Chapter 6

The Stability of Physical Theories Principle

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1. Introduction: Physical Models and Structural Stability

When models are constructed for the natural world, it is reasonable to expect that only those properties of the models that are robust have a chance to be observed. Models or theories being approximations to the natural world, it is unlikely that properties that are too sensitive to small changes (that is, that depend in a critical manner on particular values of the parameters) will be well described in the model. If a fine tuning of the parameters is needed to reproduce some natural phenomenon, then the model is basically unsound and its other predictions expected to be unreliable. For this reason a good methodological point of view, in the construction of physical theories, consists in focusing on the robust properties of the models or, equivalently, to consider only models which are stable, in the sense that they do not change, in a qualitative manner, when some parameter changes. This is what will be called the *stability of physical theories principle* (SPTP).

The stable-model point of view had a large impact in the field of nonlinear dynamics, where it led to the rigorous notion of *structural stability* [1, 2]. As already pointed out by Flato [3] and Faddeev [4] the same

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pattern seems to occur in the fundamental theories of Nature. In fact, the two physical revolutions of the last century, namely the passage from non-relativistic to relativistic and from classical to quantum mechanics, may be interpreted as transitions from two unstable theories to two stable ones.

Because a theory is a mathematical model for the natural world, stability of a theory is stability of its mathematical structure. A mathematical structure is said to be *stable* (or *rigid*) for a class of *deformations,* if any deformation in this class leads to an equivalent (isomorphic) structure. The idea of stability of the structures provides a guiding principle to test either the validity or the need for generalization of a physical theory. Namely, if the mathematical structure of a given theory is not stable, one should try to deform it until one falls into a stable one, which has a good chance of being a generalization of wider validity.

When a mathematical structure is deformed, the deformation depends on a certain number of parameters. Typically, if one starts from an unstable theory T_{α_0}, that corresponds to a particular value α_0 of the parameter α, α_0 will be an isolated point, in the sense that for any other value α of the parameter in a neighborhood of α_0, the theory T_α is not equivalent to T_{α_0}. Conversely a stable theory would be one for which α_0 has a neighborhood of theories all of them equivalent to T_{α_0}. Therefore when one deforms an unstable theory and falls into a stable one, the exact value of the deformation parameter that corresponds to the actual physical theory cannot be obtained from deformation theory because, from this point of view, all values for which the theory is stable are equivalent. The deformation parameters are therefore the natural fundamental constants that have to be obtained from experiment. In this sense *deformation theory not only is the theory of stable theories, it is also the theory that identifies the fundamental constants.*

The construction of physical theories operates at several distinct structural levels and, at each level, distinct mathematical structures are involved. Therefore the application of the ideas of stability and deformation to the distinct structural levels requires a precise formulation of deformation theory in several mathematical disciplines. Analyzing the existing physical theories one identifies an hierarchy of structural levels. In the first, which one may call the *logical level*, are the basic hypothesis about what is observable and what is not, what kind of questions can be settled by experiment and how these questions are interrelated. At this level it is that one finds the distinction between classical and quantum physics. In the literature dedicated to the foundations of science one finds, at times, some confusion concerning what distinguishes classical from quantum mechanics. For example, one finds the statement that classical mechanics is deterministic whereas non-determinism is the hallmark of quantum physics. In fact quantum mechanics is as deterministic as classical mechanics, in the sense that the Schrödinger equation is as deterministic as Hamilton's equations. Determinism is a property of the equations that define the time evolution and therefore it is a dynamical question, not a question concerning the logical structure of the theory. What happens in quantum theory is that, as in any logical structure, there are questions that can be raised and questions that cannot. As Feshbach and Weisskopf [5] said: "*If you make a silly question, you obtain a silly answer*".

