{||}\right]}, \tag{23}$$

which can be seen to become equivalent to the FDUT dispersion relation found in the literature if one further assumes $k_1 \simeq 0$ and a uniform $\phi_0^{(0)}$.[15,34] Provided D, ν, and $\sigma_{||}$ are small enough to make $D \simeq \nu \simeq \sigma_{||} \simeq 0$, (23) reads

$$\lambda_\pm \simeq -\frac{i}{2}\left(\tilde{g} + 2\partial_1\phi_0^{(0)} + \frac{\partial_1^3\phi_0^{(0)}}{k^2}\right)k_2$$
$$\pm\frac{|k_2|}{2k}\sqrt{4g(\mu^{-1}-\tilde{g}) - \left(\tilde{g}k - \frac{\partial_1^3\phi_0^{(0)}}{k}\right)^2}, \tag{24}$$

yielding the damping/growth rates

$$\gamma_\pm \simeq \pm\frac{|k_2|}{2k}\sqrt{4g(\mu^{-1}-\tilde{g}) - \left(\tilde{g}k - \frac{\partial_1^3\phi_0^{(0)}}{k}\right)^2}, \tag{25}$$

whenever the quantity under the square root is non-negative. Therefore, from (25), the solutions of (22) will not grow indefinitely (and become unstable) provided

$$4g(\mu^{-1}-\tilde{g})k^2 \le (\tilde{g}k^2 - \partial_1^3\phi_0^{(0)})^2, \tag{26}$$

which thus reads as a condition for stability. Condition (26) clearly recovers the driving effect (*vis-à-vis* instability) of a negative density gradient (a positive μ), as well as the stabilising role of the third derivative of the potential, in accordance with the previous results obtained for the interchange instability in the SOL with $\tilde{g} = 0$.[14] It is a first-order condition, secured by keeping only the leading term in the convolution-product expansions (20) or (21), and better approximations can be derived using higher-order terms in those expansions.

Now, a physical argument allows for an accurate interpretation of (26) if, referring to the form (12) for the biasing potential, one notices, from (11), that $\phi_0^{(0)} = \Lambda + U_0 e^{-Z/\lambda_D}$ is the potential at a distance Z from the polarisation probe. Hence, introducing the distance to the probe as a parameter, recalling (2) and (11), assuming a negative density gradient in the SOL, and neglecting the derivatives of the slowly varying $U(x_1, t)$ in (12), criterion (26) can be expressed in terms of physical variables as

$$4k^2\partial_1\left(\frac{1}{B}\right)\left[\frac{|\partial_1 n|}{n} - \partial_1\left(\frac{1}{B}\right)\right] \le \left[k^2\partial_1\left(\frac{1}{B}\right) - \frac{U_0 e^{-Z/\lambda_D}}{\lambda_D^3}\right]^2. \tag{27}$$

Regarding controllability, one may infer the following conclusions from (27):[35] control of unstable modes is a local effect, and becomes increasingly difficult further away from the probe (because of the e^{-Z/λ_D} term); a negative polarisation intensity U_0 is more favorable than a positive one (accounting for the fact that, in general, $\partial_1(1/B) \ge 0$ in a tokamak), whereas for positive U_0 the less controllable modes occur around $k^2 \simeq U_0 e^{-Z/\lambda_D}/\lambda_D^3 \partial_1(1/B)$; decreasing the magnetic-field inhomogeneity increases controllability, suggesting that making the magnetic field in the SOL as uniform as possible (so has to have $\partial_1(1/B) \approx 0$ there) may reduce turbulence. The simplest way of achieving this is to increase the strength B of the magnetic field, namely, of its dominant toroidal component which, by reducing the plasma normalized pressure (the so-called plasma β), is also good for its macroscopic, magnetohydrodynamic (MHD) stability, the disadvantage being a potentially higher cost for the magnetic coils.[36] In agreement with the advantage here predicted of using a negative U_0, it is worth saying that experiments have indeed shown that negative biasing leads not only to a larger improvement in particle confinement, but it also reduces the propagation of large-scale events (or blobs) and lowers the amplitude of fluctuations.[26,27] Note that such a prediction, on being more beneficial to apply a negative (as opposed to a positive) bias, could not be retrieved from the TOKAM-2D model because $\tilde{g} = 0$ there,[12–14,17,18,37] and with $\tilde{g} = 0$ (27) would read $4k^2\partial_1(1/B)|\partial_1 n|/n \le (U_0 e^{-Z/\lambda_D}/\lambda_D^3)^2$ instead, a condition where the sign of U_0 has no influence whatsoever. It could also not be retrieved from the dispersion relation previously established in connection with FDUT simulations, even if the latter properly allowed for a finite, non-vanishing $\tilde{g}$, as neither biasing nor a non-uniform unperturbed potential $\phi_0^{(0)}$ were explicitly accounted for ($\phi_0^{(0)}$ having been set to the electron temperature, obviously uniform in the FDUT model).[15] Note still that the linear growth rate for the interchange instability has been known to be an increasing function of the radial decay of the magnetic field as conveyed by g on the lhs of (26),[14] but the discussion now becomes more subtle because of the presence of $\tilde{g}$, so much so that for asymptotically large magnetic-field inhomogeneities (so as to render negligible both the density and vorticity gradients) the system does become stable, with linear stability becoming independent of $\partial_1(1/B)$. Before concluding this section, it is worth pointing out that any dependence of the biasing on x_2 (for instance, coming from using a finite number of localized probes),[18,33] and the subsequent modulation on this variable it would entail in the results derived above, would not invalidate the conclusions drawn concerning the asymmetric effect of positive and negative electric bias and the smoothing of the magnetic-field inhomogeneity.

