Stochastic solutions and singular partial differential equations

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Abstract

Stochastic solutions is a robust technique previously used to obtain new exact solutions for deterministic nonlinear partial differential equations as well as numerical algorithms suited to parallel computing. Here it is proposed as a solution method for partial differential equations driven by distribution-valued noises. Two examples are worked out in detail.

Keywords: Stochastic solutions; Singular nonlinear PDE's; Diffusion-branching processes; KPZ

1 Stochastic solutions

There is a probabilistic interpretation for the solutions of linear elliptic or parabolic equations, with Cauchy and Dirichlet boundary conditions. This is a classical result which may be traced back to the work of Courant, Friedrichs and Lewy [1] in the 1920's and is a standard tool in potential theory [2] [3] [4]. As a simple example, for the heat equation,

$$\partial_t u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) \qquad \text{with} \qquad u(0,x) = f(x), \tag{1}$$

the solution may be written either as

$$u(t,x) = \frac{1}{2\sqrt{\pi}}\int \frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{4t}\right)f(y)\,dy \tag{2}$$

or as

$$u(t,x) = \mathbb{E}_x f(X_t), \tag{3}$$

$\mathbb{E}_x$ being the expectation value, starting from x, of the process

$$dX_t = dB_t. \tag{4}$$

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B_t is the Brownian motion process.

Here Eq.(1) is a *specification* of the problem whereas (2) or (3) are *exact solutions.* Given an initial condition, they both provide an algorithm to construct a function satisfying the specification. In the first case one has an integration procedure and, in the second, the functional of a solution-independent process. An important condition for (2) or (3) to be considered as solutions is that the algorithmic tool be independent of the solution. This should be contrasted with stochastic processes constructed from given particular solutions, as has been done, for example, for the Boltzman equation [5].

For nonlinear partial differential equations explicit solutions, in terms of elementary functions or integrals, are only known in very particular cases. It is here that stochastic solutions proved to be an useful tool. Whenever a solution-independent stochastic process is found that, for arbitrary initial conditions, generates the solution in the sense of Eq.(3), a stochastic solution is obtained. In this way the set of equations for which exact solutions are known was considerably extended.

Stochastic solutions were constructed for the deterministic Navier-Stokes [6] [7] [8] [9] [10] [11], Vlasov-Poisson, Magnetohydrodynamics and other equations [12] [13] [14] [15] [16]. They are solution-independent processes for which the mean values of some functionals are solutions to these equations. They are **exact** stochastic solutions.

In the stochastic solutions approach one deals with a stochastic process that starts from the point where the solution is to be found, a functional being then computed along the sample path until it reaches the boundary. One way to construct stochastic solutions is based on a probabilistic interpretation of the Picard series. The differential equation is written as an integral equation which is rearranged in such a way that the coefficients of the successive terms in the Picard iteration obey a normalization condition. The Picard iteration is then interpreted as an evolution and branching process, the stochastic solution being equivalent to importance sampling of the normalized Picard series. This method traces its origin to McKean's paper [17] on the Kolmogorov-Petrovskii-Piskunov (KPP) equation [18].

Another method is *the method of superprocesses* developed by Dynkin [19] [20]. In this method, for each equation a superprocess on measures is generated, the solution being obtained by a scaling process and the measure on the boundary. As originally proposed, the method applies only to a limited class of equations. This limitation has however been lifted, by extending from superprocesses on measures to superprocesses on ultradistributions [21]. Nevertheless, because of the limiting scaling process needed to approach the solution, the superprocess method seems to be less efficient than McKean's for the implementation of practical solutions of PDE's. For a comparison of the two approaches refer to [22].

Finite difference or finite element solution methods are based on discrete approximations, which one hopes (or in rare cases proves) that they converge to the original continuum equation. By contrast, in stochastic approach described in this paper, once the stochastic process is correctly identified, the solution is

an exact solution in the same sense as the finding of a kernel to integrate an arbitrary initial condition is an exact solution. The approximation issue comes about only in the correct implementation of the stochastic process in particular when the nonlinear terms imply branching processes. For some details and a concrete numerical example refer to [15].

A field where stochastic solutions have been shown to be extremely useful is in the numerical construction of solutions of PDE's, being much more efficient than the direct numerical solution of the equations, especially for high dimensions. Whereas the deterministic algorithms used for numerical solutions grow exponentially with the dimension d of the space, roughly as N^d ($\frac{L}{N}$ being the linear size of the grid), the stochastic process only grows with the dimension d. They provide localized solutions and because sample paths started from the same or different points are independent from each other, they are a *natural choice for parallel and distributed computing.* Stochastic algorithms handle equally well regular and complex boundary conditions and in *domain decomposition* methods they avoid the communication time problem [23] [24].

