Stochastic solutions of nonlinear PDE's and an extension of superprocesses

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Abstract. Stochastic solutions provide new rigorous results for nonlinear PDE's and, through its local non-grid nature, are a natural tool for parallel computation. There are two different approaches for the construction of stochastic solutions: MacKean's and superprocesses. Here one shows how to extend the McKean construction to equations with derivatives and non-polynomial interactions. On the other hand, when restricted to measures, superprocesses can only be used to generate solutions for a limited class of nonlinear PDE's. A new class of superprocesses, namely superprocesses on signed measures and on ultradistributions, is proposed to extend the stochastic solution approach to a wider class of PDE's.

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1. Introduction: Stochastic solutions and measure-valued processes

A *stochastic solution* of a linear or nonlinear partial differential equation is a stochastic process which, starting from a point x in the domain generates after time t a boundary measure that, sampling the initial condition at $t=0$, provides the solution at the point x and time t. For illustration consider the McKean [1] construction of a stochastic solution for the KPP equation

$$\frac{\partial v}{\partial t}=\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+v^2-v \qquad\qquad v\left(0,x\right)=g\left(x\right) \tag{1.1}$$

Let $G\left(t,x\right)$ be the Green's operator for the heat equation $\partial_t v(t,x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}v(t,x)$

$$G\left(t,x\right)=e^{\frac{1}{2}t\frac{\partial^2}{\partial x^2}}$$

and write the KPP equation in integral form

$$v(t,x) = e^{-t} G(t,x) g(x) + \int_0^t e^{-(t-s)} G(t-s,x) v^2(s,x) ds \qquad (1.2)$$

Denoting by (ξ_t, Π_x) a Brownian motion starting from time zero and coordinate x, Eq.(1.2) may be rewritten as

$$\begin{aligned} v(t,x) &= \Pi_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-(t-s)} v^2(s, \xi_{t-s}) ds \right\} \\ &= \Pi_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-s} v^2(t-s, \xi_s) ds \right\} \qquad (1.3) \end{aligned}$$

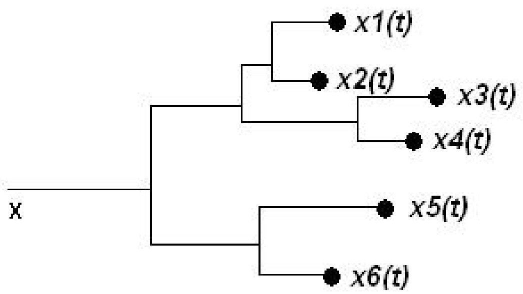

FIGURE 1. The McKean process

The *stochastic solution process* is a composite process: a Brownian motion plus a branching process with exponential holding time T, $P(T > t) = e^{-t}$(Fig.1). At each branching point the particle splits into two, the new particles going along independent Brownian paths. At time $t > 0$, if there are n particles located at $x_1(t), x_2(t), \cdots x_n(t)$, the solution of (1.1) is obtained by

$$v(t,x) = \mathbb{E}_x \{ g(x_1(t)) g(x_2(t)) \cdots g(x_n(t)) \}$$

An equivalent interpretation, that corresponds to the second equality in (1.3), is of a process starting from time t at x and propagating backwards-in-time to time zero. When it reaches $t = 0$ the process samples the initial condition, that is, it generates a measure μ at the $t = 0$ boundary which yields the solution by (1).

The construction of solutions for nonlinear equations, through the stochastic interpretation of the integral equations, has become an active field in recent years, applied for example to Navier-Stokes [2] [3] [4] [5] [6], to Vlasov-Poisson [7] [8] [9], to Euler [10] to magnetohydrodynamics [11] and to a fractional version of the KPP equation [12]. In addition to providing new exact results for nonlinear PDE's, the stochastic solutions are also a promising tool for numerical implementation, in particular for parallel computation using for example the recently develop probabilistic domain decomposition method [13] [14] [15]. This method decomposes the integration domain into subdomains, uses in each one a deterministic algorithm with Dirichlet boundary conditions, the values at the boundaries being obtained by a stochastic algorithm. This minimizes the time-consuming communication problem between subdomains and allows for extraordinary improvements in computer time.

There are basically two methods to construct stochastic solutions. The first method, which will be called the McKean method, as illustrated above, is essentially a probabilistic interpretation of the Picard series. The differential equations are written as integral equations which are rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series. The second method [16] [17] constructs the boundary measures of a measure-valued stochastic process (a superprocess) and obtains the solution of the differential equation by a scaling procedure. For a detailed comparison of the two methods refer to [18].

As developed in the past, both methods lead to boundary measure-valued processes which are used to integrate a boundary function. As representations of solutions of the nonlinear equations of physical interest both methods have serious limitations. For the McKean method it is not clear how to handle nonpolynomial interaction terms and terms with derivatives. For the measure-valued superprocesses, in addition to these problems, they can only be applied to a limited class of nonlinear partial differential equations. In this paper both problems will be addressed, namely how to handle derivatives and nonpolynomial interactions in the McKean construction and how to extend superprocesses from measure-valued to ultradistribution-valued processes.

2. Stochastic solutions with derivatives and non-polynomial terms

To extend the construction of stochastic solutions to cases more general than those dealt with in the past, techniques must be developed to handle derivatives and nonpolynomial interactions. Sometimes the direct handling of derivatives may be avoided if the derivative of the propagation kernel is smooth. This is the case in the configuration space Navier-Stokes equation [5], where by an integration by parts the derivative of the heat kernel is controlled by a majorizing kernel and absorbed in the probability measure. However, in general this is not possible.

