

Entropy and Quantum Characteristic Exponents. Steps Towards a Quantum Pesin Theory

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Abstract: Classical ergodic invariants like the Lyapunov exponent and the entropy, defined in a space of densities acted upon by the Perron-Frobenius operator, provide an intuitive framework to construct the corresponding notions in quantum mechanics.

Rigorous existence conditions obtained for quantum characteristic exponents and a definition of quantum entropy based on the Brin-Katok construction are steps towards a quantum ergodic theory, where a rigorous definition of quantum chaos may be established.

1 Characteristic Exponents

Classical mechanics deals with trajectories of dynamical systems whereas quantum mechanics deals with wave functions. This is a source of difficulty when transferring to quantum mechanics the notions of the ergodic theory of classical dynamical systems. The modulus squared of the wave function is a probability density in configuration space. Therefore we would be closer to the framework of quantum mechanics, and presumably in better conditions to develop a *quantum ergodic theory*, if classical concepts like the invariant measure, Lyapunov exponents and entropy were formulated in terms of densities rather than orbits.

Let (X, Σ, μ) be a measure space and $T : X \rightarrow X$ a measurable measure-preserving transformation, that is

- (i) $T^{-1}(A) \in \Sigma$ for all $A \in \Sigma$
- (ii) $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \Sigma$.

In the context of the ergodic theory of classical dynamical systems X is the state space and T the operator defining time-evolution.

A *density* in X is a non-negative, normalized, integrable function, the space of densities being denoted by \mathcal{D}

$$\mathcal{D} = \{ \rho \in L^1(X, \Sigma, \mu) : \rho \geq 0, \|\rho\|_1 = 1 \} \quad (1)$$

\mathcal{D} is the space of functions that, by the Radon-Nikodym theorem, characterize the measures that are absolutely continuous with respect to μ . However, for

some of the dynamical concepts to be discussed later, it will turn out that it is convenient to further restrict the space of *admissible densities*.

Time-evolution in the space of densities is described by the weakly continuous Perron-Frobenius operator, $P : L^1 \rightarrow L^1$

$$\int_A (P\rho)(x)d\mu(x) = \int_{T^{-1}(A)} \rho(x)d\mu(x) \tag{2}$$

$A \in \Sigma$, or by the Koopman operator, $U : L^1 \rightarrow L^1$

$$U\rho(x) = \rho(T(x)) \tag{3}$$

these two operators being adjoint to each other, in the sense

$$\int_X (P\rho)(x)g(x)d\mu(x) = \int_X \rho(x)Ug(x)d\mu(x) \tag{4}$$

with $\rho \in L^1$ and $g \in L^\infty$.

The basic notions of ergodic theory, usually defined in terms of the images or inverse images of sets of points in X under the action of T , may also be expressed in terms of the behavior of densities under the action of the Perron-Frobenius or the Koopman operator[1]. For example:

- μ is T -invariant if and only if $P1 = 1$.

- T is ergodic if and only if the sequence $\{P^n \rho\}$ is Cesaro convergent to 1 for all $\rho \in \mathcal{D}$.

- T is mixing if and only if $\{P^n \rho\}$ is weakly convergent to 1 for all $\rho \in \mathcal{D}$.

The mathematical framework needed to characterize the operators of time evolution and the notions of ergodicity and mixing in terms of densities is essentially the same that is needed to formulate the same notions in terms of orbits, namely a measure space and the duality of L^1 and L^∞ . The situation is different when dealing with the ergodic invariants that quantify the notions of sensitive dependence and information loss. Consider for example the existence theorem for Lyapunov exponents[2][3]. Let $X = R^d$. Then, given an ergodic invariant measure μ , there is a sequence of numbers

$$\lambda_1 > \lambda_2 > \dots > \lambda_\ell$$

$\ell \leq d$, and for μ -almost every point x , a decreasing sequence of vector spaces

$$R^d = E_1(x) \supset E_2(x) \supset \dots \supset E_\ell(x) \supset E_{\ell+1}(x) = \{0\} \tag{5}$$

such that $DT_x E_r(x) = E_r(T(x))$ and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|DT_x^n v\| = \lambda_r \tag{6}$$

for $v \in E_r(x) \setminus E_{r+1}(x)$.