Stochastic solutions constructed in McKean's style [17] rely on the formulation of the differential problem as an integral equation. If the equation contains, to start with, a nonlinear integral term, the integral equation would contain a double integral for which a stochastic version of the Picard series might also be devised. Is somewhat more complex both for simulation and for the convergence of the process.

In the past stochastic solutions have been sucessfully used to find new exact solutions of deterministic PDE's. Here this method is proposed to solve singular stochastic PDE's, in particular equations driven by distribution-valued noises.

2   Singular partial differential equations

Singular partial differential equations are partial differential equations which are singular in the sense that, for example, nonlinear functions of the driving terms, or of the solution itself, are ill-defined. As a result, even the question of what means to be a solution is a non-trivial matter. Examples of current interest are:

$$\partial_t h = \partial_x^2 h + \lambda \left(\partial_x h\right)^2 + \xi, \tag{5}$$

the Kardar-Parisi-Zhang (KPZ) model [25] for interface propagation in $d = 1$;

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi, \tag{6}$$

the stochastic quantization equation for Euclidean Φ^4 field theory [26] or a model for near mean field phase transition dynamics in dimension $d = 2$ or 3 [27], ξ being a space-time white noise and a few others [28] [29].

Past methods to deal with these equations, and in general with nonlinear partial differential equations driven by singular stochastic processes, are based either on controlled rough paths [30] [31], paraproducts [32] or a theory of regularity structures [29] [33] [34].

3

One reason why handling these SPDE's is a delicate matter is because, in an iterative process of solution, nonlinear terms, products and powers of distributions do appear and there is no canonical way to multiply arbitrary distributions. For example in the KPZ equation, if the solution has the same regularity structure as Brownian motion, the nonlinear term would involve powers of the white noise, that is, products of distributions at the same spacetime points. However, the distributions that appear in the SPDE's solutions are not arbitrary, being constrained by the structure of the equation itself, in particular by its linear part. Controlling this effect is at the basis of the previously proposed solution methods. Here, in the *stochastic solution method* that is proposed, no attempt is made to address the general problem of multiplying arbitrary distributions. Instead one uses the fact that, as already pointed out in [35], when a stochastic process is used to represent the solution of a nonlinear differential equation, products in the nonlinear terms are traded off by branchings of the process. Multiplication of distribution-valued quantities may still appear when computing the functionals, or the boundary measures, that represent the solution. However, at least when the distributions have point support or when they correspond to processes that are uncorrelated at distinct points, it turns out that, with probability one, the computation of the process solution becomes well defined. For deterministic equations the stochastic solution is an average value over all sample paths of the process, whereas for stochastic PDE's the process itself (and its functional) might be considered to be the solution.

The other methods so far used for singular equations (paraproducts, controlled rough paths or regularity structures) are appropriate to establish the existence of solutions but not very practical for its practical implementation. The main drawbacks of the stochastic solution method are the need to have a sufficiently large number of sample paths for averaging purposes and the fact that its is sometimes hard to insure convergence of the process functionals

As an illustration of the method, stochastic solutions will be constructed for the two singular PDE's mentioned above, which in the past have been treated by other means.

3   The KPZ equation

A useful transformation for the KPZ equation (5) is

$$
\begin{aligned}
Z &= e^{-t}e^{\lambda h} \\
h &= \frac{1}{\lambda}\left(t + \log Z\right),
\end{aligned}
\tag{7}
$$

that is, a Cole-Hopf transformation times the factor e^{-t}, leading to

$$
\partial_t Z = \partial_x^2 Z - Z + \lambda Z \xi.
\tag{8}
$$

Of course, this transformation does not avoid the difficulties associated to the product of distributions, because the last term still involves two distribution-valued entities. To construct a stochastic solution the first step is to rewrite (8)

as an integral equation,

$$
\begin{aligned}
Z\left(t,x\right) &= e^{-t}e^{t\partial_x^2}Z\left(0,x\right) + \int_0^t e^{-(t-s)}e^{(t-s)\partial_x^2}Z\left(s,x\right)\lambda\xi\left(s,x\right) \\
&= e^{-t}e^{t\partial_x^2}Z\left(0,x\right) + \int_0^t e^{-s}e^{s\partial_x^2}Z\left(t-s,x\right)\lambda\xi\left(t-s,x\right), \quad (9)
\end{aligned}
$$