At the second level, which may be called *the kinematical level*, one defines what are the observable quantities (*the observables*) and what are the relations between them. At this level one also defines what are the mathematical quantities that in the theory correspond to each one of the experimental apparatus. Finally, in the third level, called *the dynamics*, one includes all the hypothesis relating to time evolution of the physical systems and their interactions. The three levels of the theoretical structure define an hierarchy of hypothesis. Hence, with one *logic* several *kinematics* may be used and many different *dynamics* may be associated to each kinematics. The hypothesis of the theory include a certain number of manipulation rules which are needed to predict the results that are to be expected from the experiments. These results (in general numbers) are then compared with the corresponding results obtained in the experiments. This comparison establishes the agreement or disagreement between the theoretical predictions and the experimental results. Notice that it is only at this stage that the theory (a mathematical entity) establishes its contact with the physical world. In particular it is not essential, and sometimes not even desirable, for all the entities in the model to have a direct physical interpretation. The "external" physical world may contain many variables to which we have no direct access, or that we do not care about, when we restrict ourselves to a certain set of experiments and apparatus. Likewise the mathematical model may have parameters and internal entities which have no direct relation to external observable quantities. The only criterion of validity of the theory is the agreement of its output (that is, the measurable predictions) with the experimentally observed quantities. It is only at this level that the theory, a mathematical entity, comes into contact with what is called "reality", whatever it means. One should also bear in mind the nature of

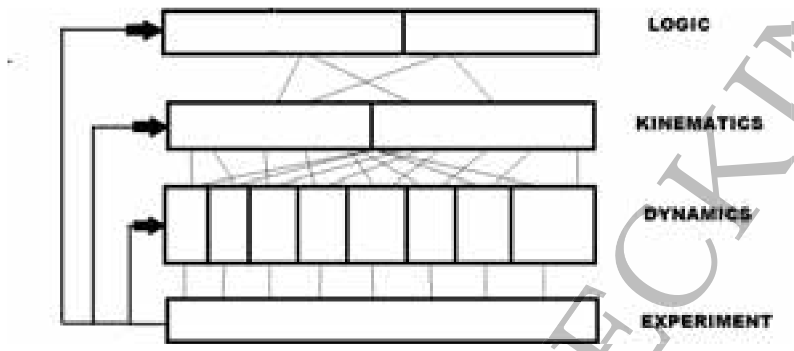

Fig. 1. The hierarchy of hypothesis in the construction of physical theories.

this precarious contact and never be misled into confusing the model with the object that is being modelled.

As suggested in the Fig. 1 above,[a] the evolution of the theoretical models operates by loops, with the signal of the theory-experiment comparison being fed back into the model, leading to changes in the dynamics which lead to new predictions, which are compared once more, etc. If after a number of such steps a reasonable agreement is not obtained, one may be led to broaden the scope of the feedback loop, that is, one might be led to change the kinematical or even the logical structure of the theory. The scientific revolutions that led from Galilean to Lorentzian mechanics and from classical to quantum mechanics are examples of a change of the kinematics and a change of the logics.

The separation between theoretical construction and experimental verification is however not so clear-cut as one might be led to believe from the discussion above. The experimental results, which serve as a control for the theoretical framework, are never pure empirical data in the sense that, when experiments are designed to test a theoretical model, they are themselves contaminated by the prejudices of the theory. The following remark by Misner, Thorne and Wheeler [6] is particularly relevant:

"*All the laws and theories of physics have this deep and subtle character, that they both define the concepts they use and make statements about these concepts. Contrariwise, the absence of some body of theory, law and principle deprives one of the means properly to define or even use concepts.*

[a]Adapted from unpublished lecture notes of Prof. F. A. Matsen at University of Texas (Austin).

Any forward step in human knowledge is truly creative in this sense: that theory, concept, law and method of measurement — forever inseparable — are born into the world in union"

The structuring effect of the theory is an important instrument in the interpretation of the experimental data. On the other hand, prejudices are thereby introduced in the analysis which may lead to neglecting some information contained in data for which there is as yet no theoretical interpretation.

Concerning the SPTP which is the main concern in this paper, one sees that to be able to discuss stability issues at all levels of the theoretical construction one has to identify the nature of the mathematical framework that is relevant at each one of the levels. For the structural stability of nonlinear dynamics the needed mathematical framework is the theory of stable vector fields and differentiable maps. To discuss stability of the kinematical level one notices that after the definition of a certain number of observables, the structure of kinematics is the structure of the algebra of these observables. For the logical level because logical questions may sometimes also be framed in an algebraic setting the mathematical framework is also an algebraic one. Notice however that to frame the logical issues in algebraic form some choice of observables is in general needed and the discussion of stability is no longer a purely logical question. It would be more appropriate to consider the lattice of propositions and discuss the stability issue in the framework of lattice theory. However, as far as I know, there is not yet a well developed deformation theory for lattices. Therefore, for the time being, it seems appropriate to discuss the stability issues both for the kinematical and the logical levels using algebraic tools.