V. STRUCTURES AND CONSERVATIVE DYNAMICS

Having up to now been concerned, in Secs. III and IV, with growth rates and the stability (both linear and nonlinear) of model solutions, it is time to try to retrieve some of the features of the solutions themselves. Many dynamical systems of physical interest have both conservative and dissipative components, having been proved that the finite-dimensional vector fields always correspond to a superposition of Hamiltonian and gradient components.[28] The identification of the Hamiltonian component is important because it frequently happens that, in some regions of phase space, the effect of the non-conservative components cancels out along a neighbourhood of some of the Hamiltonian orbits, implying the full system ends up displaying deformed versions of the latter, which led to the notion of "constants of motion in dissipative systems."[38] It is conceivable that a similar situation might apply in infinite dimensions because one of the tools used for the finite-dimensional vector field decomposition, namely, the Hodge–De Rahm theorem, can be extended to infinite dimensions.[39] For the physical problem dealt with in this paper, the implication is that, looking for solutions of the conservative part of the model, one might at least obtain the ancestors of coherent structures that might also exist in the full model.

Therefore, going back to the model in (3), its conservative part reads as the system of partial differential equations (PDE's)

$$\frac{\partial Ln}{\partial t} = -[\phi, Ln] - \tilde{g}\partial_2(Ln - \phi) \tag{28a}$$

$$\frac{\partial \nabla^2\phi}{\partial t} = -\left[\phi, \nabla^2\phi\right] - g\partial_2 Ln, \tag{28b}$$

which, upon the transformation

$$Ln(x_1, x_2, t) = \widetilde{Ln}(x_1, x_2, t) - \tilde{g}x_1 \tag{29a}$$

$$\phi(x_1, x_2, t) = \tilde{\phi}(x_1, x_2, t) - \tilde{g}x_1 \tag{29b}$$

and assuming $\tilde{g}$ to be constant (hence also g because of (2)), becomes

$$\frac{\partial \widetilde{Ln}}{\partial t} = -\left[\tilde{\phi}, \widetilde{Ln}\right] \tag{30a}$$

$$\frac{\partial \nabla^2\tilde{\phi}}{\partial t} = -\left[\tilde{\phi}, \nabla^2\tilde{\phi}\right] - g\partial_2\widetilde{Ln} + \tilde{g}\partial_2\nabla^2\tilde{\phi}. \tag{30b}$$

One now looks for travelling-wave solutions to this system by writing

$$\widetilde{Ln}(x_1, x_2, t) = \widetilde{Ln}(x_1 - v_1 t, x_2 - v_2 t) \tag{31a}$$

$$\tilde{\phi}(x_1, x_2, t) = \tilde{\phi}(x_1 - v_1 t, x_2 - v_2 t) \tag{31b}$$

so that, defining $y_i = x_i - v_i t$ and making $\partial_i = \partial/\partial y_i$ henceforth, one obtains from (30) and (31)

$$v_1\partial_1\widetilde{Ln} + v_2\partial_2\widetilde{Ln} = \partial_1\tilde{\phi}\partial_2\widetilde{Ln} - \partial_1\widetilde{Ln}\partial_2\tilde{\phi} \tag{32a}$$

$$\begin{aligned} v_1\partial_1\nabla^2\tilde{\phi} + v_2\partial_2\nabla^2\tilde{\phi} &= \partial_1\tilde{\phi}\partial_2\nabla^2\tilde{\phi} - \partial_1\nabla^2\tilde{\phi}\partial_2\tilde{\phi} \\ &\quad + g\partial_2\widetilde{Ln} - \tilde{g}\partial_2\nabla^2\tilde{\phi}. \end{aligned} \tag{32b}$$

Putting

$$\tilde{\phi}(y_1, y_2) = F(y_1, y_2) + v_2 y_1 - v_1 y_2, \tag{33}$$

(32a) reduces to

$$\partial_1 F\partial_2\widetilde{Ln} = \partial_1\widetilde{Ln}\partial_2 F \tag{34a}$$

$$\partial_1 F\partial_2\nabla^2 F - \partial_1\nabla^2 F\partial_2 F = -g\partial_2\widetilde{Ln} + \tilde{g}\partial_2\nabla^2 F, \tag{34b}$$

where (34a) can be seen to be satisfied for

$$\widetilde{Ln} = f(F), \tag{35}$$

with f an arbitrary differentiable function, so substitution in (34b) yields

$$\partial_1 F\partial_2\nabla^2 F - \partial_1\nabla^2 F\partial_2 F + g\partial_2 f(F) - \tilde{g}\partial_2\nabla^2 F = 0, \tag{36}$$

which admits a very large number of solutions, depending on the choice of $f(F)$. The symmetry exhibited by (34) allows one to identify also as solutions those of the form

$$F(y_1, y_2) = F_s(y_1, y_2) \tag{37}$$

and

$$\widetilde{Ln} = \Theta_s F_s, \tag{38}$$

with Θ_s and F_s, respectively, an operator and a function symmetric in y_1 and y_2 (meaning they remain invariant under an exchange between the two variables), so

$$F_s(y_2, y_1) = F_s(y_1, y_2) \tag{39}$$

and

$$\Theta_s F_s(y_2, y_1) = \Theta_s F_s(y_1, y_2), \tag{40}$$

in which case (34) becomes

$$g\partial_2\Theta_s F_s - \tilde{g}\partial_2\nabla^2 F_s = 0. \tag{41}$$

For example, possible solutions obeying (35) can be obtained by making

$$f(F) = \alpha F, \tag{42}$$

with α constant, and writing

$$F(y_1, y_2) = \widetilde{F}(y_1, y_2) + \tilde{g} y_1, \tag{43}$$

so (36) takes the form

$$\partial_1 \widetilde{F} \partial_2 \nabla^2 \widetilde{F} - \partial_1 \nabla^2 \widetilde{F} \partial_2 \widetilde{F} = -\alpha g \partial_2 \widetilde{F}. \tag{44}$$