In Eq.(6) the action of the tangent map on the vector v measures the rate of separation of two orbits with initial conditions at x and at x plus an "infinitesimal" in the direction of v . When dealing with densities the notion that

corresponds to this perturbation of the initial condition at x is that of a Gateaux derivative D_{Φ_x} along a (generalized) function Φ_x with support at x . Furthermore the proper definition and existence of the Lyapunov exponent in Eq.(6) requires the existence of a measure μ in the space of the initial conditions x . Likewise we will need a measure on the infinite-dimensional space of the generalized functions Φ_x . These two requirements, namely the possibility to define Gateaux derivatives along generalized functions with point support and the need for a well-defined σ -additive measure in an infinite-dimensional functional space, lead almost uniquely to the choice of mathematical framework. Namely we need to restrict the *admissible densities* to a nuclear space.

Densities are non-negative L^1 -functions. Therefore to each $\rho \in \mathcal{D}$ we may associate a non-negative square root $\rho^{\frac{1}{2}}$ and $\rho \in L^1(\mu)$ implies $\rho^{\frac{1}{2}} \in L^2(\mu)$. We now construct a Gelfand triplet[4]

$$E^* \supset L^2(\mu) \supset E \tag{7}$$

E being a nuclear space and E^* its dual. For each non-negative $f \in E$, with $\|f\|_2 = 1$, $\rho = f^2$ is an admissible density. The generalized functions Φ_x needed to define the local perturbations of the densities will be in E^* . Furthermore, nuclearity of E is important to insure the existence of σ -additive measures on E^* . For definiteness let $E = \mathcal{S}(R^d, \mu)$, the space of functions of rapid decrease topologized by the family of semi-norms $\|x^\alpha D^\beta f\|_2$, and $E^* = \mathcal{S}'(R^d, \mu)$ its dual. For simplicity μ is assumed to be a T -invariant measure absolutely continuous with respect to Lebesgue measure. Because \mathcal{S} is an algebra, $\rho^{\frac{1}{2}} \in \mathcal{S}$ implies $\rho \in \mathcal{S}$. We now consider a family of vector-valued functionals of a density $\rho \in \mathcal{S}$

$$\mathcal{F}_n(\rho) = \int d\mu(y) y P^n \rho(y) \quad y \in X \tag{8}$$

and define the Lyapunov exponent as follows

$$\lambda_v = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| -v^i D_{\partial_i \delta_x} \left(\int d\mu(y) y P^n \rho(y) \right) \right\| \tag{9}$$

v being a vector in R^d . Let X be a compact in R^d and $P^n \rho \in \mathcal{S}$. Then the Gateaux derivative of $\mathcal{F}_n(\rho)$ along $\partial_i \delta_x$ is well-defined[5] and a simple calculation leads to the same result as in Eq.(6). Existence of the limit in Eq.(9) is therefore insured in the same conditions as the limit in Eq.(6). However it is possible to prove directly the existence of the limit in Eq.(9). The proof follows essentially the same steps as used below for the quantum characteristic exponents and I leave it as an exercise for the reader.

Essential to the proofs is the existence of an appropriate measure in E^* , namely a measure for which $\cup_{x \in X} \{\partial_i \delta_x\}$ is not a null set. The following result shows how a class of such measures may be constructed.

Lemma 1.1

Let μ be a normalized measure in R^d and let $\{\Phi_x\}$, a family of elements of $\mathcal{S}'(\mu)$ indexed by the points of R^d , be integrable, that is

$$\int d\mu(x) \langle \Phi_x, \xi \rangle < \infty \quad \forall \xi \in \mathcal{S}(\mu) \tag{10}$$

Then the characteristic functional

$$C(\xi) = \int d\mu(x) e^{i\langle \Phi_x, \xi \rangle} \quad \xi \in \mathcal{S}(\mu) \tag{11}$$

defines a measure ν in \mathcal{S}' for which the set $\cup_x \Phi_x$ is not a null set. Furthermore if μ is invariant for a mapping $T : R^d \rightarrow R^d$, ν is invariant for the corresponding Perron-Frobenius operator P_T if the transformation law of Φ_x is $P_T \Phi_x = \Phi_{Tx}$.