The fact that semisimple algebras are deformation-stable, led Segal [7] to propose in 1951 that, in its evolution, physical theories would tend to be framed in terms of such algebras. However the stability principle is more general than the *simplicity criterion* because not all stable algebras are semisimple [8] and, for example, dynamical stability issues are not necessarily algebraic. Nevertheless the *algebraic simplicity* principle is a powerful one, which led to interesting developments (see Finkelstein and collaborators [9–13]).

Section 2 contains a short review of the stabilizing deformations that lead from Galilean to relativistic dynamics and from classical to quantum mechanics. Also discussed is the finite versus infinite dimensional issue when dealing with algebraic deformation questions.

Section 3 examines the stability of the algebra that is obtained by combining the algebras of relativistic and quantum mechanics, that is, the Heisenberg–Poincaré algebra. One finds that the combined algebra of relativistic quantum mechanics is not stable and its stabilization by a deformation forces the introduction of two length parameters, one of which will probably have the status of a new fundamental constant. In the new deformed algebra the space-time coordinates no longer commute and, at the scale where the effects of a non-zero fundamental length may be felt, the geometry of space-time is necessarily a non-commutative geometry. The consequences of this non-commutativity of the space-time coordinates, their geometric aspects and experimental tests have been discussed in several publications. The main results are summarized and some new consequences are explored.

Section 4 describes structural stability of maps, its use in nonlinear dynamics as well as the possible relevance to universality and critical phenomena.

Finally, Appendix A is a review of structural stability in dynamical systems theory, which is the field where the importance of stable theories was first emphasized and Appendix B contains a summary of results on deformation theory of algebras. The mathematical results contained in these appendices, which are spread over many texts, are included here to provide a first working knowledge on deformation tools for the reader interested in pursuing stability explorations in his domain.

2. From Galilean to Relativistic Dynamics and From Classical to Quantum Mechanics

Within the deformation theory of algebras, the transitions from Galilean to relativistic and from classical to quantum mechanics may be interpreted as the stabilizing deformations of two unstable theories.

The Lie algebra of the homogeneous Galilean group, the kinematical group of non-relativistic mechanics, is:

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \tag{1}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k, \tag{2}$$

$$[K_i, K_j] = 0 \tag{3}$$

the angular momenta J_i being the generators of rotations and the boosts K_i the generators of velocity transformations. The second cohomology group

(Appendix B) does not vanish because, for example, $\phi_1(K_i, K_j) = i\epsilon_{ijk}J_k$ and $\phi_1 = 0$ for all other arguments, is a 2-cocycle that is not a 2-coboundary. The deformation

$$[K_i, K_j] = -i\frac{1}{c^2}\epsilon_{ijk}J_k, \tag{4}$$

leads to the Lorentz algebra which, being semisimple, has vanishing second cohomology group and is stable. The deformation parameter $\frac{1}{c}$ (the inverse of the speed of light) is a fundamental constant.

For the deformation leading from classical to quantum mechanics, recall that the phase space of classical mechanics is a symplectic manifold $W = (T^*M, \omega)$ where T^*M is the cotangent bundle over configuration space M and ω is a symplectic form. In local (Darboux) coordinates $\{p_i, q_i\}$ the symplectic form is

$$d\omega = \sum dp_i \wedge dq_i$$

The Poisson bracket gives a Lie algebra structure to the C^∞-functions on W, namely

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \tag{5}$$

in local coordinates.

The transition to quantum mechanics is now regarded as a deformation of this Poisson algebra [14]. Let for example $T^*M = R^{2n}$. Then $\omega = \sum_{1\leq i,j\leq 2n} \omega_{ij} dx^i \wedge dx^j = \sum_{1\leq i\leq n} dx^i \wedge dx^{i+n}$.

Consider the following bidifferential operator

$$P^r(f, g) = \sum_{i_1\ldots i_r, j_1\ldots j_r} \omega^{i_1j_1}\ldots\omega^{i_rj_r}\partial_{i_1}\ldots\partial_{i_r} f\, \partial_{j_1}\ldots\partial_{j_r} g, \tag{6}$$

where $P^1(f, g)$ is the Poisson bracket. $P^3(f, g)$ is a non-trivial 2-cocycle and, barring obstructions, one expects the existence of non-trivial deformations of the Poisson algebra.