An obvious solution to the homogeneous counterpart of (44) (obtained by setting to zero the rhs of the latter) is the wave form [see (k) in the Appendix]

$$\widetilde{F}_0(y_1, y_2) = A \cos(k_1 y_1 + k_2 y_2) + B \sin(k_1 y_1 + k_2 y_2) \tag{45}$$

and, plugging into (44), the *ansatz*

$$\widetilde{F}(y_1, y_2) = \widetilde{F}_0(y_1, y_2) + H(y_1), \tag{46}$$

chosen such as not to introduce an extra term on the rhs of (44), one gets, already accounting for (45),

$$[(k^2 + \partial_1^2)\partial_1 H - \alpha g]\partial_2 \widetilde{F}_0 = 0. \tag{47}$$

To solve the ODE on the lhs of (47), one may transform it into a homogeneous ODE by setting

$$H(y_1) = \widetilde{H}(y_1) + \frac{\alpha g}{k^2} y_1, \tag{48}$$

so

$$\partial_1^3 \widetilde{H} + k^2 \partial_1 \widetilde{H} = 0, \tag{49}$$

whose solution is trivially found to be [see (l) in the Appendix]

$$\widetilde{H}(y_1) = C_0 + C_1 \cos(k y_1) + C_2 \sin(k y_1). \tag{50}$$

Retracing the calculation backwards and assembling together (29), (31), (33), (35), (42), (43), (45), (46), (48), and (50), the original quantities that are solutions to (28) are thus

$$Ln(x_1, x_2, t) = \alpha[\phi(x_1, x_2, t) - v_2(x_1 - v_1 t) + v_1(x_2 - v_2 t)] + (\alpha - 1)\tilde{g} x_1 \tag{51a}$$

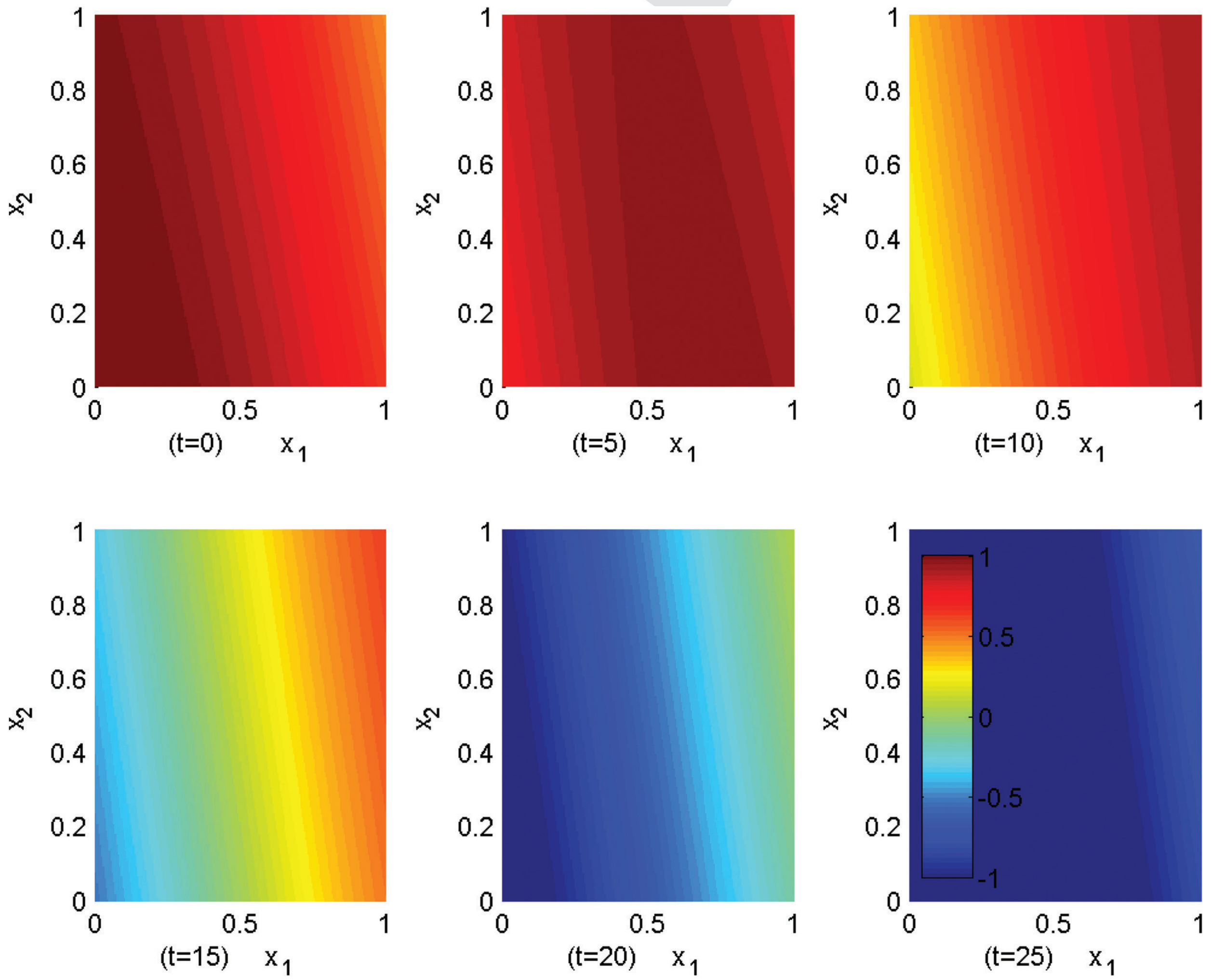

FIG. 1. Six snapshots with contour plots of the log-density function Ln for the solution (51) of the conservative system (28), with $B = C_0 = C_1 = C_2 = 0$, $\alpha = A = 1$, $v_1 = v_2 = 0.1$, $k_1 = 1$, $k_2 = 0.2$, and $g = \tilde{g} = 0.1$, the units in the colour bar being arbitrary.