Sometimes the nonpolynomial interaction case may be reduced to the polynomial case by expanding the interaction term in a Taylor series and normalizing the coefficients to obtain a probabilistic interpretation. Again, this is not always possible.

Here the solution of both problems (derivatives and non-polynomials) is illustrated in the example of the SOLEDGE2D equations [19] which describe plasma dynamics in the scrape-off layer. Other examples and details may be found in [20]. One deals with the Cauchy problem, namely the equations are defined in the full space with initial conditions at $t=0$. This is the most natural setting when the McKean approach is used. Spatial boundary conditions are easier to implement through the superprocess formulation, with or without a scaling limit (see [18]). Here the nonpolynomial and derivative terms will be treated as operator labels at the branching points of the process.

The SOLEDGE2D equations are [19]

$$\begin{aligned} \partial_t N + \frac{1}{q}\partial_\theta \Gamma + \frac{\chi}{\eta} N &= D\partial_r^2 N \\ \partial_t \Gamma + \frac{1}{q}(1-\chi)\,\partial_\theta\left(\frac{\Gamma^2}{N}+N\right) + \frac{\chi}{\eta}(\Gamma-\Gamma_0) &= \nu\partial_r^2\Gamma \end{aligned} \tag{2.1}$$

where Γ and N are the dimensionless parallel momentum and density, (r,θ) are the radial and poloidal coordinates and the mask function χ equals one in a region where an obstacle is located and zero elsewhere.

To construct a stochastic representation for the solution one needs to identify a stochastic process associated to the linear component (to the full linear component or part of it) and then, through an integral equation, construct the branching mechanism representing the nonlinear part.

2.1. The $\chi = 1$ case

In the $\chi = 1$ case the system (2.1) is linear

$$\begin{aligned} \partial_t N + \frac{1}{q}\partial_\theta \Gamma + \frac{1}{\eta} N &= D\partial_r^2 N \\ \partial_t \Gamma + \frac{1}{\eta}(\Gamma-\Gamma_0) &= \nu\partial_r^2\Gamma \end{aligned} \tag{2.2}$$

the solution being

$$\begin{aligned} \begin{pmatrix} N(t) \\ \Gamma(t) \end{pmatrix} &= e^{t\left(-\frac{1}{q}C\partial_\theta + B\partial_r^2 - \frac{1}{\eta}\right)}\left\{\begin{pmatrix} N(0) \\ \Gamma(0) \end{pmatrix}\right. \\ &\quad \left. + \int_0^t d\tau e^{-\tau\left(-\frac{1}{q}C\partial_\theta + B\partial_r^2 - \frac{1}{\eta}\right)}\begin{pmatrix} 0 \\ \frac{\Gamma_0}{\eta} \end{pmatrix}\right\} \end{aligned} \tag{2.3}$$

with B and C the matrices

$$B = \begin{pmatrix} D & 0 \\ 0 & \nu \end{pmatrix}; \qquad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

2.2. The $\chi = 0$ case

The linear part of the system for $\chi = 0$ is:

$$\begin{aligned} \partial_t N + \frac{1}{q}\partial_\theta \Gamma &= D\partial_r^2 N \\ \partial_t \Gamma + \frac{1}{q}\partial_\theta N &= \nu\partial_r^2 \Gamma \end{aligned} \tag{2.4}$$

Given the initial conditions at time zero $\begin{pmatrix} N(0,r,\theta) \\ \Gamma(0,r,\theta) \end{pmatrix}$ the solution of this system is

$$\begin{pmatrix} N(t,r,\theta) \\ \Gamma(t,r,\theta) \end{pmatrix} = \exp t\left\{-\frac{1}{q}A\partial_\theta + B\partial_r^2\right\}\begin{pmatrix} N(0,r,\theta) \\ \Gamma(0,r,\theta) \end{pmatrix} \tag{2.5}$$

A being the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

However, for the construction of a stochastic solution to the nonlinear equation, through a probabilistic interpretation of the integral equation, it is convenient to have a stochastic process that operates in a simple way on the arguments of the functions. Therefore, instead of the full linear part, only the diffusion associated to the first term in (2.4) will be used. It also provides an easier handling of the ∂_θ derivative.

For the nonlinear equations one writes

$$\begin{aligned} N(t,r,\theta) &= e^{tD\partial_r^2} N(0,r,\theta) - \frac{1}{q}\int_0^t d\tau e^{\tau D\partial_r^2}\partial_\theta \Gamma(t-\tau,r,\theta) \\ \Gamma(t,r,\theta) &= e^{t\nu\partial_r^2}\Gamma(0,r,\theta) - \frac{1}{q}\int_0^t d\tau e^{\tau\nu\partial_r^2}\partial_\theta\left\{\frac{\Gamma^2}{N} + N\right\}(t-\tau,r,\theta) \end{aligned} \tag{2.6}$$

