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Superprocesses on ultradistributions

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ABSTRACT

Stochastic solutions provide new rigorous results for nonlinear PDE's and, through its local nongrid nature, are a natural tool for parallel computation. There are two different approaches for the construction of stochastic solutions: McKean's and superprocesses. In favour of superprocesses is the fact that they handle arbitrary boundary conditions. However, when restricted to measures, superprocesses can only be used to generate solutions for a limited class of nonlinear PDE's. A new class of superprocesses, namely superprocesses on ultradistributions, is proposed to extend the stochastic solution approach to a wider class of PDE's.

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1. Stochastic solutions and measure-valued processes

A *stochastic solution* of a linear or nonlinear partial differential equation is a stochastic process which, starting from a point x in the domain, generates after time t a boundary measure that, sampling the initial condition at $t = 0$, provides the solution at the point x and time t. A classical example is the McKean [17] construction of a stochastic solution for the KPP equation,

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2} + v^2 - v; \quad v\left(0,x\right) = g\left(x\right). \tag{1}$$

Let $G\left(t,x\right)$ be the Green's operator for the heat equation $\partial_t v(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} v(t,x)$

$$G\left(t,x\right) = e^{\frac{1}{2}t\frac{\partial^2}{\partial x^2}}$$

and write the KPP equation in integral form

$$v\left(t,x\right) = e^{-t}G\left(t,x\right)g\left(x\right) + \int_0^t e^{-(t-s)}G\left(t-s,x\right)v^2\left(s,x\right)\mathrm{d}s. \tag{2}$$

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Denoting by (ξ_t, P_x), a Brownian motion starting from time zero and coordinate x, Equation (2) may be rewritten as

$$\begin{aligned} v(t,x) &= \mathbb{E}_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-(t-s)} v^2 (s, \xi_{t-s}) \, \mathrm{d}s \right\} \\ &= \mathbb{E}_x \left\{ e^{-t} g(\xi_t) + \int_0^t e^{-s} v^2 (t-s, \xi_s) \, \mathrm{d}s \right\}. \end{aligned} \tag{3}$$

The *stochastic solution process* is a composite process: a Brownian motion plus a branching process with exponential holding time T, $P(T > t) = e^{-t}$. At each branching point, the particle splits into two, the new particles going along independent Brownian paths. At time $t > 0$, if there are n particles located at $x_1(t), x_2(t), \ldots, x_n(t)$, the solution of (1) is obtained by

$$v(t,x) = \mathbb{E}_x \left\{ g(x_1(t)) \, g(x_2(t)) \ldots g(x_n(t)) \right\}. \tag{4}$$

An equivalent interpretation, that corresponds to the second equality in (3), is of a process starting from time t at x and propagating backwards-in-time to time zero. When it reaches $t = 0$, the process samples the initial condition, that is, it generates a measure μ at the $t = 0$ boundary which yields the solution by (4).

The construction of solutions for nonlinear equations, through the stochastic interpretation of the integral equations, has become an active field in recent years, applied for example to Navier–Stokes [4,15,19,20,29], to Vlasov–Poisson [11,25,28], to Euler [27] to magnetohydrodynamics [12] and to a fractional version of the KPP Equation [5]. In addition to providing new exact results for nonlinear PDE's, the stochastic solutions are also a promising tool for numerical implementation, in particular for parallel computation using, for example, the recently developed probabilistic domain decomposition method [1–3].

There are basically two methods to construct stochastic solutions. The first method, which will be called the McKean method, illustrated above, is essentially a probabilistic interpretation of the Picard series. The differential equations are written as integral equations which are rearranged in a such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series. The second method [7,8,16] constructs the boundary measure of a measure-valued stochastic process (a superprocess) and obtains the solution of the differential equation by a rescaling procedure. For a detailed comparison of the two methods, refer to [26].

Although being able to handle arbitrary boundary conditions, a limitation of measure-valued superprocesses is that they can only represent a limited class of nonlinear partial differential equations.[1] The main purpose of this paper is to extend superprocesses from measure-valued to ultradistribution-valued processes, which lead to a much wider class of stochastic solutions for partial differential equations.

2. Ultradistribution-valued superprocesses

2.1. Tempered ultradistributions

A superprocess describes the evolution of a population, without a fixed number of units, that evolves according to the laws of chance. Given a countable dense subset Q of $[0, \infty)$ and a countable dense subset F of a separable metric space E, the countable set

$$M_1 = \left\{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} : x_1 \dots x_n \in F; \alpha_1 \dots \alpha_n \in Q; n \geq 1 \right\} \tag{5}$$

is dense (in the topology of weak convergence) on the space $M\left(E\right)$ of finite Borel measures on E (Theorem 1.8 in [16]). This is at the basis of the interpretation of the limits of evolving particle systems as measure-valued superprocesses. On the other hand, the representation of an evolving measure as a collection of measures with point support is also useful for the construction of solutions of nonlinear partial differential equations as rescaling limits of measure-valued superprocesses.