$$\phi(x_1,x_2,t) = \left(v_2 + \tilde{g} + \frac{\alpha g}{k^2}\right)(x_1 - v_1 t) - v_1(x_2 - v_2 t) - \tilde{g}x_1 + A\cos[k_1(x_1 - v_1 t) + k_2(x_2 - v_2 t)] + B\sin[k_1(x_1 - v_1 t) + k_2(x_2 - v_2 t)] + C_0 + C_1\cos[k(x_1 - v_1 t)] + C_2\sin[k(x_1 - v_1 t)]. \tag{51b}$$

In Fig. 1 are shown snapshots of the propagation of the function $Ln(x_1,x_2,t)$ for one such solution with $B = C_0 = C_1 = C_2 = 0$, $\alpha = A = 1$, $v_1 = v_2 = 0.1$, $k_1 = 1$, $k_2 = 0.2$, and $g = \tilde{g} = 0.1$, and one clearly sees how the wave front propagates from the inner boundary (the core plasma region) to the outer layer. Of course, solutions of (36) such as (51) are only solutions to the conservative part of the equations, with diffusion, parallel losses, and source not being accounted for. However, inspection of (3) suggests the role of the unspecified source term in (3a) is to compensate for the parallel losses, whereas the longitudinal conductance term in (3b) will not play a determinant role as long as fluctuations away from the plasma potential are not very large. Therefore, it is not unlikely that the overall structure of the complete solutions will be mostly determined by the conservative dynamics induced by (28).

Another solution to the homogeneous counterpart of (44) is

$$\widetilde{F}_0(y_1,y_2) = Ae^{(k_1 y_1 + k_2 y_2)}, \tag{52}$$

hence, putting

$$\widetilde{F}(y_1,y_2) = \widetilde{F}_0(y_1,y_2) + \widetilde{H}(y_1) - \frac{\alpha g}{k^2}y_1, \tag{53}$$

(44) becomes

$$\partial_1^3 \widetilde{H} - k^2 \partial_1 \widetilde{H} = 0, \tag{54}$$

whose general solution reads [see (m) in the Appendix]

$$\widetilde{H}(y_1) = C_0 + C_1 e^{k y_1} + C_2 e^{-k y_1}. \tag{55}$$

On account of this, the functions

$$Ln(x_1,x_2,t) = \alpha[\phi(x_1,x_2,t) - v_2(x_1 - v_1 t) + v_1(x_2 - v_2 t)] + (\alpha - 1)\tilde{g}x_1 \tag{56a}$$