$Z\left(t,x\right)$ being distribution-valued, the integrals in Eq.(9) involve a product of distributions and, when interpreted as usual equalities, would be meaningless. However they become well-defined when interpreted as a symbolic definition of an iterated stochastic process. The product $Z\left(s,x\right)\xi\left(s,x\right)$ inside the integral simply means that the $Z-$process samples the driving term at a branching point. This case is the simplest instance of a branching. A term $Z^3\left(s,x\right)$, for example, would mean a branching at $\left(s,x\right)$ of the process into three similar processes. It is in this sense that, as stated before, undefined multiplications of distributions are traded off by well defined branchings.

Noticing that $e^{t\partial_x^2}$ is the evolution operator for a diffusion process X_t and that $e^{-t} + \int_0^t e^{-s}ds = 1$, the second equation in (9) may be written as an expectation value over a branching and diffusion process which starts at $\left(t,x\right)$ and evolves backwards in time to $t=0$.

$$
Z\left(t,x\right) = \mathbb{E}_{(t,x)}\left\{e^{-t}Z\left(0,X_t\right) + \int_0^t ds\, e^{-s}Z\left(t-s,X_s\right)\lambda\xi\left(t-s,X_s\right)\right\}. \quad (10)
$$

$p_t = e^{-t}$ is the surviving probability from time t to time zero and $\rho_s = e^{-s}$ is the probability density for branching between s and $s+ds$. Between branchings the process propagates as a pure diffusion process and at each branching point there is a sampling of the white noise at that space-time point as well as the creation of a new propagation path for the process. A typical sample path is shown in Fig.1.

The contribution to the expectation in (10) of a sample path with n branchings is

$$
F_n = \lambda^n\xi\left(t-s_1,X_s\right)\xi\left(t-s_1-s_2,X_{s_1+s_2}\right)\cdots\xi\left(t-s_1-s_2\cdots-s_n,X_{s_1+s_2\cdots+s_n}\right)\cdots Z\left(0,X_t\right).
$$

The stochastic construction of the solution to equation (8) involves two distinct probability spaces, Ω and Ω', the first being the auxiliary probability space of the Brownian motion X_t used to compute the expectation $\mathbb{E}_{(t,x)}$ and the second the probability space of the white noise that drives the equation. The expectation value (in Ω) of Eq.(10) is over the branching and diffusion process, for a particular fixed realization of the white noise $\{\xi\left(t,x\right), t \in [0,t]\}$ in Ω'. Because the driving term is a white noise, $Z\left(t,x\right)$ in Eq.(10) is also a random variable in Ω'. If one lets the ξ white noise realization change at each branching one would obtain a process $Z'(t,x)$ in $\Omega \otimes \Omega'$. The solution to (8) is a partial average over Ω.

So far one has avoided the product of distributions arising from the last term in Eq.(8) by trading off products by branchings. Then, to compute the
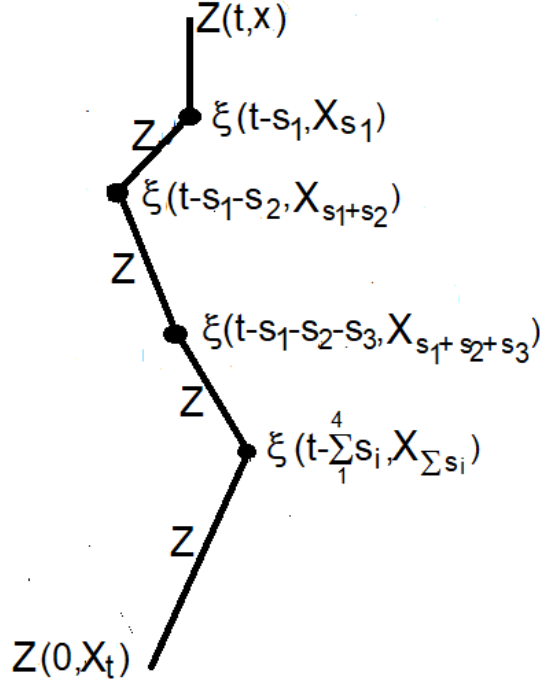
Figure 1: A sample path of the Z process

F_n functional of each sample path, one deals with the product of n white noises which, with probability one, are defined at different space-time points. Therefore one deals with products of n independent Gaussian variables, which are well-defined random variables with distribution given by Meijer G-functions [36] [37]. Hence, with probability one, the Z' process and the functionals F_n are well-defined.