Proof:

(i) $C(0) = 1$

(ii) $\sum_{j,k} \alpha_j \alpha_k^* C(\xi_j - \xi_k) = \int d\mu(x) \left| \sum_j \alpha_j e^{i\langle \Phi_x, \xi_j \rangle} \right|^2 \geq 0$

(iii) $C(\xi)$ is a continuous functional in \mathcal{S} . Follows from $\Phi_x \in \mathcal{S}'$ and (10).

Let \mathcal{B} be the Borel σ -algebra generated by the cylinder sets $A_{\xi_1, \dots, \xi_d, B}$ in \mathcal{S}'

$$A_{\xi_1, \dots, \xi_d, B} = \{f \in \mathcal{S}' : (\langle f, \xi_1 \rangle, \dots, \langle f, \xi_d \rangle) \in B\} \tag{12}$$

with B a Borel subset of R^d . From (i),(ii),(iii) and the Bochner-Minlos theorem[6] it now follows that there is in $(\mathcal{S}', \mathcal{B})$ a unique σ -additive measure ν such that

$$C(\xi) = \int_{\mathcal{S}'} d\nu(\chi) e^{i\langle \chi, \xi \rangle} \tag{13}$$

To show that $\cup_x \Phi_x$ has positive measure consider an orthonormal basis $\{\zeta_n\}$ of generalized eigenvectors in \mathcal{S}' [7]. Eq.(11) becomes

$$C(\xi) = \int d\mu(x) e^{i \sum_n \Phi_{x,n} \xi_n} \tag{14}$$

$\{\Phi_{x,n}\}$ and $\{\xi_n\}$ being the coefficient sequences for the expansions of Φ_x and ξ , convergence of the sum being insured by the fact that $\xi \in \mathcal{S}$ and $\Phi_x \in \mathcal{S}'$. The measure ν in \mathcal{S}' is then a measure in the space of tempered sequences and from the definition of the characteristic functional one sees that the corresponding measure has support on the $\{\Phi_{x,n}\}$ sequences. Informally

$$d\nu(\chi) = \int d\mu(x) \prod_n \delta(\chi_n - \Phi_{x,n}) d\chi_n \tag{15}$$

Invariance of ν for P_T is an easy consequence of the T-invariance of μ and the transformation law for Φ_x . Notice that invariance of ν for P_T is equivalent to invariance of $C(\xi)$ for the Koopman operator U . \square

Actually there is, for the classical case, a simpler way to construct measures in $\mathcal{S}'(\mu)$ with support in $\cup_x \Phi_x$. However the construction through characteristic functions has the merit of being generalizable to the quantum case.

So far I have been concerned with the formulation of the Lyapunov exponents in terms of densities. Notice however that by replacing in Eq.(9) the Gateaux derivative $D_{\partial_i \delta_x}$ by a Gateaux derivative D_Φ along other $\Phi \in E^*$, other ergodic invariants may be obtained. To prove the existence of such invariants relies on the construction of the appropriate measure, which in some cases may be obtained

from lemma 1.1. For example for the higher-order Lyapunov exponents discussed by Farmer and Sidorovich[8] and Taylor[9], we would have

$$\Phi_x = \partial_{i_1} \cdots \partial_{i_n} \delta_x \tag{16}$$

Other choices of Φ may lead to other ergodic invariants[10].

An approach that would bring us closer to the formulation of quantum mechanics would be to consider the functional $\mathcal{F}_n(\rho)$ in Eqs.(8,9) as a functional of $\rho^{\frac{1}{2}}$ rather than of ρ , with the same Gateaux derivative. This means that the differential perturbation at x is performed on the square root of the density, which is like a "classical wave function". However, because of the regularity conditions on $\rho^{\frac{1}{2}} \in \mathcal{S}$ and the $\frac{1}{n}$ limit, the result is exactly the same.