However, as far as representations of solutions of nonlinear PDE's, superprocesses constructed in the space $M\left(E\right)$ of finite measures have serious limitations. The set of interaction terms that can be handled is limited (essentially to $u^{\alpha}\left(x\right)$ with $\alpha \leq 2$) and derivative interactions cannot be included as well. The first obvious generalization would be to construct superprocesses on distributions of point support, because any such distribution is a finite sum of deltas and their derivatives [23]. However, because in a general branching process, the number of branches is not bounded, one really needs a framework that can handle arbitrary sums of deltas and their derivatives. This requirement leads naturally to the space of ultradistributions of compact support. Let $E = \mathbb{R}$. The space of *ultradistributions* $\mathcal{Z}'$ is the topological dual of $\mathcal{Z}$, a space of test functions for which the Fourier transform is in $\mathcal{D}$, the space of infinitely differentiable functions of compact support. The fact that the Fourier transform of $\mathcal{Z}$ has compact support endows ultradistributions with a rich analytical structure which makes these "generalized functions" more convenient than distributions in many applications. An important dense subspace of $\mathcal{Z}'$ is the space of *tempered ultradistributions* $\mathcal{U}'$ which may be characterized as Fourier transforms of distributions of exponential type[2] [14,21,22].

However, the representation of tempered ultradistributions by analytical functions is the most convenient one for practical calculations. Let $\mathcal{S}$ be the Schwartz space of functions of rapid decrease and $\mathcal{U} \subset \mathcal{S}$ those functions in $\mathcal{S}$ that may be extended into the complex plane as entire functions of rapid decrease on strips. More precisely

$$\mathcal{U} = \cap_{p=0}^{\infty} \mathcal{U}_p$$

with $\mathcal{U}_p$ a space of entire functions topologized by the norm $\left\|\varphi\right\|_p = \sup_{z \in \Lambda_p} \left\{ \left(1 + \left|z\right|^p\right) \left| \varphi\left(z\right) \right| \right\}$, Λ_p being the open strip $\Lambda_p = \left\{ z \in \mathbb{C} : \left|\mathrm{Im}\left(z\right)\right| < p \right\}$. Each $\mathcal{U}_p$ space may also be characterized by the Fourier transform $\mathcal{F}$

$$\mathcal{U}_p = \left\{ \varphi : \mathcal{F}\left\{\varphi\right\} \in \mathcal{K}_p \right\},$$

$\mathcal{K}_p$ being the completion of C^∞ for the norm $\|\varphi\| = \max_{0\le q\le p}\left\{\sup\left|e^{p|x|}\varphi^{(q)}\right|\right\}$.

$\mathcal{U}'$, the topological dual of $\mathcal{U}$, is Silva's space of tempered ultradistributions [21,22]. Define B_η as the complement in $\mathbb{C}$ of the strip $\mathrm{Im}\left(z\right) \le \eta$

$$B_\eta = \left\{z : \mathrm{Im}\left(z\right) > \eta\right\} \tag{6}$$

and H_η the set of functions which are holomorphic and of polynomial growth in B_η

$$\varphi\left(z\right) \in H_\eta \Longrightarrow \exists M, \alpha : \left|\varphi\left(z\right)\right| < M\left|z\right|^\alpha, \forall z \in B_\eta. \tag{7}$$

$H_\eta = \underrightarrow{\lim} H_k$ is the inductive limit of the spaces H_k with topology defined by the norms

$$\|\varphi\|_\eta = \sup_{z\in B_\eta}\left|\frac{\varphi\left(z\right)}{1+|z|^k}\right|$$

Let now H_ω be the union

$$H_\omega = \bigcup_{\eta\ge 0} H_\eta \tag{8}$$

and in H_ω define the equivalence relation Ξ by

$$\varphi \overset{\Xi}{\simeq} \psi \text{ if } \varphi - \psi \text{ is a polynomial.}$$

Then, the space of tempered ultradistribution is

$$\mathcal{U}' = H_\omega / \Xi \tag{9}$$

and $\left[\phi\left(z\right)\right]$ will denote the equivalence class. The vectorial operations as well as derivation and multiplication by polynomials, defined on H_ω, are compatible with the equivalence relation and $\mathcal{U}'$ becomes a vector space with these operations.