Denote by $\xi_s^{(N)}$ and $\xi_s^{(\Gamma)}$ two Brownian motions in the $r-$coordinate with diffusion coefficients $\sqrt{2D}$ and $\sqrt{2\nu}$. Then the equations (2.6) may be reinterpreted as defining a probabilistic processes for which the expectation values are the functions $N(t,r,\theta)$ and $\Gamma(t,r,\theta)$, that is

$$\begin{aligned} N(t,r,\theta) &= \mathbb{E}_{(t,r,\theta)}\left[p\frac{1}{p}N\left(0,\xi_t^{(N)},\theta\right)-\frac{t}{q(1-p)}\int_0^t\frac{1-p}{t}d\tau\partial_\theta\Gamma\left(t-\tau,\xi_\tau^{(N)},\theta\right)\right] \\ \Gamma(t,r,\theta) &= \mathbb{E}_{(t,r,\theta)}\left[p\frac{1}{p}\Gamma\left(0,\xi_t^{(\Gamma)},\theta\right)-\frac{2t}{q(1-p)}\int_0^t\frac{1-p}{t}d\tau\partial_\theta\left\{\frac{1}{2}\frac{\Gamma^2}{N}+\frac{1}{2}N\right\}\left(t-\tau,\xi_\tau^{(\Gamma)},\theta\right)\right] \end{aligned} \tag{2.7}$$

$\mathbb{E}_{(t,r,\theta)}$ denotes the expectation value of a backwards-in-time stochastic process started from (t,r,θ). The processes that construct the solution at the point (t,r,θ) are backwards-in-time processes that start from time t and propagate to time zero. With probability p the process reach time zero and the contribution to the expectation value is $\frac{1}{p}N\left(0,\xi_t^{(N)},\theta\right)$ (or $\frac{1}{p}\Gamma\left(0,\xi_t^{(\Gamma)},\theta\right)$). With probability $(1-p)$ the process is interrupted at a time τ chosen with uniform probability in the interval $(t,0)$. For the process associated to N, the process changes its nature, becomes a Γ process and picks up a factor $-\frac{t}{q(1-p)}$. For the case of the process Γ, with probability $\frac{1}{2}$, this process either changes to a N process or branches into a N and a Γ process. In both cases it picks up a factor $-\frac{2t}{q(1-p)}$.

Notice that the propagation process acts only on the $r-$coordinate. Therefore the derivative ∂_θ, the square in Γ^2 and the quotient in $\frac{\Gamma^2}{N}$ may all be treated as operators which are kept as labels at each branching point. When all the lines of the process reach time zero, the initial condition is sampled at the arrival r_0-coordinate. This initial condition is not simply a number but a function of θ ($\Gamma(0,r_0,\theta)$ or $N(0,r_0,\theta)$). It implies that both the initial condition and all its derivatives at the argument θ must be provided. This initial functions are then backtracked throughout the sample lines, the multiplicative factors are picked up at each τ interrupt and the operators applied whenever a labelled branching point is reached. This provides the contribution of each sample path to the expectation value. Figure 2 displays an example of a sample path, where the operators picked up along the way are denoted by flags.

Notice the order of the operators at each branching point. For example, at the leftmost $\delta-$ labelled point the operation is

$$\partial_\theta\left\{\frac{\Gamma^2\left(0,r_0^{(1)},\theta\right)}{N\left(0,r_0^{(2)},\theta\right)}\right\}$$

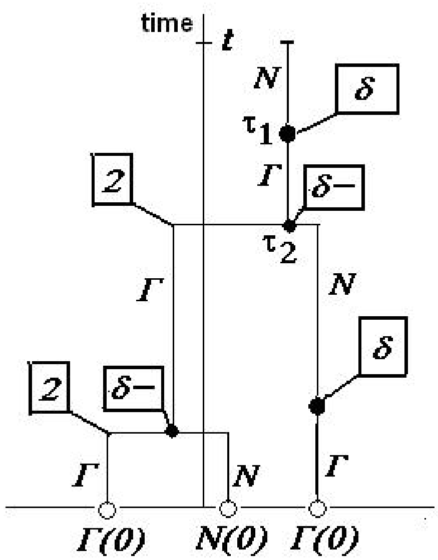

FIGURE 2. A sample path of the $N-\Gamma$ stochastic process

and the whole contribution of this sample path to the $N-$expectation value is

$$\partial_\theta^2\left\{\left(\partial_\theta\left\{\frac{\Gamma^2\left(0,r_0^{(1)},\theta\right)}{N\left(0,r_0^{(2)},\theta\right)}\right\}\right)^2\left\{\partial_\theta\Gamma\left(0,r_0^{(3)},\theta\right)\right\}^{-1}\right\}$$

times the factor $\left(\frac{1}{p}\right)^3\frac{4t\tau_1\tau_2^2}{q^4(1-p)^4}$.

The branching, being identical to a Galton-Watson process, has a finite number of branches almost surely. Therefore, with a uniform unit bound on the quantities at each branching vertex, one obtains almost sure convergence of the expectation value. In conclusion:

Proposition 2.1 - If the initial conditions $\left|\frac{\Gamma^2}{N},N,\Gamma\right|$ and all its derivatives are bound by a constant M, the above described process provides a solution to the SOLEDGE2D equations up to time $t\leq\frac{q}{M}$ a.s.

This bound, which is obtained by a worst case analysis, is in practice too severe because the probability of the generated trees is a fast decreasing function of the number of branches.