We now turn to quantum mechanics. Let U^k (k continuous or discrete) be the unitary operator of quantum time evolution acting on an Hilbert space \mathcal{H} . \mathcal{H} is taken to be the space of functions on the configuration manifold M and \tilde{Y} is a coordinate operator in M . The quantum version of the functional $\mathcal{F}_n(\rho)$ is

$$\mathcal{F}_n(\Psi) = \left(U^n \Psi, \tilde{Y} U^n \Psi \right) \tag{17}$$

and taking the Gateaux derivative

$$v^i D_{\partial_i \delta_x} \left(U^n \Psi, \tilde{Y} U^n \Psi \right) = 2v^i Re \langle U^n \partial_i \delta_x, \tilde{Y} U^n \Psi \rangle \tag{18}$$

Proper definition of the right-hand side of (18) requires the construction of a Gelfand triplet as in (7). It is convenient to obtain E as a countably normed space, projective limit of a sequence of Hilbert spaces \mathcal{H}_p

$$E^* \supset \dots \supset \mathcal{H}_{-p} \supset \dots \supset \mathcal{H}_{-1} \supset \mathcal{H} \supset \mathcal{H}_1 \supset \dots \supset \mathcal{H}_p \supset \dots \supset E$$

with ordered norms

$$\|\cdot\| \leq \|\cdot\|_1 \leq \dots \leq \|\cdot\|_p \leq \dots$$

The standard way to construct these spaces is to use a self-adjoint operator A to define the norms

$$\|f\|_p^2 = (A^p f, A^p f) \tag{19}$$

If A is bounded from below and there is a $k \in N$ for which A^{-k} is Hilbert-Schmidt, the projective limit space is a nuclear space. Notice that for a density $\rho \in \mathcal{S}$ the limit in (9) for the classical Lyapunov exponent turns out to be independent of ρ . To make the corresponding limit in the quantum characteristic exponent to be independent of Ψ , except for regularity conditions, we will need a norm for the functionals.

One now writes the *quantum characteristic exponent* as follows

$$\lambda_{v,x} = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left\| U^{-n} \tilde{Y} U^n v^i \partial_i \delta_x \right\|_{-p} \tag{20}$$

$\|\cdot\|_{-p}$ being the norm in \mathcal{H}_{-p}

$$\|\cdot\|_{-p} = \sup_{\xi \in \mathcal{E}, \|\xi\|_p \leq 1} |(\cdot, \xi)| \tag{21}$$

p must be greater than or equal to the order of the functional $U^{-n}\tilde{Y}U^n v^i \partial_i \delta_x$. Notice that U^n , being an element of the infinite-dimensional unitary group, has a natural action in E^* [6]. Let \mathcal{B} be the Borel σ -algebra generated by the cylinder sets of E^* and ν a probability measure on (E^*, \mathcal{B}) , invariant under the element U of the infinite-dimensional unitary group. Then, existence of the limit in Eq.(20) is a corollary of the following theorem, stated in a weaker form in[11]

Theorem 1.1

Let $\Phi \in \mathcal{H}_{-p}$, \mathcal{H}_p invariant under $U^{-k}\tilde{Y}U^n$ and ν a U -invariant probability measure on (E^*, \mathcal{B}) such that

$$G(\Phi) = \log \frac{\| \tilde{Y}U\Phi \|_{-p}}{\| \tilde{Y}\Phi \|_{-p}} \quad \text{and} \quad F_n(\Phi) = \frac{\| U^{-n}\tilde{Y}U^n\Phi \|_{-p}}{\| U^{-n+1}\tilde{Y}U^n\Phi \|_{-p}} \quad (22)$$

are $L^1(\nu)$ functions, with $F_n(\Phi)$ converging in $L^1(\nu)$. Then the limit

$$\lambda_\Phi = \lim_{n \rightarrow \infty} \frac{1}{n} \log \| U^{-n}\tilde{Y}U^n\Phi \|_{-p}$$

exists ν -almost everywhere in \mathcal{H}_{-p} .

The proof[22] uses Birkhoff's theorem and one of its corollaries[24]. The conditions on the functions in (22) impose restrictions on the quantum evolution operator U . Much weaker conditions are sufficient if one uses limsup in the definition of the characteristic exponent or a norm adapted to the dynamics[22].

The quantum characteristic exponent defined above may be explicitly computed in simple cases[13] and its physical meaning seems to be very close to the classical notion of Lyapunov exponent.