The Schwartz space $\mathcal{S}'$ of tempered distributions may be identified with a subspace of $\mathcal{U}'$ by the Stieltges transform, that is, a linear mapping of $\mathcal{S}'$ on a subspace $\mathcal{U}'^*$ of $\mathcal{U}'$. Namely, given $\mu\left(x\right) \in \mathcal{S}'$

$$\varphi\left(z\right) = \frac{p\left(z\right)}{2\pi i}\int \frac{\mu\left(x\right)}{p\left(x\right)\left(x-z\right)}\mathrm{d}x + P\left(z\right) \tag{10}$$

$\left[\varphi\left(z\right)\right] \in \mathcal{U}'$. Here $p\left(z\right)$ is a polynomial such that $\mu/p \sim O\left(x^{-1}\right)$ and $P\left(z\right)$ is an arbitrary polynomial. Whereas a general tempered ultradistribution is characterized by a pair of functions in H_η (up to a polynomial) for some $\eta > 0$, to a tempered distribution one associates a pair of functions which are holomorphic respectively in the upper and lower complex plane.

Operations on tempered ultradistributions $f \in \mathcal{U}'$ are performed using their analytical images $\varphi\left(z\right)$. For example f is integrable in $\mathbb{R}$ if there is an $y_0 \in \mathbb{R}$ and a $\varphi\left(z\right)$ in $\left[\varphi\left(z\right)\right] \in \mathcal{U}'$ such that $\varphi\left(x+iy_0\right) - \varphi\left(x-iy_0\right)$ is integrable in $\mathbb{R}$ in the sense of distributions. Then

$$\langle\varphi|g\rangle = \oint_{\Gamma_{y_0}} \varphi\left(z\right)g\left(z\right)\mathrm{d}z \tag{11}$$

$\varphi \in \mathcal{U}', g \in \mathcal{U}$ and the integral runs around the boundaries of the strip $\mathrm{Im}\left(z\right) \leq y_0$.

An ultradistribution vanishes in an open set $A \in \mathbb{R}$ if $\varphi\left(x+iy\right) - \varphi\left(x-iy\right) \to 0$ for $x \in A$ when $y \to 0$ or, equivalently, if there is an analytical extension of φ to the vertical strip $Re\, z \in A$. The support of $\left[\varphi\left(z\right)\right]$ is the complement in $\mathbb{R}$ of the largest open set where $\left[\varphi\left(z\right)\right]$ vanishes.

All these notions are easily generalized to $\mathbb{R}^n$ [13,22] by considering products of semiplans as in (6) and the corresponding polynomial bounds. For the equivalence relation Ξ, one uses pseudopolynomials, that is, functions of the form

$$\sum_{j,k} \rho\left(z_1, \ldots, \overset{\wedge}{z_j}, \ldots, z_n\right) z_j^k,$$

ρ is a function in some H_η and $\overset{\wedge}{z_j}$ means that this variable is absent from the arguments of ρ.

An ultradistribution in $\mathbb{R}^n$ has compact support if there is a disk D such that any φ in $\left[\varphi\left(z\right)\right] \in \mathcal{U}'$ has an analytic extension to $\left(\mathbb{C}/D\right)^n$. Then the integral in (11) is around a closed contour containing the support of the ultradistribution.

For the purposes of this paper, the most important property of ultradistributions of compact support is the fact that any such ultradistribution has a representation as a series of multipoles, of the form [18,22]

$$\nu\left(x\right) = \sum_{r_1=0}^{\infty} \cdots \sum_{r_n=0}^{\infty} p_{r_1,\ldots,r_n} \delta^{\left(r_1,\ldots,r_n\right)}\left(x-a\right). \tag{12}$$

with the $p_{r_1,\ldots,r_n}$ being constants and the $\delta^{\left(r_1,\ldots,r_n\right)}$'s derivatives of the delta distribution. This result follows from the fact that for compact support one may apply to the Stieltjes image the Cauchy theorem over a closed contour. The space of *tempered ultradistributions of compact support* will be denoted $\mathcal{U}_0'$.

2.2. Superprocesses

Let the underlying space of the superprocess be $\mathbb{R}^n$. Denote by $\left(X_t, P_{0,\mu}\right)$ a branching stochastic process with values in $\mathcal{U}_0'$ and transition probability $P_{0,\mu}$ starting from time 0 at a point in $\mathbb{R}^n$ and $\mu \in \mathcal{U}_0'$. The process is assumed to satisfy the *branching property,* that is, given $\mu = \mu_1 + \mu_2$

$$P_{0,\mu} = P_{0,\mu_1} * P_{0,\mu_2}. \tag{13}$$

After the branching $\left(X_t^1, P_{0,\mu_1}\right)$ and $\left(X_t^2, P_{0,\mu_2}\right)$ are independent and $X_t^1 + X_t^2$ has the same law as $\left(X_t, P_{0,\mu}\right)$. In terms of the *transition operator* V_t operating on functions on $\mathcal{U}$, this would be