3. Superprocesses

A superprocess describes the evolution of a population, without a fixed number of units, that evolves according to the laws of chance. Given a countable dense subset Q of $[0,\infty)$ and a countable dense subset F of a separable metric space E, the countable set

$$M_1 = \left\{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} : x_1 \cdots x_n \in F; \alpha_1 \cdots \alpha_n \in Q; n \geq 1 \right\} \qquad (3.1)$$

is dense on the space $M(E)$ of finite Borel measures on E (theorem 1.8 in [21]). This is at the basis of the interpretation of the limits of evolving particle systems as measure-valued superprocesses. On the other hand the representation of an evolving measure as a collection of measures with point support is also useful for the construction of solutions of nonlinear partial differential equations as scaling limits of measure-valued superprocesses.

However, as far as representations of solutions of nonlinear PDE's, superprocesses constructed in the space $M(E)$ of finite measures have serious limitations. The set of interaction terms that can be handled is limited (essentially to $u^{\alpha}(x)$ with $\alpha \leq 2$) and derivative interactions cannot be included as well. The first obvious generalization would be to construct superprocesses on distributions of point support, because any such distribution is a finite sum of deltas and their derivatives [22]. However, because in a general branching process the number of branches is not bounded, one really needs a framework that can handle arbitrary sums of deltas and their derivatives. This requirement leads naturally to the space of ultradistributions of compact support.

Ultradistributions may be characterized as Fourier transforms of distributions of exponential type [23]. However, the representation of ultradistributions by analytical functions is actually simpler and also more convenient for practical calculations. Let $\mathcal{S}$ be the Schwartz space of functions of rapid decrease and $\mathcal{U} \subset \mathcal{S}$ those functions in $\mathcal{S}$ that may be extended into the complex plane as entire functions of rapid decrease on strips. $\mathcal{U}'$, the dual of $\mathcal{U}$, is Silva's space of tempered ultradistributions [24] [25].

Let first $E = \mathbb{R}$. Define B_η as the complement in $\mathbb{C}$ of the strip $\mathrm{Im}(z) \leq \eta$

$$B_\eta = \{ z : \mathrm{Im}(z) > \eta \} \qquad (3.2)$$

and H_η the set of functions which are holomorphic and of polynomial growth in B_η

$$\varphi(z) \in H_\eta \Longrightarrow \exists M, \alpha : |\varphi(z)| < M |z|^\alpha, \forall z \in B_\eta \tag{3.3}$$

Let H_ω be the union of all such spaces

$$H_\omega = \bigcup_{\eta \geq 0} H_\eta \tag{3.4}$$

and in H_ω define the equivalence relation Π by

$$\varphi \overset{\Pi}{\simeq} \psi \text{ if } \varphi - \psi \text{ is a polynomial}$$

Then, the space of tempered ultradistribution is

$$\mathcal{U}' = H_\omega / \Pi \tag{3.5}$$

The relation to ultradistributions as entities $f(x)$ in $\mathbb{R}$ is obtained by the generalized Stieltjes transform

$$\varphi(z) = \frac{p(z)}{2\pi i} \int \frac{f(x)}{p(x)(x-z)} dx + P(z) \tag{3.6}$$

$p(z)$ being a polynomial such that $f/p \sim O\left(t^{-1}\right)$ and $P(z)$ an arbitrary polynomial. In this sense one may say that $[\varphi] \in H_\omega/\Pi$ is the Stieltjes image of the ultradistribution f. Operations with $f(x)$ are performed using their analytical images. For example f is integrable in $\mathbb{R}$ if there is an y_0 and a $\varphi(z)$ in the Stieltjes image such that $\varphi(x+iy_0) - \varphi(x-iy_0)$ is integrable in $\mathbb{R}$ in the sense of distributions.

An ultradistribution vanishes in an open set $A \in \mathbb{R}$ if $\varphi(x+iy) - \varphi(x-iy) \to 0$ for $x \in A$ when $y \to 0$ or, equivalently, if there an analytical extension of φ to the vertical strip $\mathrm{Re} z \in A$. The support of f is the complement in $\mathbb{R}$ of the largest open set where f vanishes.

All these notions are easily generalized to $\mathbb{R}^n$ [25] by considering products of semiplans as in (3.2) and the corresponding polynomial bounds. For the equivalence relation Π one uses pseudopolynomials, that is, functions of the form

$$\sum_{j,k} \rho\left(z_1, \cdots, \overset{\wedge}{z_j}, \cdots, z_n\right) z_j^k$$

$\overset{\wedge}{z_j}$ meaning that this variable is absent from the arguments of ρ.

An ultradistribution f in $\mathbb{R}^n$ has compact support if there is a disk D such that any φ in the Stieltjes image has an analytic extension to $(\mathbb{C}/D)^n$.

For our purposes, the most important property of ultradistributions of compact support is the fact that any such ultradistribution has a representation as a series of multipoles

$$f(x)=\sum_{r_1=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}p_{r_1,\cdots,r_n}\delta^{(r_1,\cdots,r_n)}(x-a)$$

a being a point in the support of f. This result follows from the fact that for compact support one may apply to the Stieltjes image the Cauchy theorem over a closed contour. The space of tempered ultradistributions of compact support will denoted as $\mathcal{U}_0'$.