Existence of non-trivial deformations has been proved in a very general context [15–18]. They always exist if W is finite-dimensional and for a flat Poisson manifold they are all equivalent to the Moyal bracket [19]

$$[f, g]_M = \frac{2}{\hbar} sin\left(\frac{\hbar}{2}P\right)(f, g) = \{f, g\} - \frac{\hbar}{4.3!}P^3(f, g) + \cdots. \tag{7}$$

Moreover $[f,g]_M = \frac{1}{i\hbar}(f *_\hbar g - g *_\hbar f)$ where $f *_\hbar g$ is an associative star-product

$$f *_\hbar g = \exp\left(i\frac{\hbar}{2}P(f,g)\right). \tag{8}$$

Correspondence with quantum mechanics formulated in Hilbert space is obtained by the Weyl quantization prescription. Let $f(p,q)$ be a function in phase space and $\widetilde{f}$ its Fourier transform. Then, if to the function f we associate the Hilbert space operator

$$\Omega(f) = \int \widetilde{f}(x_i, y_i) \exp\left(-i\frac{\sum x_i Q_i + y_i P_i}{\hbar}\right) dx_i dy_i$$

with $Q_i\Psi = x_i\Psi$ and $P_i\Psi = -i\hbar\frac{\partial}{\partial x_i}\Psi$, one finds

$$[\Omega(f), \Omega(g)] = -i\hbar\Omega([f,g]_M).$$

In the left-hand side is the usual commutator of Hilbert space operators. Therefore quantum mechanics may be described either by associating self-adjoint operators in Hilbert space to the observables or, equivalently, by staying in the classical setting of phase-space functions but deforming their product to a $*_\hbar$product and the Poisson bracket to the Moyal bracket.

The quantization-by-deformation program initiated in [14] was later on considerably extended to general Poisson manifolds which are not necessarily sympletic manifolds [20–22]. One of the main results states that there is a canonical correspondence between deformations of an algebra A of C^∞ functions on a Poisson manifold M and formal Poisson structures $(\pi_t = t\pi_1 + t^2\pi_2 + \cdots)$ on A [23]. Furthermore an explicit deformation formula is provided for $M = \mathbb{R}^n$ and the product of the deformed algebra is a *star product*, that is, in $* = \Sigma t^n B_n$ the B_n's are bidifferential operators.

There is a basic difference in the deformations leading from non-relativistic to relativistic and from classical to quantum mechanics. In the first case one deals with the deformation of a finite-dimensional algebra and, in the second, with the more complex case of deformation of an infinite-dimensional algebra of functions. With the benefit of hindsight one may simplify the presentation by using for classical mechanics, instead of the Poisson algebra in phase space, a formulation in Hilbert space. Then the transition appears in both cases as a deformation of a finite-dimensional Lie algebra. This not only simplifies the presentation but is the appropriate setting for further analysis of the stability of relativistic quantum mechanics.

A description of classical mechanics by operators in Hilbert space was proposed, soon after the discovery of quantum mechanics, by Koopman [24] and von Neumann [25]. A constant energy surface Ω_E in the phase space of N particles carries an invariant measure μ_E, which is the restriction of the Liouville measure $d^{3N}xd^{3N}p$ to Ω_E. In the space of square-integrable functions $L^2(\Omega_E, \mu_E)$, the Hamiltonian flow T_t induces an unitary operator by

$$(U_t f)(w) = f(T_t w), \tag{9}$$

where $w \in \Omega_E$ and $f \in L^2(\Omega_E, \mu_E)$. Unitarity is a consequence of the invariance of the measure, that is $\mu(T_t^{-1}F) = \mu(F)$ for a measurable set $F \in \Omega_E$.

In the Hilbert space $L^2(\Omega_E, \mu_E)$, classical mechanics has an operator formulation. The time evolution is induced by the unitary operator U_t as in quantum mechanics and the observables are smooth functions on Ω_E, which act as multiplicative operators in $L^2(\Omega_E, \mu_E)$.