$$\left\langle V_t f, \mu_1 + \mu_2\right\rangle = \left\langle V_t f, \mu_1\right\rangle + \left\langle V_t f, \mu_2\right\rangle \tag{14}$$

with V_t defined by $e^{-\left\langle V_t f, \mu\right\rangle} = P_{0,\mu} e^{-\left\langle f, X_t\right\rangle}$ or

$$\left\langle V_t f, \mu\right\rangle = -\log P_{0,\mu} e^{-\left\langle f, X_t\right\rangle} \tag{15}$$

$f \in \mathcal{U}, \mu \in \mathcal{U}_0'$.

Underlying the usual construction of superprocesses, in the form that is useful for the representation of solutions of PDE's, there is a stochastic process with paths that start from a particular point in $\mathbb{R}^n$, then propagate and branch, but the paths preserve the same nature after the branching. In terms of measures it means that one starts from an initial δ_x $\left(x \in \mathbb{R}^n\right)$ which at the branching point originates other $\delta' s$ with at most some scaling factors. It is to preserve this pointwise interpretation that, in this larger setting, one considers ultradistributions in $\mathcal{U}_0'$, because, as stated above, any ultradistribution in $\mathcal{U}_0'$ may be represented as a multipole expansion. Therefore to define the process, it suffices to specify how the branching acts on arbitrary delta derivatives. The construction may now proceed as in the measure-valued case [7,8], only with a more general branching function.

In $M = [0, \infty) \times \mathbb{R}^n$, consider a process $\xi = \left(\xi_\tau, \Pi_{0,x}\right)$ with parameter k defining the lifetime and an open regular set $Q \subset M$. A regular set is a set for which the regular points $\left(r, y\right)$ in ∂Q are a total set, a regular point being such that

$$\Pi_{0,x}\left(\tau = t\right) = 1$$

τ being the first exit time.

The process starts from $x \in \mathbb{R}^n$ carrying along an ultradistribution $\mu \in \mathcal{U}_0'$ indexed by the path coordinate. At each branching point (ruled by $\Pi_{0,x}$) of the ξ_t-process, there is a transition ruled by a $P_{0,x}$ probability in $\mathcal{U}_0'$ leading to one or more elements in $\mathcal{U}_0'$. These $\mathcal{U}_0'$ elements are then carried along by the new paths of the ξ_t-process. By construction, in each path, the process never leaves $\mathcal{U}_0'$. The whole process stops at the boundary ∂Q, finally defining an exit process $\left(X_Q, P_{0,x}\right)$ on $\mathcal{U}_0'$. If the initial μ is δ_x and $f \in \mathcal{U}$ a function on ∂Q one writes

$$u\left(r, x\right) = \left\langle V_Q f, \delta_x\right\rangle = -\log P_{0,x} e^{-\left\langle f, X_Q\right\rangle} \tag{16}$$

$\left\langle f, X_Q\right\rangle$ being computed on the (space–time) boundary with the exit ultradistribution generated by the process.

The connection with nonlinear PDE's is established by defining the whole process to be a $\left(\xi, \psi\right)-$*superprocess* if $u\left(r, x\right)$ in (16) satisfies equation

$$u + G_Q \psi\left(u\right) = K_Q f \tag{17}$$

where G_Q is the Green operator,

$$G_Q f\left(\tau, x\right) = \Pi_{0,x} \int_0^\tau f\left(s, \xi_s\right) \mathrm{d}s \tag{18}$$

and K_Q the Poisson operator

$$K_Q f\left(x\right) = \Pi_{0,x} 1_{\tau<\infty} f\left(\xi_\tau\right) \tag{19}$$

$\psi\left(u\right)$ means $\psi\left(r, x; u\left(r, x\right)\right)$ and τ is the first exit time from Q.

A remark on notation: t denotes the time of the process, whereas $\left(r, x\right)$ is the time–space point at which the solution of (17) is computed, meaning that the boundary ∂Q is $r \times \partial\Lambda$, $\Lambda \in \mathbb{R}^n$. In an alternative formulation one might consider a backwards-in-time process starting from $x \in \mathbb{R}^n$ at time r and the boundary ∂Q at $0 \times \partial\Lambda$.

Equation (17) is recognized as the integral version of a nonlinear partial differential equation with the Green operator determined by the linear part of the equation and $\psi(u)$ by the nonlinear terms. If the equation does not possess a natural Poisson clock for the branching (like the $-v$ term in KPP, Equation 1), we have to introduce an artificial lifetime for the particles in the process (e^{-k}), which in the end must vanish ($k \to \infty$) through a rescaling method.