Let the underlying space E be $\mathbb{R}^n$. Denote by $(X_t, P_{0,\nu})$ a branching stochastic process with values in $\mathcal{U}_0'$ and transition probability $P_{0,\nu}$ starting from time 0 and $\nu \in \mathcal{U}_0'$. The process is assumed to satisfy the *branching property,* that is, given $\nu=\nu_1+\nu_2$

$$P_{0,\nu}=P_{0,\nu_1}*P_{0,\nu_2} \tag{3.7}$$

After the branching $\left(X_t^1, P_{0,\nu_1}\right)$ and $\left(X_t^2, P_{0,\nu_2}\right)$ are independent and $X_t^1+X_t^2$ has the same law as $(X_t, P_{0,\nu})$. In terms of the *transition operator* V_t operating on functions on $\mathcal{U}$ this would be

$$\langle V_t f, \nu_1+\nu_2\rangle=\langle V_t f, \nu_1\rangle+\langle V_t f, \nu_2\rangle \tag{3.8}$$

with V_t defined by $e^{-\langle V_t f,\nu\rangle} \stackrel{\circ}{=} P_{0,\nu}e^{-\langle f, X_t\rangle}$ or

$$\langle V_t f, \nu\rangle=-\log P_{0,\nu}e^{-\langle f, X_t\rangle} \tag{3.9}$$

$f\in\mathcal{U}, \nu\in\mathcal{U}_0'$.

Underlying the usual construction of superprocesses, in the form that is useful for the representation of solutions of PDE's, there is a stochastic process with paths that start from a particular point in E, then propagate and branch, but the paths preserve the same nature after the branching. In terms of measures it means that one starts from an initial δ_x which at the branching point originates other $\delta's$ with at most some scaling factors. It is to preserve this pointwise interpretation that, in this larger setting, one considers ultradistributions in $\mathcal{U}_0'$, because, as seen above, any ultradistribution in $\mathcal{U}_0'$ may be represented as a multipole expansion at any point of its support. Therefore an arbitrary transition in the process X_t in $\mathcal{U}_0'$ may be associated to a branching of paths in E and along these new paths new distributions with point support will propagate. As a result the construction now proceeds as in the measure-valued case.

In $M=[0,\infty)\times E$ consider a set $Q\subset M$ and the associated exit process $\xi=(\xi_t, \Pi_{0,x})$ with parameter k defining the lifetime. The process stars from $x\in E$ carrying along an ultradistribution in $\mathcal{U}_0'$ with support on the path. At each branching point of the ξ_t-process there is a transition ruled by the P probability in $\mathcal{U}_0'$ leading to one or more elements in $\mathcal{U}_0'$. These $\mathcal{U}_0'$ elements are then carried along by the new paths of the ξ_t−process. The whole process stops at the boundary ∂Q, finally defining a exit process $(X_Q, P_{0,\nu})$ on $\mathcal{U}_0'$. If the initial ν is δ_x one writes

$$u(x) = \langle V_Q f, \nu \rangle = -\log P_{0,x} e^{-\langle f, X_Q \rangle} \tag{3.10}$$

$\langle f, X_Q \rangle$ being computed on the (space-time) boundary with the exit ultradistribution generated by the process.

The connection with nonlinear PDE's is established by defining the whole process to be a (ξ, ψ) −*superprocess* if $u(x)$ satisfies the equation

$$u + G_Q \psi(u) = K_Q f \tag{3.11}$$

where G_Q is the Green operator,

$$G_Q f(r, x) = \Pi_{0,x} \int_0^\tau f(s, \xi_s)\, ds \tag{3.12}$$

and K_Q the Poisson operator

$$K_Q f(x) = \Pi_{0,x} 1_{\tau<\infty} f(\xi_\tau) \tag{3.13}$$

$\psi(u)$ means $\psi(0, x; u(0, x))$ and τ is the first exit time from Q.

The superprocess is constructed as follows: Let $\varphi(s, x; z)$ be the branching function at time s and point x. Then for $e^{-w(0,x)} \circeq P_{0,x} e^{-\langle f, X_Q \rangle}$ one has

$$P_{0,x} e^{-\langle f, X_Q \rangle} \circeq e^{-w(0,x)} = \Pi_{0,x} \left[e^{-k\tau} e^{-f(\tau,\xi_\tau)} + \int_0^\tau ds k e^{-ks} \varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right) \right] \tag{3.14}$$

τ is the first exit time from Q and $f(\tau, \xi_\tau) = \langle f, X_Q \rangle$ is computed with the exit boundary ultradistribution. For measure-valued superprocesses the branching function would be

$$\varphi(s, y; z) = c \sum_0^\infty p_n(s, y) z^n \tag{3.15}$$

with $\sum_n p_n = 1$ and c the branching intensity, but now it may be a more general function.

For the interpretation of the superprocesses as generating solutions of PDE's, an essential role is played by a transformation of Eq.(3.14) that uses $\int_0^\tau k e^{-ks} ds = 1 - e^{-k\tau}$ and the Markov property $\Pi_{0,x} 1_{s<\tau} \Pi_{s,\xi_s} = \Pi_{0,x} 1_{s<\tau}$. This is lemma 1.2 in ch.4 of Ref.[16]. Because it only depends on the Markov properties of the $(\xi_t, \Pi_{0,x})$ process it also holds in this more general context. A proof is included in the Appendix with the notations used in this paper.