Considered as multiplicative operators in Hilbert space, the functions of coordinates and momenta are an infinite-dimensional abelian algebra. However, in the Hilbert space formulation we need not consider explicitly the infinite-dimensional algebra because the full content of the theory is obtained by selecting a finite set of paired observables (p_i, x_i) or $(p_i, y_i = e^{ix_i})$ and defining its transformation properties under U_t and its algebraic properties which, in classical mechanics, are

$$[p_i, x_j] = [p_i, p_j] = [x_i, x_j] = [p_i, y_j] = 0. \tag{10}$$

The transition to quantum mechanics is now done by the replacement of this Abelian algebra by the Heisenberg algebra

$$[p_i, p_j] = [x_i, x_j] = 0, \tag{11}$$

$$[x_i, p_j] = i\hbar\Im\delta_{ij}, \tag{12}$$

$\Im$ is the identity operator, a trivial center of the algebra of observables. The infinite-dimensional Moyal algebra is therefore replaced by the simpler finite-dimensional Heisenberg algebra. The role of the Heisenberg algebra, in the context of deformation theory, has however to be discussed carefully. Consider the one-dimensional case of a classical abelian algebra $[x, p] = 0$. This abelian algebra is clearly not stable and in its neighborhood there is the algebra

$$[x, p] = i\epsilon x \tag{13}$$

or the Heisenberg algebra

$$[x, p] = i\hbar\Im \tag{14}$$

which is the central extension of the abelian algebra. The algebra (13) is a stable algebra. Indeed the only stable algebra in two dimensions is isomorphic to [26]

$$[Y, X_1] = X_1 \tag{15}$$

but the Heisenberg algebra itself is not stable.

There are two ways of looking at the instability of the Heisenberg algebra. First if we consider it as a tridimensional algebra, $[X_2, X_3] = X_1$ (all the other commutators being zero), the complete structure of its neighborhood, in the space of Lie algebra laws, is known [27]. Namely, the Heisenberg algebra is a contraction of any algebra of the same dimension that carries a linear contact form. Conversely any perturbation of the Heisenberg algebra supports a linear contact form. For example from the Lie algebra of $SO(3)$

$$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2$$

which is semisimple and therefore stable, with the following linear change of coordinates

$$Y_1 = \epsilon X_1,\ Y_2 = \sqrt{\epsilon} X_2,\ Y_3 = \sqrt{\epsilon} X_3,$$

one obtains

$$[Y_1, Y_2] = \epsilon Y_3,\ [Y_2, Y_3] = Y_1,\ [Y_3, Y_1] = \epsilon Y_2$$

and in the $\epsilon \to 0$ limit one obtains the Heisenberg algebra.

We could also have considered the Heisenberg algebra as a two-dimensional algebra with a trivial center. That is, we restrict our deformations to those that preserve the zero commutator of X_1 with the other two elements. Consider in this case the deformation

$$[X_2, X_3] = X_1 + \alpha X_2 + \beta X_3.$$

With the linear change of variables

$$Y_2 = \alpha X_2 + X_1 + \beta X_3,\ Y_3 = \alpha^{-1} X_3,$$

we now fall on the stable two-dimensional algebra (15), $[Y_2, Y_3] = Y_2$.

We conclude in both cases that the Heisenberg algebra is unstable and has a stable algebra in its neighborhood. Therefore it would seem, at first sight, that the Hilbert space construction leads to conclusions different from the phase space construction described before, which interprets the transition from classical to quantum mechanics as a deformation from an unstable Poisson algebra to the stable Moyal-Vey algebra. A simple reasoning shows however that this is not the case and that the constructions are indeed equivalent and they are both the transition from an unstable classical algebra to a stable quantum algebra. The apparent difference is merely an artifact of the singling out of x as the observable, when in fact the observables are all the smooth functions of x (and p). Consider the explicit representation

$$p = \frac{\hbar}{i}\frac{d}{dx}, \; x = x.$$

The physical content of the theory will be the same if instead of the coordinate x we consider any linear or nonlinear function of x. In particular considering $y = \exp(ix)$ one obtains the algebra

$$[p, y] = \hbar y$$

which is isomorphic to the stable two-dimensional algebra (15). Hence the Heisenberg algebra is equivalent, through a nonlinear coordinate transformation that preserves the physical content, to a stable algebra. In this sense the transition from classical to quantum mechanics is again seen to be a stabilizing deformation of an unstable algebra. The main reason why the coordinate choice leading to the Heisenberg algebra is physically convenient is that the observable p has then a simple interpretation as the generator of translations in x. This example also shows that, when selecting a finite subset of observables rather than an infinite-dimensional space of functions, the notion of linear equivalence of algebras, in the sense of (81), is not sufficient for the stability analysis and one should also consider nonlinear transformations preserving the physical content of the theory.