The superprocess is then constructed as follows: let $\varphi(s, x; z)$ be the branching function at time s and point x. Then denoting $P_{0,x}e^{-\langle f, X_Q\rangle}$ as $e^{-w(0,x)}$ one has

$$P_{0,x}e^{-\langle f, X_Q\rangle} = e^{-w(0,x)} = \Pi_{0,x}\left[e^{-k\tau}e^{-f(\tau,\xi_\tau)} + \int_0^\tau \mathrm{d}ske^{-ks}\varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right)\right] \quad (20)$$

where τ is the first exit time from Q and $f(\tau, \xi_\tau) = \langle f, X_Q\rangle$ is computed with the exit boundary ultradistribution. Existence of $\langle f, X_Q\rangle$ and hence of $e^{-w(0,x)}$ is insured if $f \in \mathcal{U}$ and the branching function is such that the exit $X_Q \in \mathcal{U}_0'$.

For measure-valued superprocesses, the branching function would be

$$\varphi(s, y; z) = c\sum_0^\infty p_n(s, y)z^n \quad (21)$$

with $\sum_n p_n = 1$ and c the branching intensity, but now it may be a more general function.

For the interpretation of the superprocesses as generating solutions of PDE's, an important role is played by the following transformation that uses $\int_0^\tau ke^{-ks}\mathrm{d}s = 1 - e^{-k\tau}$ and the Markov property $\Pi_{0,x}1_{s<\tau}\Pi_{s,\xi_s} = \Pi_{0,x}1_{s<\tau}$:

$$\begin{aligned} u(t, x) &= \Pi_{0,x}\left\{e^{-kt}u(\xi_t, 0) + \int_0^t ke^{-ks}\Phi(\xi_s, t-s)\,\mathrm{d}s\right\} \\ &= \Pi_{0,x}\left\{u(\xi_t, 0) + k\int_0^t \left(\Phi(\xi_s, t-s) - u(\xi_s, t-s)\right)\mathrm{d}s\right\}. \end{aligned} \quad (22)$$

Proof of this result is sketched in Chapter 4 of Ref. [7]. A detailed proof, with the notations used in this paper may be found in [26]. Because (22) only depends on the Markov properties of the $(\xi_t, \Pi_{0,x})$ process it also holds in the ultradistribution context.

Equation (22) converts Equation (20) for $e^{-w(0,x)}$ into

$$e^{-w(0,x)} = \Pi_{0,x}\left[e^{-f(\tau,\xi_\tau)} + k\int_0^\tau \mathrm{d}s\left[\varphi\left(s, \xi_s; e^{-w(\tau-s,\xi_s)}\right) - e^{-w(\tau-s,\xi_s)}\right]\right]. \quad (23)$$

Equation (17) is now obtained by a limiting process. Let in (23) replace $w(0, x)$ by $\beta w_\beta(0, x)$ and f by βf. β is interpreted as the mass of the particles and when the $\mathcal{U}_0'$-valued process βX_Q replaces X_Q then P_μ becomes $P_{\frac{\mu}{\beta}}$,

$$e^{-\beta w(0,x)} = \Pi_{0,x}\left[e^{-\beta f(\tau,\xi_\tau)} + k_\beta\int_0^\tau \mathrm{d}s\left[\varphi_\beta\left(s, \xi_s; e^{-\beta w(\tau-s,\xi_s)}\right) - e^{-\beta w(\tau-s,\xi_s)}\right]\right] \quad (24)$$

Two rescaling limits will be used in this paper. The first one, called here as *type I*, is the one used in the past for superprocesses on measures, namely it defines

$$u_\beta^{(1)} = \left(1 - e^{-\beta w_\beta}\right)/\beta \quad ; \quad f_\beta^{(1)} = \left(1 - e^{-\beta f}\right)/\beta \tag{25}$$

and

$$\psi_\beta^{(1)}\left(s, x; u_\beta^{(1)}\right) = \frac{k_\beta}{\beta}\left(\varphi\left(s, x; 1 - \beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)}\right) \tag{26}$$

one obtains from (24)

$$u_\beta^{(1)}(r, x) + \Pi_{0,x}\int_0^\tau \mathrm{d}s\psi_\beta^{(1)}\left(s, \xi_s; u_\beta^{(1)}\right) = \Pi_{0,x} f_\beta^{(1)}\left(\tau, \xi_\tau\right) \tag{27}$$

that is

$$u_\beta^{(1)} + G_Q\psi_\beta^{(1)}\left(u_\beta^{(1)}\right) = K_Q f_\beta^{(1)}. \tag{28}$$

One sees from (25) that when $\beta \to 0, f_\beta^{(1)} \to f$ and if $\psi_\beta^{(1)}$ goes to a well defined limit ψ then $u_\beta^{(1)}$ tends to a limit u solution of (17) associated to a superprocess. Also one sees from (25) that in the $\beta \to 0$ limit

$$u_\beta^{(1)} \to w_\beta = -\log P_{0,x} e^{-\langle f, X_Q\rangle} \tag{29}$$

as in Equation (16). The superprocess corresponds to a cloud of ultradistribution 'particles' for which both the mass and the lifetime tend to zero.