As follows from the preceding theorem, existence of a quantum characteristic exponent, analogous to the Lyapunov exponent, requires the existence of a U -invariant measure with support in the set $\{\Phi_x\} = \{\partial_i \delta_x\}$. When the quantum evolution is induced by a substitution operator as in one of the sectors of the four-dimensional Arnold's cat[13], the construction of the measure is the same as in lemma 1.1. In more general cases one may use a construction similar to the classical Bogolyubov-Krylov theorem. Let

$$C(\xi) = \int d\gamma(x) e^{i \langle \Phi_x, \xi \rangle} \quad (23)$$

where $\gamma(x)$ is an arbitrary normalized measure in the configuration space M such that

$$\int d\gamma(x) \langle \Phi_x, \xi \rangle < \infty \quad \forall \xi \in E$$

Because $|C(U^k \xi)| \leq 1 \forall k$, the sequence of partial sums

$$C_n(\xi) = \frac{1}{n} \sum_{k=0}^{n-1} C(U^k \xi)$$

has limit points and one defines

$$\overline{C}(\xi) = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} C(U^k \xi) \tag{24}$$

$\overline{C}(\xi)$ is continuous and

$$\overline{C}(U\xi) = \overline{C}(\xi)$$

Hence $\overline{C}(\xi)$ is the characteristic function of an U -invariant measure in E^* with support in the set $\{\Phi_x\}$.

Before moving to a discussion of quantum entropy let me point out that other definitions of characteristic exponents in infinite-dimensional spaces have been proposed in the past by several authors[25][18][26][27][28]. They characterize several aspects of the dynamics of linear and non-linear systems. The definition discussed in this paper, and that was proposed for the first time in [13], attempts however to be as close as possible to the spirit of the classical definition of Lyapunov exponent. The essential point is to measure, in a suitable norm, the time evolution of a singular perturbation of the initial condition and, keeping this in mind, the definition (20) may even be further simplified[22].

2 Entropy

The notion of entropy is used in several different senses in mathematics and physics. In the ergodic theory of dynamical systems, the metric (or Kolmogoroff-Sinai) entropy is an asymptotic characterization of the number of different orbits. Usually the Kolmogoroff-Sinai (K-S) entropy is defined by means of the entropy of a state space partition, refined by the time evolution. Carrying over this construction to quantum mechanics, the role of the partition is played by a finite-dimensional Abelian subalgebra of a von Neuman algebra or an Abelian sublattice in the lattice of projections[14][15][16][17]. However, the refinement of the quantum partition defined in this way has a nature different from the classical case. In the classical case the assignment of a trajectory, at a certain time, to an element of the phase-space partition does not change the orbit. Therefore the rate of refinement of the partition does not depend on the measurement process, only on the diversity of orbit behavior. In the quantum mechanical construction, the projections change the dynamical state, hence the construction may depend as much on the reduction features of the quantum measurement process as on the intrinsic dynamical complexity. This point is discussed in more detail elsewhere[18]. Notice also that the definition of entropy proposed in a recent paper by Słomczyński and Życzkowski[19] also involves in an essential way both a dynamical system and a measurement instrument.

As I have already pointed out[18] there is, in the classical case, a definition of entropy, alternative to the one of K-S, which may be generalized to quantum mechanics without involving the measurement process in the intermediate steps of the quantum evolution. This is the Brin-Katok local entropy[20] which, for a compact metric space K is equivalent to the Kolmogoroff-Sinai entropy.

Let $T : K \rightarrow K$ be continuous and preserving the measure μ . Define

$$B_\epsilon(T, n, x) = \{y \mid d(T^k(x), T^k(y)) \leq \epsilon, 0 \leq k \leq n\} \tag{25}$$

$$h_\epsilon^+(T, x) = \limsup_{n \rightarrow \infty} \left(-\frac{1}{n} \log \mu(B_\epsilon(T, n, x)) \right) \tag{26}$$

$$h_\epsilon^-(T, x) = \liminf_{n \rightarrow \infty} \left(-\frac{1}{n} \log \mu(B_\epsilon(T, n, x)) \right) \tag{27}$$

Then

$$\lim_{\epsilon \rightarrow 0} h_\epsilon^+(T, x) = \lim_{\epsilon \rightarrow 0} h_\epsilon^-(T, x) = h(T, x) \tag{28}$$

for μ -almost every x and $h(T, x)$ is integrable

$$h_\mu(T) = \int_K h(T, x) d\mu(x) \tag{29}$$

$h_\mu(T)$ coinciding with the Kolmogoroff-Sinai entropy.