Proposition 1 *The functionals F_n over the branching, diffusion and white noise sample paths generate a well-defined process Z' in $\Omega \otimes \Omega'$. A partial expectation over Ω, when it exists, is the $\Omega'-$process solution of Eq.(8). The process solution of the KPZ equation is obtained from the second equation in (7).*

Here, the second equation in (9) was used to generate a backwards in time process, from time t to time zero. Similarly, using the first equation in (9) one obtains a forward in time process, for which a similar construction may be performed.

The construction of the Z' process associated to Eq.(8) is an exact result. In this case however, in virtue of the simple structure of the branching process, an heuristic derivation of the expectation value may be performed. Assuming a

single realization of the ξ white noise, the expectation value over the branching may be performed explicitly by by summing all possible sample paths of the branching process, noticing that in each case the composition of the branching probability $e^{-s}e^{-s_1}e^{-s_2}\cdots e^{-(t-\sum s_i)}$ is e^{-t}. The last factor is the final backwards propagation to time zero. The result is

$$
\begin{aligned}
Z(t,x) &= \mathbb{E}_{(t,x)}\left\{e^{-t}Z(0,X_t)\left(1+\lambda\int_0^t dB(t-s,X_s)+\lambda^2\int_0^t\int_0^s dB(t-s,X_s)\,dB(t-s-s',X_{s+s'})\right.\right. \\
&\qquad\left.\left.+\lambda^3\int_0^t\int_0^s\int_0^{s'} dB(t-s,X_s)\,dB(t-s-s',X_{s+s'})\,dB(t-s-s'-s'',X_{s+s'+s''})+\cdots\right)\right\} \\
&= \mathbb{E}_{(t,x)}\left\{e^{-t}Z(0,X_t)e^{\lambda\int_0^t dB(t-s,X_s)}\right\},
\end{aligned}
\tag{11}
$$

$B(t,x)$ being the Brownian motion for which the white noise $\xi(t,x)$ is the distributional derivative. Finally

$$
h(t,x)=\frac{1}{\lambda}\log\mathbb{E}_{(t,x)}\left\{e^{\lambda h(0,X_t)}e^{\lambda\int_0^t dB(t-s,X_s)}\right\},
\tag{12}
$$

which in the $\lambda\to 0$ limit converges to the known solution of the stochastic heat equation

$$
\begin{aligned}
h_{\lambda\to 0}(t,x) &= \lim_{\lambda\to 0}\frac{1}{\lambda}\log\left\{1+\sum_{k=1}^{\infty}\frac{\lambda^k}{k}\mathbb{E}_{(t,x)}\left(h(0,X_t)+\int_0^t dB(t-s,X_s)\right)^k\right\} \\
&= \mathbb{E}_{(t,x)}\left\{h(0,X_t)+\int_0^t dB(t-s,X_s)\right\}.
\end{aligned}
\tag{13}
$$

$e^{\lambda\int_0^t dB(t-s,X_s)}$ and $\int_0^t dB(t-s,X_s)$ are random variables in Ω' for each sample path of the process in Ω.

So far, as in the work of other authors [38] [39] [40], a version of the Cole-Hopf transformation was used to solve the KPZ equation. However the stochastic solution technique may also be applied directly to the equation (5). Because in that equation there is no natural branching clock, one adds and subtracts a linear term to the equation

$$
\partial_t h(t,x)=\partial_x^2 h(t,x)-\mu h(t,x)+\lambda\left(\partial_x h\right)^2+\mu h(t,x)-\xi(t,x),
\tag{14}
$$

which allows to write the integral equation as

$$
h(t,x)=e^{-t\mu}e^{t\partial_x^2}h(0,x)+\int_0^t ds\,e^{-\mu s}e^{s\partial_x^2}\left\{\lambda\left(\partial_x h(t-s,x)\right)^2+\mu h(t-s,x)-\xi(t-s,x)\right\}.
\tag{15}
$$

The simultaneous addition and subtraction of the linear term may seem an arbitrary procedure. However, as will be seen in the next section, it is a most natural procedure, whenever one wants to provide a probabilistic interpretation to this type of equation. In this case it plays the same role as the multiplication factor e^{-t} in the Cole-Hopf transformation of Eq.(7).