In both the Galilean and the Poisson algebra cases, the deformed algebras are all equivalent for non-zero values of $\frac{1}{c^2}$ and of $\hbar$. This means that although we could have derived relativistic and quantum mechanics purely from the stability of their algebras, the exact values of the deformation parameters cannot be obtained from algebraic considerations. The *deformation parameters* are therefore the natural *fundamental constants* to be obtained from experiment. It is in this sense that deformation theory not only is the theory of stable theories, it also is the theory that identifies the fundamental constants.

3. Stabilizing the Heisenberg–Poincaré algebra

In Section 2 both the transition from Galilean to Lorentzian and the transition from classical to quantum mechanics are cast as deformations of finite-dimensional Lie algebras of operators in Hilbert space. A trivial point in this construction, which however has non-trivial consequences, is the fact that, to have both constructions in a finite-dimensional algebra setting, it is essential to include the coordinates as basic operators in the defining (kinematical) algebra of relativistic quantum mechanics. The full algebra of relativistic quantum mechanics will contain the Lorentz algebra (1, 2, 4), the Heisenberg algebra for the momenta and space-time coordinates (P_μ, x_ν) in Minkowski space and also the commutators that define the vector nature (under the Lorentz group) of P_μ and x_ν. Defining

$$M_{ij} = \epsilon_{ijk} J_k,\ M_{0i} = K_i$$

and measuring velocities and actions in units of c and $\hbar$ (that is $c = \hbar = 1$) one obtains

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \tag{16}$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu\eta_{\nu\lambda} - P_\nu\eta_{\mu\lambda}), \tag{17}$$

$$[M_{\mu\nu}, x_\lambda] = i(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}), \tag{18}$$

$$[P_\mu, P_\nu] = 0, \tag{19}$$

$$[x_\mu, x_\nu] = 0, \tag{20}$$

$$[P_\mu, x_\nu] = i\eta_{\mu\nu}\Im \tag{21}$$

with $\eta_{\mu\nu} = (1, -1, -1, -1)$. This algebra, the Heisenberg–Poincaré algebra, is the algebra of relativistic quantum mechanics $\Re_0 = \{M_{\mu\nu}, P_\mu, x_\mu, \Im\}$.

We know that the Lorentz algebra, $\{M_{\mu\nu}\}$, being semi-simple, is stable and that each one of the two-dimensional Heisenberg algebras $\{P_\mu, x_\mu\}$ is also stable in the nonlinear sense discussed in Section 2. When the algebras are combined through the covariance commutators (17–18), the natural question to ask is whether the whole algebra is stable or whether there are any non-trivial deformations.

The answer is that the algebra $\Re_0 = \{M_{\mu\nu}, P_\mu, x_\mu, \Im\}$ defined by Eqs. (16–21) is not stable [28]. This is shown by exhibiting a two-parameter deformation of $\Re_0$ to a simple algebra which itself is stable. To understand the role of the deformation parameters consider first the Poincaré subalgebra $P = \{M_{\mu\nu}, P_\mu\}$. It is well known that already this subalgebra is not stable and may be deformed [3] [29] to the stable simple algebras of the De Sitter groups $O(4,1)$ or $O(3,2)$. Writing

$$P_\mu = \frac{1}{R} M_{\mu 4} \tag{22}$$

the commutation relations $[M_{\mu\nu}, M_{\rho\sigma}]$ and $[M_{\mu\nu}, P_\lambda]$ are the same as before, that is (16–17), and $[P_\mu, P_\nu]$ becomes

$$[P_\mu, P_\nu] = -i\frac{\epsilon_4}{R^2} M_{\mu\nu}. \tag{23}$$

Equations (16), (17), and (23), all together, are the algebra

$$[M_{ab}, M_{cd}] = i(-M_{bd}\eta_{ac} - M_{ac}\eta_{bd} + M_{bc}\eta_{ad} + M_{ad}\eta_{bc}) \tag{24}$$

of the five-dimensional pseudo-orthogonal group with metric

$$\eta_{aa} = (1, -1, -1, -1, \epsilon_4),\ \epsilon_4 = \pm 1.$$

That is, the Poincaré algebra deforms to the stable algebras of $O(3,2)$ or $O(4,1)$, according to the sign of ϵ_4.