An equivalent result is obtained with a *rescaling of type II*

$$u_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta w_\beta} - e^{-\beta w_\beta}\right) \quad ; \quad f_\beta^{(2)} = \frac{1}{2\beta}\left(e^{\beta f} - e^{-\beta f}\right). \tag{30}$$

Notice that, as before, $u_\beta^{(2)} \to w_\beta$ and $f_\beta^{(2)} \to f$ when $\beta \to 0$.

2.3. *Existence of the superprocess*

Existence of the superprocess is existence of a unique solution for the Equation (24) and its rescaling limit (28). It will depend on the appropriate choice of the branching function $\varphi\left(s, y; z\right)$. For measure-valued processes this function is a polynomial in z, which corresponds to a branching particle system where the offspring of each particle has the same nature as the parent or, in terms of point measures, to branching of δ into other deltas with a positive coefficient. For ultradistributions of compact support, it suffices to specify the probabilities of branching from an arbitrary delta derivative $\delta^{(n)}$ to other delta derivatives with a positive or negative coefficient. Because of the multipole representation of ultradistributions of compact support (Equation (12)) any ultradistribution branching may be obtained by a linear combination of elementary branchings of this type.

Suppose that such a ultradistribution branching is specified. Associated to the ultradistribution superprocess Γ with branching function φ, there is an *enveloping measure superprocess* $\widetilde{\Gamma}$ with branching function $\widetilde{\varphi}$ that has the same branching topology but without any derivative change in the original delta measure at time zero nor on its sign.

General existence conditions for measure-valued superprocesses have been found in the past [6,9,10]. Namely $\widetilde{\varphi}$ should have the form

$$\widetilde{\varphi}\left(s,y:z\right)=-b\left(s,y\right)z-c\left(s,y\right)z^{2}+\int_{0}^{\infty}\left(e^{-\lambda z}+\lambda z-1\right)n\left(s,y;\mathrm{d}\lambda\right). \tag{31}$$

Suppose that the branching $\widetilde{\varphi}$ for the process $\widetilde{\Gamma}$ is of the form (31). This insures almost sure existence of $e^{-\langle g,\widetilde{X}\rangle}$, $\widetilde{X}$ being the exit measure generated by the $\widetilde{\Gamma}$ process. Decomposing the $\widetilde{X}$ measure into the (measure) components associated to the each one of the delta derivatives of each sign in the corresponding ultradistribution Γ process

$$\langle g,\widetilde{X}\rangle=\sum_{n=0}\left(\left\langle g,\widetilde{X}_{n}^{(+)}\right\rangle+\left\langle g,\widetilde{X}_{n}^{(-)}\right\rangle\right). \tag{32}$$

(C1) Let for each outgoing branch, the transition in $\mathcal{U}_0'$ be restricted to changes of sign ($\delta^n\rightarrow\pm\delta^n$) or to changes of derivative order ($\delta^n\rightarrow\pm\delta^m$).

Then the same computation as in (32) for the correspondent exit ultradistribution X yields

$$\begin{aligned}\langle f,X\rangle&=\sum_{n=0}\left((-1)^{n}\left\langle f^{(n)},X_{n}^{(+)}\right\rangle+(-1)^{n-1}\left\langle f^{(n)},X_{n}^{(-)}\right\rangle\right)\\&\leq\sum_{n=0}\left|\left\langle f^{(n)},X_{n}^{(+)}\right\rangle\right|+\left|\left\langle f^{(n)},X_{n}^{(-)}\right\rangle\right|\\&\leq M\int_{\partial Q}\sum_{n=0}\left|f^{(n)}\right|.\end{aligned} \tag{33}$$

$f^{(n)}$ being $n-$order derivatives of the function $f\in\mathcal{U}$.

Hence, one has the following sufficient condition for the existence of a ultradistribution-valued superprocess:

Proposition 1: A $\mathcal{U}_0'$ ultradistribution-valued exit superprocess Γ exists if the branching function $\widetilde{\varphi}$ of the associated enveloping exit measure process $\widetilde{\Gamma}$ is as in Equation (31), the condition C1 is satisfied and the function f is such that the integral over the exit boundary of $\Sigma_n\left|f^{(n)}\right|$ is finite.