7

The solution will then be an expectation value of a diffusion and branching process, denoted $DB1$

$$
\begin{aligned}
h\left(t,x\right) \;=\; & \mathbb{E}^{DB1}\left\{e^{-t\mu}h\left(0,X_t\right)+\gamma\int_0^t ds\mu e^{-\mu s}\left[\frac{\lambda}{\gamma}\left(\partial_{X_s}h\left(t-s,X_s\right)\right)^2\right.\right. \\
& \left.\left.+\frac{\mu}{\gamma}h\left(t-s,X_s\right)-\frac{1}{\gamma}\xi\left(t-s,X_s\right)\right]\right\},
\end{aligned}
\tag{16}
$$

with $\gamma=\frac{\lambda+\mu+1}{\mu}$. The $DB1-$process starts from (t,x) and diffuses backwards-in-time either to time zero with probability $e^{-t\mu}$ or to a branching point at time s with probability $ds\mu e^{-\mu s}$. At the branching point, with probability $\frac{1}{\gamma}$ the process samples the white noise, with probability $\frac{\mu}{\gamma}$ the process proceeds undisturbed and with probability $\frac{\lambda}{\gamma}$ two new $DB1-$processes are started from $(t-s,X_s)$ and a derivative label ∂_{X_s} is assigned to the branching point. In the subsequent processes both $h\left(t-s,X_s\right)$ and the label ∂_{X_s} are transported by the processes

$$
e^{s\partial_x^2}\partial_x h\left(t-s,x\right)=e^{s\partial_x^2}\partial_x e^{-s\partial_x^2}e^{s\partial_x^2}h\left(t-s,x\right).
$$

Because the propagation process here is a simple diffusion, the transport of the labels is very straightforward and at the final step, when the processes reach time zero, the derivative is applied at the same point as the point reached by the processes. For more complex propagation processes the situation is more delicate. This is discussed in the Appendix. To use this method of labelled branching trees allows for a simple construction of stochastic solutions for equations involving derivatives or even nonpolynomial interactions. This method was first used in [14] (see also [41]), being later rediscovered by other authors [42]. Fig.2 shows one sample path of the $DB1-$process.

The contribution of this sample path to the expectation in Eq.(16) would be

$$
-\gamma^4\partial^2 h\left(0,X_t^{(3)}\right)\partial^3 h\left(0,X_t^{(5)}\right)\partial^3 h\left(0,X_t^{(6)}\right)\xi\left(t-s'',X_{s''}^{(2)}\right).
$$

Notice that at each branching point there is a coupling constant γ. Also when more than one white noise contribution appears in the multiplicative functional, with probability one they sample the white noises at different space-time points. Therefore one has a well-defined product of independent Gaussian random variables.

As before one deals with two probability spaces Ω and Ω'. The expectation value in Eq.(16) is an expectation value over the $DB1-$process for each fixed realization of the white noise process, hence an expectation in Ω. When a different realization of the white noise is sampled whenever it appear in the branching tree one obtains a $DB1'-$ process in $\Omega\otimes\Omega'$.

Proposition 2 *The partial expectation (in Ω) of the multiplicative functional of the $DB1'-$process in $\Omega\otimes\Omega'$ (diffusion, branching and independent white noises at each branching) generates a solution of the KPZ equation (5)*
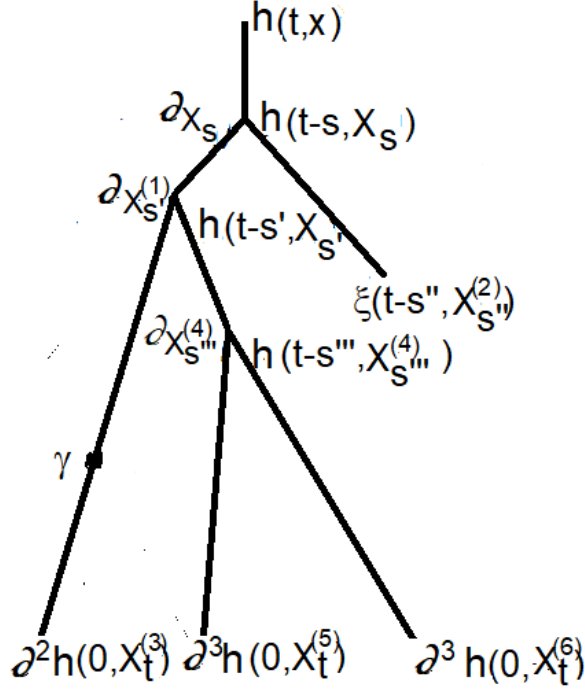
Figure 2: A sample path of the DB1-process