Using the lemma, Eq.(3.14) for $e^{-w(0,x)}$ is converted into

$$e^{-w(0,x)} = \Pi_{0,x} \left[e^{-f(\tau,\xi_\tau)} + k \int_0^\tau ds \left[\varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right) - e^{-w(\tau-s,\xi_s)} \right] \right] \tag{3.16}$$

Eq.(3.11) is now obtained by a limiting process. Let in (3.16) replace $w\,(0,x)$ by $\beta w_\beta\,(0,x)$ and f by βf. β is interpreted as the mass of the particles and when the $\mathcal{U}_0'$-valued process $X_Q \rightarrow \beta X_Q$ then $P_\mu \rightarrow P_{\frac{\mu}{\beta}}$.

$$e^{-\beta w(0,x)} = \Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta \int_0^\tau ds\left[\varphi_\beta\left(s,\xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right] \tag{3.17}$$

Two scaling limits will be used in this paper. The first one, which is the one used in the past for superprocesses on measures, defines

$$u_\beta^{(1)} = \left(1-e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta^{(1)} = \left(1-e^{-\beta f}\right)/\beta \tag{3.18}$$

and

$$\psi_\beta^{(1)}\left(0,x;u_\beta^{(1)}\right) = \frac{k_\beta}{\beta}\left(\varphi\left(0,x;1-\beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right) \tag{3.19}$$

one obtains from (3.17)

$$u_\beta^{(1)}\,(0,x) + \Pi_{0,x}\int_0^\tau ds\psi_\beta^{(1)}\left(s,\xi_s;u_\beta^{(1)}\right) = \Pi_{0,x} f_\beta^{(1)}\,(\tau,\xi_\tau) \tag{3.20}$$

that is

$$u_\beta^{(1)} + G_Q\psi_\beta^{(1)}\left(u_\beta^{(1)}\right) = K_Q f_\beta^{(1)} \tag{3.21}$$

When $\beta \rightarrow 0$, $f_\beta^{(1)} \rightarrow f$ and if ψ_β goes to a well defined limit ψ then u_β tends to a limit u solution of (3.11) associated to a superprocess. Also one sees from (3.18) that in the $\beta \rightarrow 0$ limit

$$u_\beta^{(1)} \rightarrow w_\beta = -\log P_{0,x} e^{-\langle f, X_Q\rangle} \tag{3.22}$$

as in Eq.(3.10). The superprocess corresponds to a cloud of ultradistribution "particles" for which both the mass and the lifetime tend to zero.

3.1. Measure-valued superprocesses and nonlinear PDE's

Here one restricts oneself to measure-valued superprocesses, that is, in terms of paths, to $\delta's$ propagating along the paths of the $(\xi_t, \Pi_{0,x})$ process and simply branching to new δ measures at each branching point. Let us construct a superprocess providing a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^\alpha \tag{3.23}$$

for $1 < \alpha \leq 2$. Comparing with (3.11) one should have

$$\psi\,(0,x;u) = u^\alpha$$

Then from (3.19) and (3.15), with $z=1-\beta u_\beta^{(1)}$ one has

$$\begin{aligned}\varphi(0,x;z) &= \sum_n p_n z^n = z+\frac{\beta}{k_\beta}u_\beta^{(1)\alpha} = z+\frac{\beta}{k_\beta}\frac{(1-z)^\alpha}{\beta^\alpha} \\ &= z+\frac{1}{k_\beta\beta^{\alpha-1}}\left(1-\alpha z+\frac{\alpha(\alpha-1)}{2}z^2-\frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3+\cdots\right)\end{aligned} \tag{3.24}$$

Choosing $k_\beta=\frac{\alpha}{\beta^{\alpha-1}}$ the terms in z cancel and for $1<\alpha\leq 2$ the coefficients of all the remaining z powers are positive and may be interpreted as branching probabilities. It would not be so for $\alpha>2$. Then

$$p_0=\frac{1}{\alpha};\quad p_1=0;\quad \cdots\quad p_n=\frac{(-1)^n}{\alpha}\binom{\alpha}{n}\quad n\geq 2 \tag{3.25}$$

with $\sum_n p_n=1$. With this choice of probabilities p_n for branching into new δ measures and with $k_\beta=\frac{\alpha}{\beta^{\alpha-1}}$ and $\beta\to 0$ one obtains a superprocess which, through (3.10), provides a solution to the Eq.(3.23). $\alpha=2$ is an upper bound for this representation, because for $\alpha>2$ some of the $p_n's$ would be negative and would not be interpretable as branching probabilities.

For the particular case

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-u^2 \tag{3.26}$$

$$p_1=0;\quad p_0=p_2=\frac{1}{2};\quad k_\beta=\frac{2}{\beta} \tag{3.27}$$

When $\beta\to 0$, the solutions are given by (3.10) and the superprocesses correspond to the scaling limit of processes where both the mass and the lifetime of the particles tend to zero and at each bifurcation point one has probability p_0 of dying without offspring or creating n new δ measures with probabilities p_n.

Superprocesses are usually associated with nonlinear PDE's in the scaling limit $\beta\to 0$ of (3.19)-(3.20). However other limits may also be useful. For example with with $p_n=\delta_{n,2}$, $\beta=1$ and $k_\beta=1$ one obtains

$$\begin{aligned}\psi_\beta^{(1)}\left(0,x;u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\varphi\left(0,x;1-\beta u_\beta^{(1)}\right)-1+\beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\sum p_n\left(1-\beta u_\beta^{(1)}\right)^n-1+\beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\beta^2 u_\beta^{(1)2}-\beta u_\beta^{(1)}\right) \\ &\to u^2-u\end{aligned} \tag{3.28}$$

Therefore, in this case, one is led to the KPP equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u^2 + u \tag{3.29}$$

However in this case, because $\beta = 1$ instead of $\beta \to 0$, the solution is given by $(1 - e^{-w})$ instead of (3.10). Because of the natural stochastic clock provided by the linear u term, a stochastic solution for the Cauchy problem of the KPP equation may be constructed by the McKean method [1], as seen before. However, the interpretation as an exit measure, allows for the construction of solutions with arbitrary boundary conditions.