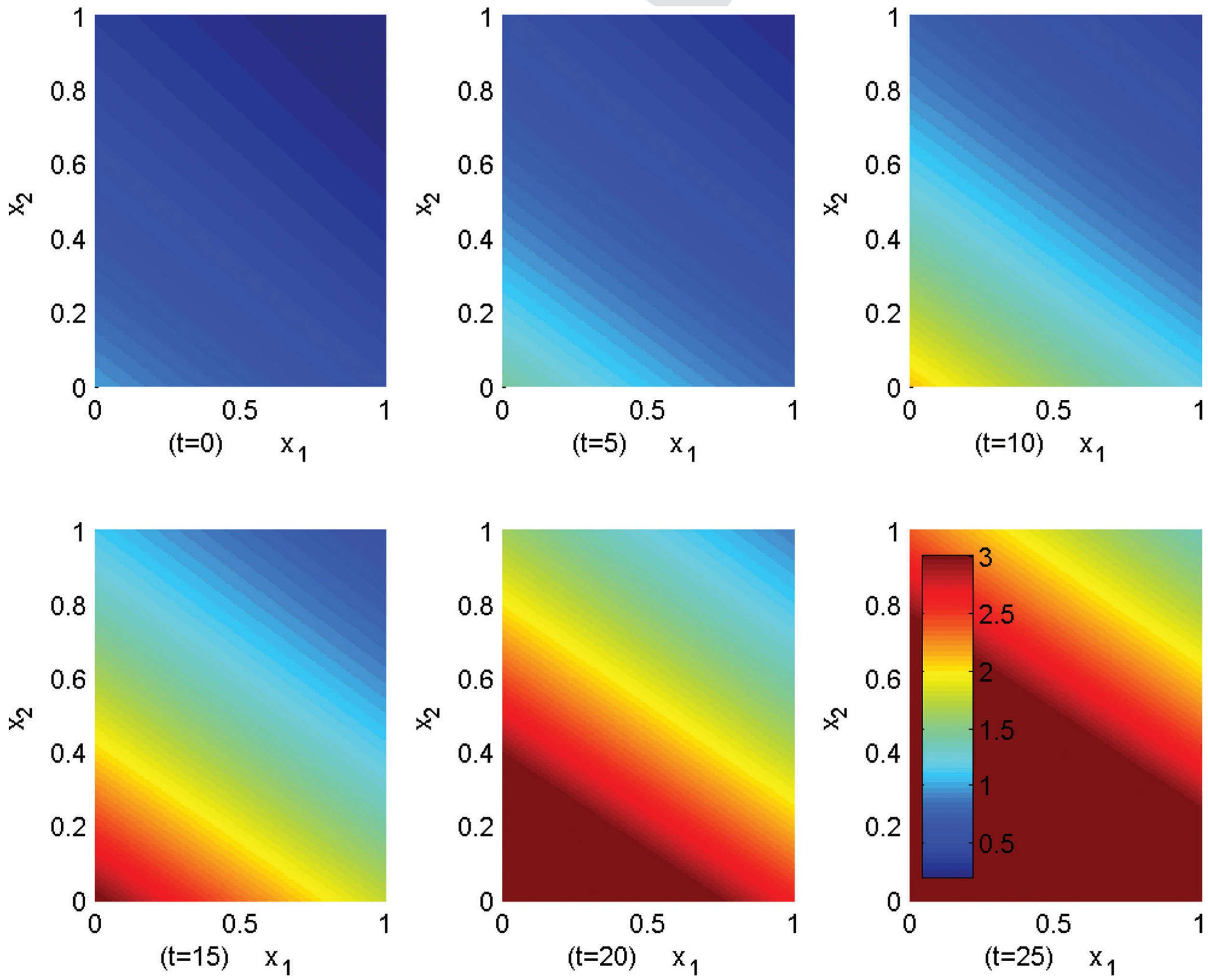

FIG. 2. Six snapshots with contour plots of the log-density function Ln for the solution (56) of the conservative system (28), for $C_0 = C_1 = C_2 = 0$, $\alpha = A = 1$, $v_1 = v_2 = 0.05$, $k_1 = -0.5$, $k_2 = -1$, and $g = \tilde{g} = 0.1$, the units in the colour bar being arbitrary.

$$\phi(x_1,x_2,t) = \left(v_2 + \tilde{g} - \frac{\alpha g}{k^2}\right)(x_1 - v_1 t) - v_1(x_2 - v_2 t) - \tilde{g}x_1 + Ae^{[k_1(x_1-v_1t)+k_2(x_2-v_2t)]} + C_0 + C_1 e^{k(x_1-v_1t)} + C_2 e^{-k(x_1-v_1t)} \tag{56b}$$

are also solutions to (28), snapshots of $Ln(x_1,x_2,t)$ hereabove being given in Fig. 2 for $C_0 = C_1 = C_2 = 0$, $\alpha = A = 1$, $v_1 = v_2 = 0.05$, $k_1 = -0.5$, $k_2 = -1$, and $g = \tilde{g} = 0.1$. One sees, also in this case, a large-scale structure that is initially located at the inner SOL region and subsequently starts moving essentially outwards. The important point to retain here is the existence of solutions that move concentrations of particles and energy along both (radial- and poloidal-like) directions in a 2D cross-section, eventually ejecting them from the core to the wall, hence mimicking the basic, gross behavior depicted by blobs that is observed in experiments and numerical simulations.[2,4,6,9–11,14–16,19–22]

Eventually more localized solutions to (28) can be derived from (41) by noting that the Laplacian operator ∇^2 is symmetric in y_1 and y_2, so one can put

$$\Theta_s F_s = \frac{\tilde{g}}{g}\nabla^2 F_s \tag{57}$$

and choose (amongst very many possibilities)

$$F_s(y_1,y_2) = Ae^{-\gamma(y_1^2+y_2^2)/2}, \tag{58}$$

which leads to

$$Ln(x_1,x_2,t) = -\frac{\tilde{g}}{g}\gamma A e^{-\gamma[(x_1-v_1t)^2+(x_2-v_2t)^2]/2} \times \left\{2 - \gamma\left[(x_1-v_1t)^2 + (x_2-v_2t)^2\right]\right\} - \tilde{g}x_1 \tag{59a}$$

$$\phi(x_1,x_2,t) = Ae^{-\gamma[(x_1-v_1t)^2+(x_2-v_2t)^2]/2} + v_2(x_1-v_1t) - v_1(x_2-v_2t) - \tilde{g}x_1 \tag{59b}$$

by back substituting through (29), (31), (33), (37), (38), (57), and (58). The time evolution of $Ln(x_1,x_2,t)$ given by (59) is shown in Fig. 3 for $A = -1.0$, $v_1 = 0.048$, $v_2 = 0.028$, $\gamma = 15$, and $g = \tilde{g} = 0.1$, and in Fig. 4 for the same set of parameters but for $\gamma = 50$. The shape of the blob-like structures depicted in Figs. 3 and 4 can be modulated by multiplying the Gaussian packet in (58) by any other function symmetric in y_1 and y_2, which can take a multitude of