4 A near mean field phase transition dynamics equation

Equation (6) written as an integral equation is

$$\Phi(t, x) = e^{t\Delta}\Phi(0, x) - \int_0^t e^{s\Delta}\left(\Phi^3 - \xi\right)(t - s, x)\,ds, \tag{17}$$

which may be rewritten as an expectation over the Brownian process X_t generated by the Laplacian Δ,

$$\Phi(t, x) = \mathbb{E}_{(t,x)}\left\{\Phi(0, X_t) - \int_0^t \left(\Phi^3 - \xi\right)(t - s, X_s)\,ds\right\}. \tag{18}$$

As has been seen in the KPZ example, the replacing of the ill-defined distributional products by branchings relies on the existence of a representation of the equation as a branching process. In the Cole-Hopf transformed KPZ equation (8), the branching clock is supplied by the linear term $-Z$. In the equation (18) there is no such term and multiplying Φ by e^{-t} is also not convenient because of the Φ^3 term. There is a rather trivial way to obtain a branching

9

clock. Nevertheless it is useful to explain why, although trivial, it is natural and rather general. Take a very general approach: One wants to find a functional of Φ, $\mathcal{F}_\beta(\Phi)$, depending on a parameter β, which has a diffusion and branching representation and is such that in some limit ($\beta \to 0$, for example) obeys the equation (18) and $\lim_{\beta \to 0} \mathcal{F}_\beta(\Phi) \to \Phi$. By hypothesis $\mathcal{F}_\beta(\Phi)$ obeys a branching and diffusion equation. Therefore

$$
\begin{aligned}
\mathcal{F}_\beta(\Phi)(t,x) &= e^{-\lambda t} e^{t\Delta} \mathcal{F}_\beta(\Phi)(0,x) - \int_0^t \lambda e^{-\lambda s} e^{s\Delta} \Psi_\beta(x, \Phi(t-s,x))\, ds \\
&= \mathbb{E}_{(t,x)} \left\{ e^{-\lambda t} \mathcal{F}_\beta(\Phi)(0,X_t) - \int_0^t \lambda e^{-\lambda s} \Psi_\beta(X_t, \Phi(t-s,X_t))\, ds \right\},
\end{aligned}
$$
(19)

where $\Psi_\beta(X_t, \Phi(t-s,X_t))$ is the branching kernel.

Compute

$$
\begin{aligned}
&\mathbb{E}_{(t,x)} \left\{ \int_0^t \lambda \mathcal{F}_\beta(\Phi)(t-s,X_s)\, ds \right\} \\
&= \mathbb{E}_{(t,x)} \left\{ \int_0^t \lambda e^{-\lambda(t-s)} \mathcal{F}_\beta(\Phi)(0,X_{s+t-s})\, ds \right. \\
&\qquad \left. - \int_0^t \lambda ds \int_0^{t-s} \lambda e^{-\lambda s'} \Psi_\beta(X_{s+s'}, \Phi(t-s-s',X_{s+s'}))\, ds' \right\}.
\end{aligned}
$$
(20)

Now summing Eqs.(19) and (20) one obtains

$$
\mathcal{F}_\beta(\Phi)(t,x) = \mathbb{E}_{(t,x)} \left\{ \mathcal{F}_\beta(\Phi)(t,X_t) - \lambda \int_0^t ds\, (\Psi_\beta(X_s, \Phi(t-s,X_s)) - \mathcal{F}_\beta(\Phi)(t-s,X_s)) \right\}.
$$
(21)

This is an useful identity which relies on the Markov nature of the process (for a detailed proof refer to [19] or [22]).