This result imposes some restrictions on the boundary conditions of the associated nonlinear differential equations, which however are not too serious. It suffices, for example that f in the boundary ∂Q be well approximated by an arbitrary polynomial. On the other hand, the branching condition C1 can also be easily generalized.

2.4. Examples: superprocesses on signed measures and ultradistributions

As stated before, because of the multipole expansion property of ultradistributions of compact support, it suffices to specify how the branching operates on general delta derivatives.

The variable z that appears in $\varphi_\beta\left(s,x;z\right)$ is in fact $z=e^{-\beta w\left(\tau-s,\xi_s\right)}=P_{0,x}e^{-\langle\beta f,X\rangle}$. When restricting the superprocess to measures, the delta measure, at each branching point, may at most branch into other deltas (with positive coefficients) and therefore

$\varphi\left(s,x;z\right)$ must be a sum of monomials in z, with positive coefficients to have a probability interpretation. When one generalizes to $\mathcal{U}_0'$, changes of sign and transitions from deltas to their derivatives are allowed. In the end, the exponential $e^{-\langle \beta f, X\rangle}$ will be computed by evaluation of the function on the ultradistribution that reaches the boundary. To find out the equation that is represented by the process one then computes $\psi_\beta^{(1)}\left(s,x;u_\beta\right)$ of Equation (26) for the corresponding $\varphi\left(s,x;z\right)$ in the $\beta \to 0$ limit. Recalling that $\varphi\left(s,x;z\right)=\varphi_\beta\left(s,\xi_s;e^{-\beta w\left(\tau-s,\xi_s\right)}\right)$ and $z=e^{-\beta w_\beta}$, one concludes that there are basically two new transitions at the branching points:

(1) A change of sign in the point support ultradistribution

$$e^{\langle \beta f,\delta_x\rangle}=e^{\beta f(x)}\to e^{\langle \beta f,-\delta_x\rangle}=e^{-\beta f(x)} \tag{34}$$

which corresponds to

$$z\to\frac{1}{z} \tag{35}$$

and

(2) A change from $\delta^{(n)}$ to $\pm\delta^{(n+1)}$, for example

$$e^{\langle \beta f,\delta_x\rangle}=e^{\beta f(x)}\to e^{\langle \beta f,\pm\delta_x'\rangle}=e^{\mp\beta f'(x)} \tag{36}$$

which corresponds to

$$z\to e^{\mp\partial_x \log z}. \tag{37}$$

Case (1) corresponds to an extension of superprocesses on measures to superprocesses on signed measures and the second to superprocesses in $\mathcal{U}_0'$. Another possible transformation would be one decreasing the order of the derivatives in the δ's. This might be useful to generate solutions of integrodifferential equations, but will not be dealt with here.

Now, referring back to Equation (17), one knows that to obtain a superprocess that generates solutions of a particular nonlinear partial differential equation amounts to finding a branching function $\varphi\left(s,x;z\right)$ which, in the scaling limit, generates a $\psi\left(r,x;u\right)$ identical to the nonlinear term of equation. How this provides stochastic representations of solutions for a larger class of PDE's, is illustrated by two results[3]:

Proposition 2: The superprocess with branching function

$$\varphi\left(s,x;z\right)=p_1e^{\partial_x \log z}+p_2e^{-\partial_x \log z}+p_3z^2 \tag{38}$$

provides a solution to equation

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-2u^2-\frac{1}{2}\left(\partial_x u\right)^2 \tag{39}$$

whenever the boundary function $u|_{\partial Q}$ satisfies the condition of Proposition 1.

***Proof*:** The branching function $\varphi\left(s,x;z\right)$ means that at each branching point, with probability p_1 a derivative is added to the propagating ultradistribution, with probability p_2

a derivative is added plus a change of sign and with probability p_3 the ultradistribution branches into two identical ones. The branching function $\widetilde{\varphi}$ of the associated enveloping measure process is $(p_1 + p_2)\, z + p_3 z^2$, therefore belonging to the class of branchings in Equation (31).