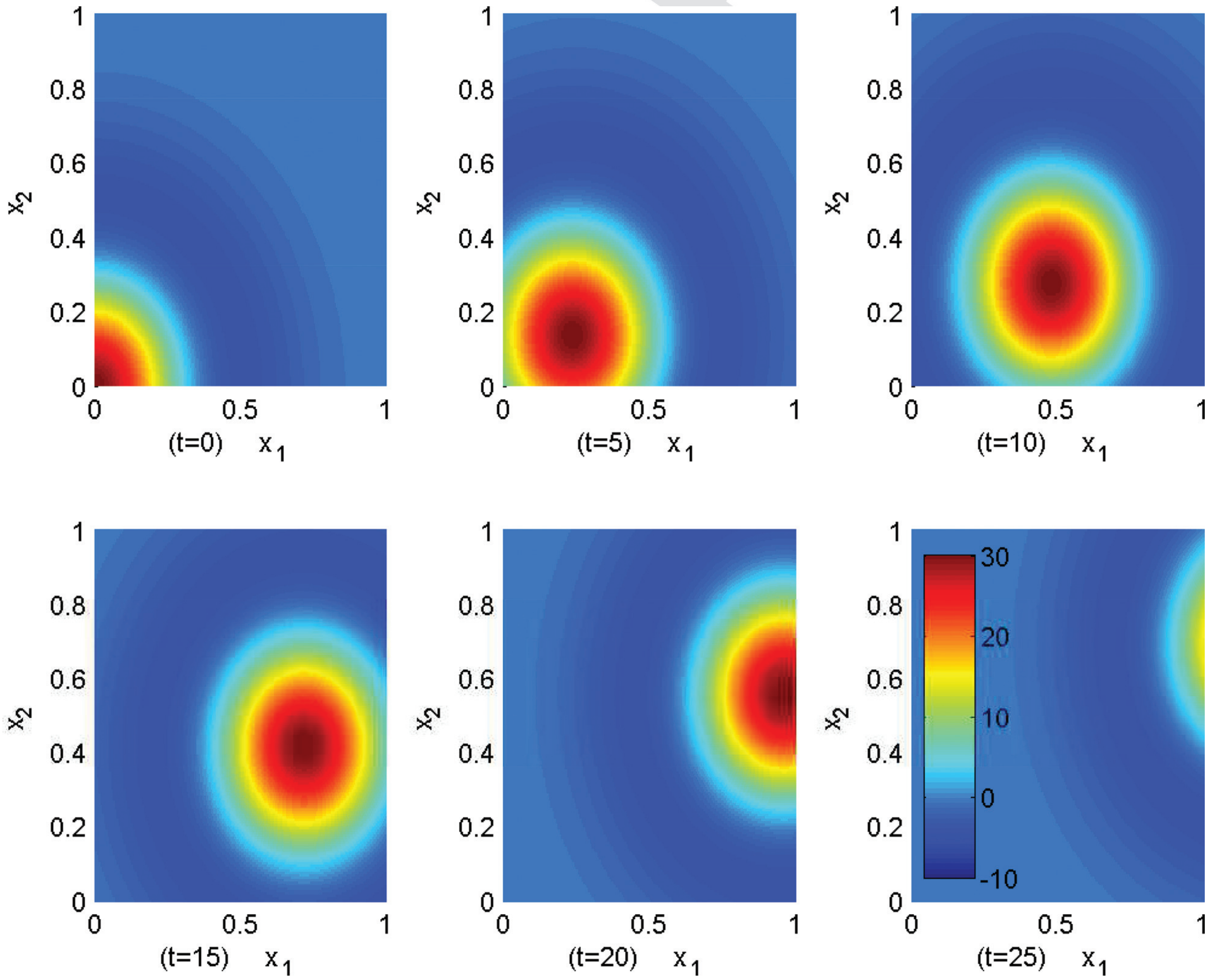

FIG. 3. Six snapshots with contour plots of the log-density function Ln for the solution (59) of the conservative system (28), for $A = -1.0$, $v_1 = 0.048$, $v_2 = 0.028$, $\gamma = 15$, and $g = \tilde{g} = 0.1$, the units in the colour bar being arbitrary.

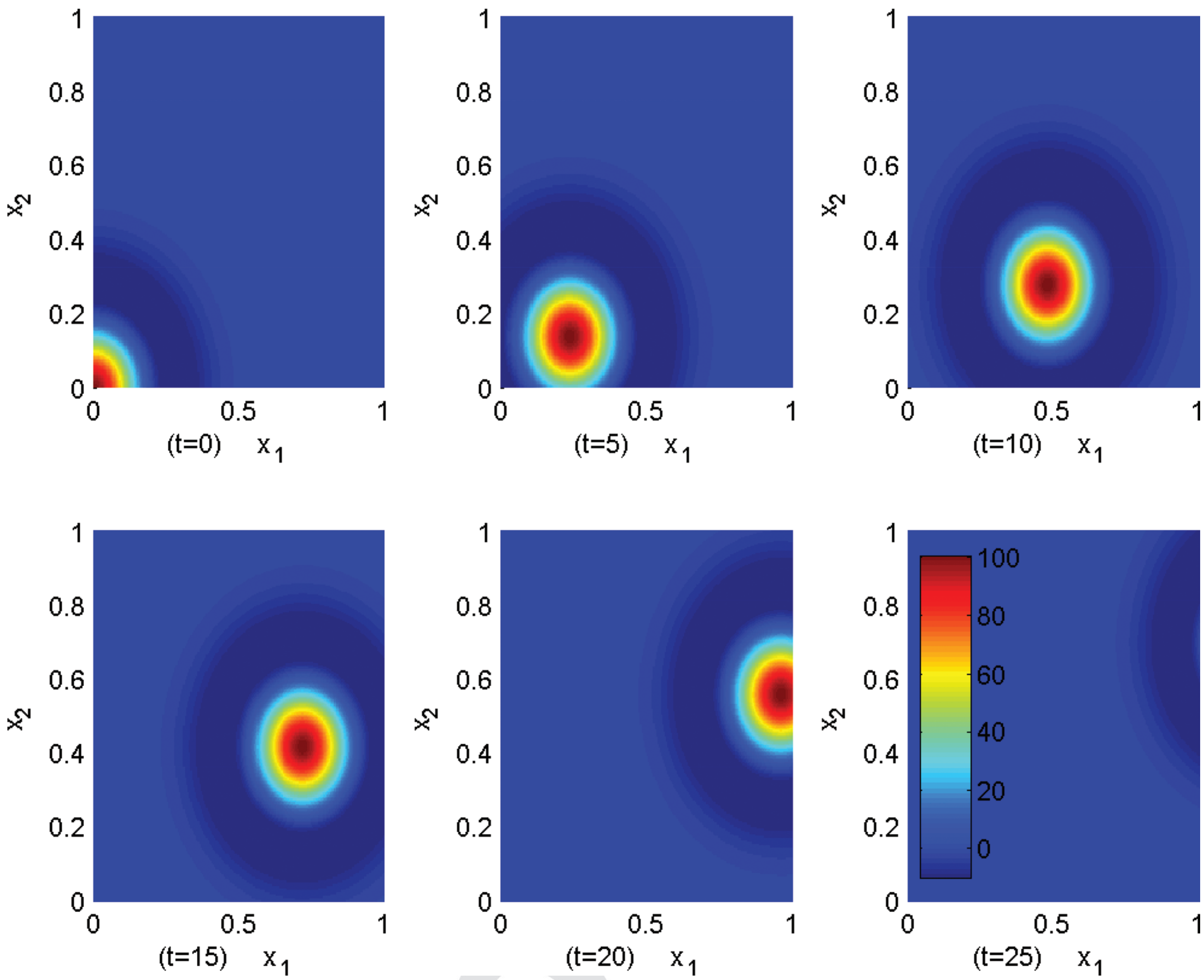

FIG. 4. Six snapshots with contour plots of the log-density function Ln for the solution (59) of the conservative system (28)), for $A = -1.0$, $v_1 = 0.048$, $v_2 = 0.028$, $\gamma = 50$, and $g = \tilde{g} = 0.1$, the units in the colour bar being arbitrary.