Comparing now (18) with (21) and requiring $\mathcal{F}_\beta(\Phi) \underset{\beta \to 0}{\to} \Phi$ one concludes

$$
\lambda\left(\Psi_\beta(X_s, \Phi(t-s,X_s)) - \mathcal{F}_\beta(\Phi)(t-s,X_s)\right) \underset{\beta \to 0}{\to} \left(\Phi^3 - \xi\right)(t-s,X_s), \quad (22)
$$

that is, the branching kernel Ψ_β must be of the form

$$
\frac{1}{\lambda}\left(\lambda\Phi + \Phi^3 - \xi\right).
$$
(23)

It must contain a branching to Φ^3, a sampling of the white noise ξ and a linear term $\lambda\Phi$. The linear term simply means that when the branching clock rings, with probability $\frac{\lambda}{\lambda+2}$ the path is not interrupted. The two other possibilities have probabilities $\frac{1}{\lambda+2}$. The minus sign is a coupling constant that appears in the contribution to the expectation value when the white noise is sampled. Alternatively, in the language of superprocesses, it means that one is dealing not

10

with a superprocess in measures, but with a superprocess in signed measures [21].

Coming back to the original equation one sees that it all amounts to add and subtract a term linear in Φ to the equation. One term generates the branching clock, the other goes to the branching kernel. The result is

$$
\begin{aligned}
\Phi\left(t,x\right) &= e^{-t}e^{t\Delta}\Phi\left(0,x\right) - \int_0^t e^{-s}e^{s\Delta}\left(\Phi^3 + \Phi - \xi\right)\left(t-s,x\right)ds \\
&= \mathbb{E}_{(t,x)}^{BD2}\left\{e^{-t}\Phi\left(0,X_t\right) - 3\int_0^t e^{-s}\frac{1}{3}\left(\Phi^3 + \Phi - \xi\right)\left(t-s,X_s\right)ds\right\},
\end{aligned}
$$

$$(24)$$

where again the equations should be interpreted as a symbolic definition of a process. This backwards-in-time diffusion and branching process ($DB2$), starting from (t,x), generates the solution. The probability to reach time zero is e^{-t} and the probability density to branch at time s is e^{-s}. At the branching point there are 3 possibilities with equal probability $\frac{1}{3}$. Either the process branches into 3 similar processes, or the process proceeds undisturbed, or it samples the white noise. A sample path is shown in Fig.3.
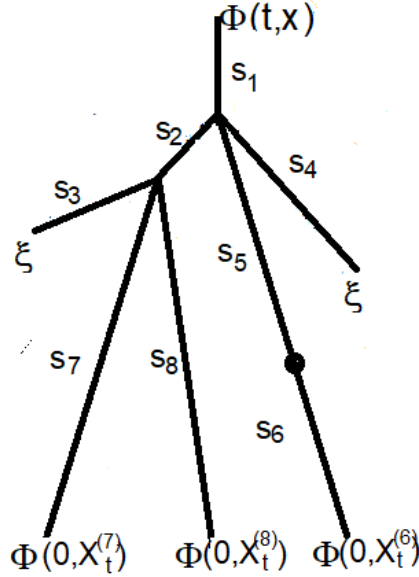

Figure 3: A sample path of the DB2-process

The contribution of this sample path to the expectation in Eq.(24) would be

$$-3^5 \Phi\left(0, X_t^{(7)}\right) \Phi\left(0, X_t^{(8)}\right) \Phi\left(0, X_t^{(6)}\right) \xi\left(t - s_1 - s_4, X_{s_1+s_4}\right) \xi\left(t - s_1 - s_2 - s_3, X_{s_1+s_2+s_3}\right).$$

As before, when more than one white noise contribution appears in the multiplicative functional, with probability one they sample the white noises at different space-time points. Hence a well-defined product of independent Gaussian random variables is obtained.

The expectation value in Eq.(24) is an expectation value over the $DB2-$process (in Ω) for each fixed realization of the white noise process. If a different realization of the white noise is sampled whenever it appear in the branching tree, a $DB2'-$ process is obtained in $\Omega \otimes \Omega'$.

Proposition 3 *Partial expectations of the multiplicative functionals associated to the $DB2'-$process generate the $\Omega'-$process solution of the equation (17)*

5   Conclusions

The technique of stochastic solutions, trading products by branchings, avoids some of the problems associated to the product of distributions. Because in each sample path the white noises are, with probability one, sampled at different space-time points, the functionals for each sample path of the $\Omega \otimes \Omega'-$processes are well-defined. However the computation and existence of the partial expectation in Ω may be a delicate matter. In the case of the deterministic equations studied in the past, existence of the expectation was proved by imposing boundedness conditions on the multiplicative terms that appear in the functionals. That procedure is not applicable here because of the unbounded nature of the white noises. For practical purposes one might impose some truncation on the stochastic driving term, but this is not satisfactory from a mathematical point of view.