3.2. Superprocesses on signed measures and ultradistributions

When superprocesses are generalized from measures to ultradistributions of compact support, a general aim would be, of course, to characterize all the admissible transition kernels and branching mechanisms compatible with the new formulation. I will leave this for future work and just present a few examples of transitions and branchings which provide stochastic representations for a wider class of nonlinear PDE's.

Although the scaling limit $\beta \to 0$ of measure-valued superprocesses allows the construction of solutions for equations which do not possess a natural Poisson clock, it has the severe limitation of requiring a polynomial branching function $\varphi(s, x; z)$. This automatically restricts the nonlinear terms in the pde's to be powers of u. In addition, these terms must be such that all coefficients in the z^n expansion in Eq.(3.15) are positive to be interpretable as branching probabilities. As seen before, it was this requirement that led to the restriction $1 < \alpha \leq 2$ in (3.23).

The variable z that appears in $\varphi_\beta(s, x; z)$ is in fact $z = e^{-\beta w(\tau - s, \xi_s)} = P_{0,x} e^{-\langle \beta f, X \rangle}$. When restricting the superprocess to measures, the delta measure, at each branching point, may at most branch into other deltas (with positive coefficients) and therefore $\varphi(s, x; z)$ must be a sum of monomials in z. When one generalizes to $\mathcal{U}_0'$ ultradistributions of compact support, changes of sign and transitions from deltas to their derivatives are allowed. In the end, the exponential $e^{-\langle \beta f, X \rangle}$ will be computed by evaluation of the function on the ultradistribution that reaches the boundary. To find out the equation that is represented by the process one needs to compute the $\psi_\beta(0, x; u_\beta)$ of Eq.(3.19) for the corresponding $\varphi(s, x; z)$ in the $\beta \to 0$ limit. Recalling that $\varphi(s, x; z) = \varphi_\beta\left(s, \xi_s; e^{-\beta w(\tau - s, \xi_s)}\right)$ and $z = e^{-\beta w_\beta}$, one concludes that there are basically two new transitions at the branching points:

1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \to e^{\langle \beta f, -\delta_x \rangle} = e^{-\beta f(x)} \tag{3.30}$$

which corresponds to

$$z \to \frac{1}{z} \tag{3.31}$$

and

2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle \beta f, \delta_x \rangle} = e^{\beta f(x)} \to e^{\left\langle \beta f, \pm\delta'_x \right\rangle} = e^{\mp\beta f'(x)} \tag{3.32}$$

which corresponds to

$$z \to e^{\mp\partial_x \log z} \tag{3.33}$$

Case 1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and the second to superprocesses in $\mathcal{U}'_0$.

How these transformations provide stochastic representations of solutions for other classes of pde's, will be illustrated by two examples:

First, let

$$\varphi^{(1)}\left(0, x; z\right) = p_1 e^{\partial_x \log z} + p_2 e^{-\partial_x \log z} + p_3 z^2 \tag{3.34}$$

This branching function means that at the branching point, with probability p_1 a derivative is added to the propagating ultradistribution, with probability p_2 a derivative is added plus a change of sign and with probability p_3 the ultradistribution branches into two identical ones. Using the transformation and scaling limit (3.18) one has, for small β

$$z \to e^{\mp\partial_x \log z} = e^{\mp\partial_x \log\left(1-\beta u_\beta^{(1)}\right)} = 1 \pm \beta\partial_x u_\beta^{(1)} + \frac{\beta^2}{2}\left\{\left(\partial_x u_\beta^{(1)}\right)^2 \pm \partial_x u_\beta^{(1)2}\right\} + O\left(\beta^3\right) \tag{3.35}$$

$$z \to z^2 = \left(1 - \beta u_\beta^{(1)}\right)^2 = 1 - 2\beta u_\beta^{(1)} + \beta^2 u_\beta^{(1)2} \tag{3.36}$$

Then, computing $\psi_\beta\left(0, x; u_\beta^{(1)}\right)$ with $p_1 = p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$ one obtains

$$\begin{aligned} \psi_\beta^{(1)}\left(0, x; u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta}\left(\varphi^{(1)}\left(0, x; z\right) - z\right) \\ &= \frac{k_\beta}{\beta}\left(\varphi^{(1)}\left(0, x; 1 - \beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right) \\ &= \frac{k_\beta}{\beta}\left(\frac{1}{8}\beta^2\left(\partial_x u_\beta^{(1)}\right)^2 + \frac{1}{2}\beta^2 u_\beta^{(1)2} + O\left(\beta^3\right)\right) \end{aligned} \tag{3.37}$$

meaning that, with $k_\beta = \frac{4}{\beta}$, the superprocess provides, in the $\beta \to 0$ limit, a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - 2u^2 - \frac{1}{2}\left(\partial_x u\right)^2 \tag{3.38}$$

For the second example a different scaling limit will be used, namely

$$u_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta w_\beta} - e^{-\beta w_\beta}\right) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta f} - e^{-\beta f}\right) \tag{3.39}$$