different forms including, for instance, polynomial or trigonometric. As exemplified in Figs. 1–4 for solutions (51), (56), and (59), the blob-, jet-like behavior of large-scale structures arising from solving for the conservative part of the model can take very many distinct forms, in fact, as many as afforded by the very large number of possible solutions to (28), combined with the multiplicity in the choice of the various parameters characterizing them. Before concluding, note the plots in Figs. 1–4 have no pretension, at this stage, to exactly mimic or reproduce any actual experimental observations, and some of them may even end up by not being easily associated with a physical picture of what happens in the SOL. Strictly speaking, they represent exact solutions of a system of PDE's, some of which may not have an obvious physical interpretation, a direct confrontation with experiments demanding eventually a more judicious choice of the parameters entering them which, probably, have to be chosen taking into account appropriate initial or boundary conditions, as well as the relevant machine data.

VI. SUMMARY AND CONCLUSIONS

In this article, a theoretical analysis has been provided on various mathematical aspects of a two-fluid model describing SOL turbulence in slab geometry, similar to the TOKAM-2D model,[12–14,17,18] but retaining the magnetic-field inhomogeneity terms in the continuity equation as in previous FDUT simulations.[15,16] The model equations have a conservative kernel governing transport across magnetic-field lines, plus extra terms that account for diffusion, longitudinal losses (along the magnetic field) and a plasma core source. It has been shown that an upper bound for the growth rate of the vorticity gradient depends on (hence is partially controlled by) the density gradient and that, inversely, an upper bound for the growth rate of the latter depends on the former, which seems to indicate the presence in the model of a nonlinear instability, with both quantities working together to pull their gradients further and further up. The possibility of controlling the turbulent fluctuations in model quantities by means of a biasing potential has also been assessed, having one confirmed the stabilising role played by the third derivative of this potential,[14] but also having one demonstrated that negative is more favorable than positive bias, thus providing a theoretical explanation for the experimental observations pointing in the same direction.[26,27] In relation to the benefits of applying a negative versus a positive polarization, it is worth mentioning that it has quite recently been found that a negative electrode biasing stabilizes the $m/n = 2/1$ tearing mode, by accelerating its rotation, whereas a positive biasing leads to opposite effects.[40] It has further been checked that reducing the inhomegeneity of the magnetic field in the SOL helps in stabilizing fluctuations there. Keep in mind that it is not within the scope of this paper to address the eventual use of finer, "smarter" control techniques, nor their eventual energetic cost,[32,33] its purpose regarding control of fluctuations being limited to the more "brute-force" approach of identifying how one could act on the stability of possible turbulent modes and on their growth rates, so as to compare with the previous linear-stability-analysis results.[14]

Finally, the model equations have been analytically solved for their conservative part and exact solutions of the travelling-wave type have been derived, some of which propagate from the inner to the outer plasma layers and might, therefore, be interpreted as the conservative ancestors of the collective, large-scale structures named blobs observed both in experiments and in computations. Of course, this plausible conjecture that the exact travelling-wave solutions of the conservative model are the ancestors of the collective local structures that develop in the SOL should be confirmed for the full model by numerical simulations. This is a project under development for which, as stated in Sec. I, to have already some analytical control over model solutions will allow one to distinguish between the actual model behavior and the numerical artefacts or nonsense.

ACKNOWLEDGMENTS

The authors thank Rogério Jorge and Patrick Tamain for useful comments on the manuscript, and are also grateful to Luís Venâncio for identifying some errors in the original plots.

This work was carried out within the framework of the EUROfusion Consortium and received funding from the Euratom research and training programme 2014–2018 under Grant Agreement No. 633053, including enabling the research project CfP-WP14-ER-01/CEA-09. Instituto Superior Técnico (IST) activities also received financial support from Fundação para a Ciência e a Tecnologia (FCT, Lisboa) through project No. UID/FIS/50010/2013. The views and opinions expressed herein do not necessarily reflect those of the European Commission, of FCT, of IST or of their services.

APPENDIX: COMPLEMENTARY MATHEMATICAL DETAILS

(a) Note that the diffusion related term $D|\nabla Ln|^2$ in (3), which is often missing in SOL literature, comes from the usual $D\nabla^2 n$ term in the equation for $\partial n/\partial t$, which transforms according to $D\nabla^2 n/n = D(\nabla^2 Ln + |\nabla Ln|^2)$ when passing to the equation for $\partial Ln/\partial t$.

(b) Note that $\int_D \phi\partial\nabla^2\phi/\partial t = -\int_D \nabla\phi\cdot\partial\nabla\phi/\partial t = -(1/2)\int_D \partial|\nabla\phi|^2/\partial t$ and $\int_D \phi\nabla\nabla^2\phi\times\nabla\phi = -\int_D \nabla\times(\phi\nabla\phi)\nabla^2\phi = -\int_D(\phi\nabla\times\nabla\phi + \nabla\phi\times\nabla\phi)\nabla^2\phi = 0$.