The computation of the averages may involve an arbitrary large number of white noise factors. Therefore it is natural to think that the $\Omega'-$process solution will not be distribution -valued, because such unbounded sums are known not to be in general in the distribution space. A reasonable conjecture is that the $\Omega'-$process would be ultradistribution-valued[1].

Appendix: The transport of the labels

For PDE's with derivatives or nonpolynomial interactions in the nonlinear terms [14] [41] the branching and diffusion processes that generate the solution, receive, at each branching point, labels that represent the derivatives or the nonpolynomial functions. As the solution is transported by the diffusion process between

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[1]The ultradistribution space $\mathcal{Z}'$ is the Fourier image of the Schwartz distribution space $\mathcal{D}'$[44]. The rich analytic structure of $\mathcal{Z}'$ makes it very useful for the applications. In particular, multipole series are defined in $\mathcal{Z}'$ but not in $\mathcal{D}'$[45].

the branching points, so are the labels and, in general, both the solution h and the labels may be modified by this transport. Let us take as an example a derivative label that is transported from (x_0, t_0) to (x_1, t_1) and consider a propagation by a general diffusion and drift process where both the drift and the diffusion coefficients may be space-time dependent.

$$dW = \bar{b}dt + \bar{\sigma}dB$$

Then given the value $\partial_x h(x_0)$ at t_0, one obtains the expectation value of the derivative at $t_0 + \Delta t$ by partial integration with the Green's function

$$
\begin{aligned}
\mathbb{E}\left(\partial_x h(t_0 + \Delta t)\right) &= \int dy \partial_x h(x_0 + y) \frac{1}{\sqrt{2\pi\bar{\sigma}\Delta t}} e^{-\frac{(y-\bar{b}\Delta t)^2}{2\bar{\sigma}\Delta t}} \\
&= \mathbb{E}\left(h(x_0 + W_{\Delta t}) \frac{\left(W_{\Delta t} - \bar{b}\Delta t\right)}{\bar{\sigma}\Delta t}\right).
\end{aligned}
$$

Therefore keeping track of the successive increments $W_{\Delta t}$ of the process one may obtain the value of the derivative at (x_1, t_1) from the value of the function that is sampled at (x_0, t_0). This is what some authors [43] have called *automatic differentiation.* Likewise for the second derivative one obtains

$$
\begin{aligned}
\mathbb{E}\left(\partial_x^2 h(t_0 + \Delta t)\right) &= \int dy \partial_x^2 h(x_0 + y) \frac{1}{\sqrt{2\pi\bar{\sigma}\Delta t}} e^{-\frac{(y-\bar{b}\Delta t)^2}{2\bar{\sigma}\Delta t}} \\
&= \mathbb{E}\left(h(x_0 + W_{\Delta t}) \frac{\left(W_{\Delta t} - \bar{b}\Delta t\right)^2 - \bar{\sigma}\Delta t}{(\bar{\sigma}\Delta t)^2}\right).
\end{aligned}
$$

However, whenever $\bar{\sigma}$ and $\bar{b}$ are space independent constants, one sees from the linear evolution that takes place between the branching points,

$$\frac{\partial h}{\partial t} = \frac{1}{2}\bar{\sigma}^2 \partial_x^2 h + \bar{b}\partial_x h,$$

that all derivatives of h evolve according to the same process as h. Therefore it is sufficient to apply the derivatives to the sampled initial condition when this is reached in the process, without any need to do the backtrack of automatic differentiation. However this no longer applies if at some intermediate step in the branching a dependency on the x coordinate is introduced.

For the functional labels $f(h)$,

$$\frac{\partial f(h)}{\partial t} = \frac{1}{2}\bar{\sigma}^2 \partial_x^2 f(h) + \bar{b}\partial_x f(h) - \frac{1}{2}\bar{\sigma}^2 \left(\partial_x f(h)\right)^2 \frac{\partial^2 f(h)}{\partial h^2},$$

hence $f(h)$ evolves with the same process as h only if $f(h)$ is a linear function of h. If it is not, the calculation of the values at the branching points should be computed with a backtracking similar to automatic differentiation, namely,

$$\mathbb{E}\left(f(h(t_0 + \Delta t))\right) = \mathbb{E}\left(f(x_0 + W_{\Delta t})\right).$$

Therefore, the simplest situation occurs when $\overline{\sigma}$ and $\overline{b}$ are chosen to be constants in the linear part of the equation together with a polynomial approximation for $f(h)$ because then it can always be decomposed into a product of first order polynomials.

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