I will use the Brin-Katok form to generalize the entropy for the quantum case. As for the characteristic exponent, the first step will be to express the Brin-Katok formulas in terms of densities. Consider a compact $K \subset R^d$ and the Euclidean distance. Then

$$B_\epsilon(T, n, x) = \left\{ y : \left| D_{(\delta_x - \delta_y)} \left(\int d\mu(z) z P^k \rho(z) \right) \right| \leq \epsilon, 0 \leq k \leq n \right\} \tag{30}$$

where linearity in ρ of the functional is used.

The measure to be considered is now a measure ν in S' with support on the $\{\delta_x\}$ set. It will be the measure induced from μ by the construction of lemma 1.1.

For the quantum case the ϵ -ball may be defined using a wave function

$$B_\epsilon(U, n, \delta_x, \Psi) = \left\{ \delta_y : \left| Re \langle U^k(\delta_x - \delta_y), \tilde{Y} U^k \Psi \rangle \right| \leq \epsilon, 0 \leq k \leq n \right\} \tag{31}$$

or using norms as in Eq.(21)

$$B_\epsilon(U, n, \delta_x) = \left\{ \delta_y : \left\| U^{-k} \tilde{Y} U^k (\delta_x - \delta_y) \right\|_{-p} \leq \epsilon, 0 \leq k \leq n \right\} \tag{32}$$

and we may now define the quantum local entropy as

$$h(U, x) = - \lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \log \nu(B_\epsilon(U, n, \delta_x)) \tag{33}$$

ν being a measure in S' invariant for the quantum evolution induced by the unitary operator U . Existence of the limit in Eq.(33) depends on the properties of the measure. I state a few conditions for the existence of the limit.

- (i) There is an p such that $U^{-k} \tilde{Y} U^k (\delta_x - \delta_y) \in \mathcal{H}_{-p}$

(ii) There is an U -invariant measure in \mathcal{S}' such that

$$F_{\epsilon,n}(\delta_x) = \log \frac{\nu(B_\epsilon(U, n, \delta_x))}{\nu(B_\epsilon(U, n-1, U\delta_x))}$$

is an integrable sequence of functionals converging to an integrable functional $F_\epsilon(\delta_x)$. Writing

$$\frac{1}{n} \log \nu(B_\epsilon(U, n, \delta_x)) = \frac{1}{n} \sum_{j=0}^{n-1} F_{\epsilon,n-j}(U^j \delta_x) + \frac{1}{n} \log \nu(B_\epsilon(U, 0, U^n \delta_x))$$

existence of $\lim_{n \rightarrow \infty} \frac{1}{n} \log \nu(B_\epsilon(U, n, \delta_x))$ follows from condition (ii) by the application of Birkhoff's theorem.

An intuitive derivation of the Pesin formula, relating the entropy and the sum over the positive Lyapunov exponents, follows from using the multipole expansion

$$\delta_y - \delta_x = \sum_{\{i_1 \dots i_n\}} \frac{1}{(\sum i_k)!} (y_1 - x_1)^{i_1} (y_2 - x_2)^{i_2} \dots (y_n - x_n)^{i_n} \partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n} \delta_x$$

which requires an extension of the mathematical setting to ultradistribution spaces[21]. The conclusion is that a theory for the quantum entropy and a quantum analogue of the Pesin formula may be established in this framework. What is needed however is to relate the properties of the quantum evolution operators to the nature of the measures in \mathcal{S}' . For this I refer to work in progress[22].

3 Conclusions

From the rigorous results in Section 1 concerning the quantum characteristic exponents and the proposed definition of a quantum entropy in Section 2, I conclude that it seems possible to develop an ergodic theory for quantum mechanics that closely resembles the corresponding construction in classical mechanics. This might provide a rigorous framework to define quantum chaos. Furthermore, as explained elsewhere[13], this framework may also provide an explanation for the puzzling time reversibility in numerical simulations of quantum systems that are classically chaotic.

The Lyapunov exponent and the entropy play an important role in characterizing chaotic classical dynamical systems. However, except for the simplest systems, it is in general not possible to compute these numbers analytically. Therefore the existence of accurate numerical algorithms for the ergodic invariants plays an important role in the study of such systems. If the quantum characteristic exponents and the quantum entropy are to play a similar role in the study of quantum systems, developing adequate numerical algorithms will also be an important requirement[23].

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