(c) Note that $\nu(\nabla^2\phi)^2 \geq 0$, $(g\phi - \partial_2 Ln)^2 \geq 0$, so $2\phi g\partial_2 Ln \leq (g\phi)^2 + (\partial_2 Ln)^2$, and $(\partial_2 Ln)^2 \leq (\partial_2 Ln)^2 + (\partial_1 Ln)^2 = |\nabla Ln|^2$.

(d) Note that $\int_D \nabla Ln\cdot\partial\nabla Ln/\partial t = (1/2)\int_D \partial|\nabla Ln|^2/\partial t$, $\int_D \nabla Ln\cdot\nabla(\widehat{\mathbf{b}}\cdot\nabla Ln\times\nabla\phi) = -\widehat{\mathbf{b}}\cdot\int_D(\nabla Ln\times\nabla\phi)\nabla^2 Ln$, $\int_D \nabla Ln\cdot\nabla[\tilde{g}\partial_2(Ln-\phi)] = -\int_D \tilde{g}\partial_2(Ln-\phi)\nabla^2 Ln$, and $\int_D \nabla Ln\cdot\nabla[D(\nabla^2 Ln + |\nabla Ln|^2)] = -\int_D D(\nabla^2 Ln + |\nabla Ln|^2)\nabla^2 Ln$.

(e) Note that $(\nabla^2 Ln)^2 \geq 0$, $-2\widehat{\mathbf{b}}\cdot(\nabla Ln\times\nabla\phi)\nabla^2 Ln \leq |\nabla\phi|^2 + |\widehat{\mathbf{b}}\times\nabla Ln|^2(\nabla^2 Ln)^2 = |\nabla\phi|^2 + |\nabla Ln|^2(\nabla^2 Ln)^2$, $-2|\nabla Ln|^2\nabla^2 Ln \leq |\nabla Ln|^4 + (\nabla^2 Ln)^2$, and $2\tilde{g}\partial_2(Ln-\phi)\nabla^2 Ln \leq [\partial_2(Ln-\phi)]^2 + (\tilde{g}\nabla^2 Ln)^2 \leq |\nabla(Ln-\phi)|^2 + (\tilde{g}\nabla^2 Ln)^2$.

(f) Note that $|\nabla(Ln-\phi)|^2 \leq (|\nabla Ln| + |\nabla\phi|)^2 = 2(|\nabla Ln|^2 + |\nabla\phi|^2) - (|\nabla Ln| - |\nabla\phi|)^2 \leq 2(|\nabla Ln|^2 + |\nabla\phi|^2)$.

(g) Note that $[f+\delta f, h+\delta h] = [f,h] + [f,\delta h] + [\delta f, h] + [\delta f, \delta h]$.

(h) Note that $f(x_1,x_2,t) = \sum_{n=0}^{\infty}(x_1\partial_1 + x_2\partial_2)^n f(0,0,t)/n!$, $\widehat{h}(k_1-q_1,k_2-q_2,t) = \sum_{n'=0}^{\infty}(-1)^{n'}(q_1\partial_{k1} + q_2\partial_{k2})^{n'}\widehat{h}(k_1,k_2,t)/n'!$, and $(1/4\pi^2)\int\int(q_1\partial_{k1} + q_2\partial_{k2})^{n'}e^{-i(q_1x_1+q_2x_2)}dq_1dq_2 = i^{n'}(\partial_1\partial_{k1} + \partial_2\partial_{k2})^{n'}\delta(x_1)\delta(x_2)$, where Taylor and binomial expansions have been used, and $\delta(x_i)$ is a Dirac delta function.

(i) Note that $(x_1\partial_1 + x_2\partial_2)^n f(0,0,t) = \sum_{m=1}^{n}[n!/m!(n-m)!]x_1^m\partial_1^m x_2^{n-m}\partial_2^{n-m}f(0,0,t)$, $(\partial_1\partial_{k1} + \partial_2\partial_{k2})^{n'}\widehat{h}(k_1,k_2,t) = \sum_{m'=1}^{n'}[n'!/m'!(n'-m')!]\partial_1^{m'}\partial_{k1}^{m'}\partial_2^{n'-m'}\partial_{k2}^{n'-m'}\widehat{h}(k_1,k_2,t)$, and $\int\int x_1^m\partial_1^{m'}x_2^{n-m}\partial_2^{n'-m'}\delta(x_1)\delta(x_2)dx_1dx_2 = (-1)^{n'}m'!(n'-m')!\int\int x_1^{m-m'}x_2^{n-n'-m+m'}\delta(x_1)\delta(x_2)dx_1dx_2 = (-1)^{n'}m'!(n'-m'))\delta_{mm'}\delta_{nn'}$, where Taylor and binomial expansions have been used, and $\delta_{nn'}$ is the Kronecker delta symbol.

(j) Note that the characteristic equation for the system of ODE's $\partial\widehat{\delta Ln}/\partial t = A\widehat{\delta Ln} + B\widehat{\delta\phi}$ and $\partial\widehat{\delta\phi}/\partial t = C\widehat{\delta Ln} + D\widehat{\delta\phi}$ is the eigenvalue equation $(A-\lambda)(D-\lambda) - BC = 0$, whose solutions are $\lambda_\pm = (A+D)/2 \pm\sqrt{(A-D)^2 + 4BC}/2$, from which the damping/growth rates can be extracted as $\gamma_\pm = \mathrm{Re}(\lambda_\pm)$.

(k) Note that $\partial_i\widetilde{F}_0\partial_j\nabla^2\widetilde{F}_0 = -k^2\partial_i\widetilde{F}_0\partial_j\widetilde{F}_0$.

(l) Note that the corresponding characteristic equation is simply $\lambda^3 + k^2\lambda = 0$.

(m) Note that the corresponding characteristic equation is now $\lambda^3 - k^2\lambda = 0$.

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