Notice that, as before, $u_\beta^{(2)} \to w_\beta$ and $f_\beta^{(2)} \to f$ when $\beta \to 0$. In this case with $z = e^{\beta w_\beta}$ one has

$$\begin{aligned} z &= -2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 - 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} \frac{1}{z} &= 2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 + 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned} \tag{3.41}$$

For the integral equation, instead of (3.20), one has

$$u_\beta^{(2)}(0,x) + \Pi_{0,x}\int_0^\tau ds\psi_\beta^{(2)}\left(s,\xi_s;u_\beta^{(2)}\right) = \Pi_{0,x} f_\beta^{(2)}(\tau,\xi_\tau) \tag{3.42}$$

with

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right) = k_\beta\left(\frac{1}{2\beta}\left(\varphi(0,x;z) - \varphi\left(0,x;\frac{1}{z}\right)\right) - u_\beta^{(2)}\right) \tag{3.43}$$

Let now

$$\varphi^{(2)}(0,x;z) = p_1 z^2 + p_2\frac{1}{z} \tag{3.44}$$

This branching function means that with probability p_1 the ultradistribution branches into two identical ones and with probability p_2 it changes its sign. Therefore, in this case, one is simply extending the superprocess construction to signed measures. Using (3.40) and (3.41) one computes $\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right)$ obtaining

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right) = k_\beta\left\{-p_1 8u_\beta^{(2)}\left(1 + \frac{1}{2}\beta^2 u_\beta^{(2)2}\right) + p_2 u_\beta^{(2)} - u_\beta^{(2)} + O\left(\beta^4\right)\right\} \tag{3.45}$$

and with $p_1 = \frac{1}{10}; p_2 = \frac{9}{10}$ and $k_\beta = \frac{5}{2\beta^2}$ one obtains in the in the $\beta \to 0$ limit

$$\psi_\beta^{(2)}\left(0,x;u_\beta^{(2)}\right) \to -u_\beta^{(2)3} \tag{3.46}$$

meaning that this superprocess provides a solution to the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u^3 \tag{3.47}$$

In conclusion: Extending the superprocess construction to signed measures and ultradistributions, stochastic solutions are obtained for a much larger class of partial differential equations.

4. Final remarks

Stochastic solutions are powerful tools both to construct new exact solutions of nonlinear PDE's and to develop faster numerical algorithms for parallel computing. Two related methods are used to construct the stochastic solutions. Both use limiting processes, as in branching particle systems, to generate boundary measures which sample the initial (boundary) conditions. The limitations in the classical constructions, are the handling of derivatives, nonpolynomial terms and negative branching coefficients which cannot be interpreted as probabilities. To overcome these limitations one used either operator labels at the branching vertices or an extension of superprocesses from measures to ultradistributions. In reality these methods are closely related. In the first one keeps the propagating entities as delta measures but then has to backtrack the initial conditions from the final boundary time to apply the operators at each vertex. In the second the propagating entities are modified at each vertex, the final generated entity being directly applied to the initial conditions without any backtracking.

Notice however that there are cases where backtracking of the initial conditions through the tree is needed even in case without derivatives or nonpolynomial terms. This is, for example, the case of Navier-Stokes or magnetohydrodynamics [11], because of the Leray product at each vertex.

The simple superprocess examples treated here deal with the kind of terms that will appear in PDE's with local interactions. More general nonlocal interactions or integral equations would require a more general treatment of the ultradistribution superprocesses, where the allowed transitions are not simply changes of sign and derivatives.

5. Appendix: Proof of a lemma

Let

$$u(x,t)=\Pi_{0,x}\left\{e^{-kt}u(\xi_t,0)+\int_0^t ke^{-ks}\Phi(\xi_s,t-s)\,ds\right\} \qquad (5.1)$$

Then

$$\Pi_{0,x}\int_0^t ku(\xi_s,t-s)\,ds = \Pi_{0,x}\left\{\int_0^t ke^{-k(t-s)}u(\xi_{s+t-s},0)\,ds + \int_0^t kds\int_0^{t-s}kds'e^{-ks'}\Phi(\xi_{s+s'},t-s-s')\right\} \tag{5.2}$$

Summing (5.1) and (5.2)

$$\begin{aligned} & u(x,t)+\Pi_{0,x}\int_0^t ku(\xi_s,t-s)\,ds \\ = \; & \Pi_{0,x}\left\{\left(e^{-kt}+\int_0^t ke^{-k(t-s)}ds\right)u(\xi_t,0)\right. \\ & \left. +k\int_0^t e^{-ks}\Phi(\xi_s,t-s)\,ds+k\int_0^t ds\int_0^{t-s}kds'e^{-ks'}\Phi(\xi_{s+s'},t-s-s')\,ds'\right\} \end{aligned} \tag{5.3}$$

Changing variables in the last integral in (5.3) from (s,s') to $(s,\sigma=s+s')$ one obtains for the last term

$$k\int_0^t d\sigma\int_0^\sigma kdse^{-k(\sigma-s)}\Phi(\xi_\sigma,t-\sigma)\,ds$$

and finally

$$\begin{aligned} & u(x,t)+\Pi_{0,x}k\int_0^t u(\xi_s,t-s)\,ds \\ = \; & \Pi_{0,x}\left\{u(\xi_t,0)+k\int_0^t\Phi(\xi_s,t-s)\,ds\right\} \end{aligned} \tag{5.4}$$

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