Using now the transformation and rescaling (25) one has, for small β

$$z \to e^{\mp \partial_x \log z} = e^{\mp \partial_x \log\left(1 - \beta u_\beta^{(1)}\right)} = 1 \pm \beta \partial_x u_\beta^{(1)} + \frac{\beta^2}{2} \left\{ \left(\partial_x u_\beta^{(1)} \right)^2 \pm \partial_x u_\beta^{(1)2} \right\} + O\left(\beta^3\right) \tag{40}$$

$$z \to z^2 = \left(1 - \beta u_\beta^{(1)}\right)^2 = 1 - 2\beta u_\beta^{(1)} + \beta^2 u_\beta^{(1)2}. \tag{41}$$

Then, computing $\psi_\beta \left(s, x; u_\beta^{(1)}\right)$ with $p_1 = p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{2}$ one obtains

$$\begin{aligned} \psi_\beta \left(s, x; u_\beta^{(1)}\right) &= \frac{k_\beta}{\beta} \left(\varphi^{(1)} \left(s, x; z\right) - z \right) \\ &= \frac{k_\beta}{\beta} \left(\varphi^{(1)} \left(s, x; 1 - \beta u_\beta^{(1)}\right) - 1 + \beta u_\beta^{(1)} \right) \\ &= \frac{k_\beta}{\beta} \left(\frac{1}{8} \beta^2 \left(\partial_x u_\beta^{(1)} \right)^2 + \frac{1}{2} \beta^2 u_\beta^{(1)2} + O\left(\beta^3\right) \right) \end{aligned} \tag{42}$$

meaning that, with $k_\beta = \frac{4}{\beta}$, the superprocess provides, in the $\beta \to 0$ limit, a solution to Equation (39) □

Proposition 3: The superprocess associated to the branching function

$$\varphi\left(s, x; z\right) = p_1 z^2 + p_2 \frac{1}{z} \tag{43}$$

provides a solution to equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u^3 \tag{44}$$

whenever the boundary function $u|_{\partial Q}$ satisfies the condition of Proposition 1.

***Proof*:** The branching function φ means that with probability p_1 the ultradistribution branches into two identical ones and with probability p_2 it changes its sign. Therefore, in this case, one is simply extending the superprocess construction to signed measures. The $\widetilde{\varphi}$ branching function is $p_2 z + p_1 z^2$.

Here the rescaling of Equation (30) is used. With $z = e^{\beta w_\beta}$ one has

$$\begin{aligned} z &= -2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 - 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right) \end{aligned} \tag{45}$$

and

$$\begin{aligned}\frac{1}{z} &= 2\beta u_\beta^{(2)} + 2\sqrt{\beta^2 u_\beta^{(2)2} + 1} \\ &= 2 + 2\beta u_\beta^{(2)} + \beta^2 u_\beta^{(2)2} + O\left(\beta^4\right).\end{aligned} \tag{46}$$

For the integral equation, instead of (27), one has

$$u_\beta^{(2)}(r,x) + \Pi_{0,x}\int_0^\tau ds\,\psi_\beta^{(2)}\left(s,\xi_s;u_\beta^{(2)}\right) = \Pi_{0,x} f_\beta^{(2)}\left(\tau,\xi_\tau\right) \tag{47}$$

with

$$\psi_\beta^{(2)}\left(s,x;u_\beta^{(2)}\right) = k_\beta\left(\frac{1}{2\beta}\left(\varphi(s,x;z) - \varphi\left(s,x;\frac{1}{z}\right)\right) - u_\beta^{(2)}\right). \tag{48}$$

Let now the branching function $\varphi(s,x;z)$ be as stated in (43)

$$\varphi(s,x;z) = p_1 z^2 + p_2\frac{1}{z}.$$

Using (45) and (46) one computes $\psi_\beta^{(2)}\left(r,x;u_\beta^{(2)}\right)$ obtaining

$$\psi_\beta^{(2)}\left(r,x;u_\beta^{(2)}\right) = k_\beta\left\{-p_1 8u_\beta^{(2)}\left(1+\frac{1}{2}\beta^2 u_\beta^{(2)2}\right) + p_2 u_\beta^{(2)} - u_\beta^{(2)} + O\left(\beta^4\right)\right\} \tag{49}$$

and with $p_1 = \frac{1}{10}$; $p_2 = \frac{9}{10}$ and $k_\beta = \frac{5}{2\beta^2}$ one obtains in the in the $\beta \to 0$ limit

$$\psi_\beta^{(2)}\left(r,x;u_\beta^{(2)}\right) \to -u_\beta^{(2)3} \tag{50}$$

meaning that this superprocess provides a solution to Equation (44)

□

In conclusion, extending the superprocess construction to signed measures and ultra-distributions, stochastic solutions are obtained for a much larger class of partial differential equations.

Notes

1. For a detailed account of the nature of the limitations of superprocesses on measures as related to the positivity of the coefficients in the offspring generating function, see [26].
2. Distributions which locally are $\mu(x) = D^k\left(e^{a|x|}f\right)$, f bounded and continuous.
3. These two examples had been described at an heuristic level in the conference paper [24]. However, their rigorous proof depended on the establishment of Proposition 1.

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