This instability of the Poincaré algebra is well understood. It simply means that flat space is an isolated point in the set of arbitrarily curved spaces. Faddeev [4] points out that the Einstein theory of gravity may also be considered as a deformation in a stable direction. This theory is based on curved pseudo Riemann manifolds. Therefore, in the set of Riemann spaces, Minkowski space is a kind of degeneracy whereas a generic Riemann manifold is stable in the sense that in its neighborhood all spaces are curved. However, as long as the Poincaré group is used as the kinematical group of the tangent space to the space-time manifold, and not as a group of motions in the manifold itself, it is perfectly consistent to take $R \to \infty$ and this deformation would be removed.

For the full algebra $\Re_0 = \{M_{\mu\nu}, P_\mu, x_\mu, \Im\}$ the situation is more interesting. In this case the stabilizing deformation [28] is obtained by setting

$$P_\mu = \frac{1}{R} M_{\mu 4}, \tag{25}$$

$$x_\mu = \ell M_{\mu 5}, \tag{26}$$

$$\Im = \frac{\ell}{R} M_{45}, \tag{27}$$

to obtain

$$[P_\mu, P_\nu] = -i\frac{\epsilon_4}{R^2}M_{\mu\nu}, \tag{28}$$

$$[x_\mu, x_\nu] = -i\epsilon_5\ell^2 M_{\mu\nu}, \tag{29}$$

$$[P_\mu, x_\nu] = i\eta_{\mu\nu}\Im, \tag{30}$$

$$[P_\mu, \Im] = -i\frac{\epsilon_4}{R^2}x_\mu, \tag{31}$$

$$[x_\mu, \Im] = i\epsilon_5\ell^2 P_\mu \tag{32}$$

with $[M_{\mu\nu}, M_{\rho\sigma}]$, $[M_{\mu\nu}, P_\lambda]$ and $[M_{\mu\nu}, x_\lambda]$ being the same as before.

The stable algebra $\Re_{\ell,R}$ to which $\Re_0$ has been deformed is the algebra of the six-dimensional pseudo-orthogonal group with metric

$$\eta_{aa} = (1, -1, -1, -1, \epsilon_4, \epsilon_5),\ \epsilon_4, \epsilon_5 = \pm 1.$$

In addition to the signs ϵ_4 and ϵ_5, two deformation parameters, R and ℓ, with dimensions of length, characterize this stabilizing deformation. R, associated to the non-commutativity of the generators of translations, must be related to the local curvature. Therefore, because the curvature is not a constant, R cannot have the status of a fundamental constant. However, the other constant ℓ might be a *fundamental length*, a new fundamental physical constant.

As in the case of the Poincaré algebra discussed above, if one is mostly concerned with the algebra of observables in the tangent space, one may take the limit $R \to \infty$ and finally obtain

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} - M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \tag{33}$$

$$[M_{\mu\nu}, P_\lambda] = i(P_\mu\eta_{\nu\lambda} - P_\nu\eta_{\mu\lambda}), \tag{34}$$

$$[M_{\mu\nu}, x_\lambda] = i(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}), \tag{35}$$

$$[P_\mu, P_\nu] = 0, \tag{36}$$

$$[x_\mu, x_\nu] = -i\epsilon_5\ell^2 M_{\mu\nu}, \tag{37}$$

$$[P_\mu, x_\nu] = i\eta_{\mu\nu}\Im, \tag{38}$$

$$[P_\mu, \Im] = 0, \tag{39}$$

$$[x_\mu, \Im] = i\epsilon_5\ell^2 P_\mu, \tag{40}$$

$$[M_{\mu\nu}, \Im] = 0, \tag{41}$$

as the *stable algebra of relativistic quantum mechanics*. The main features are the non-commutativity of the x_μ coordinates and the fact that $\Im$, previously a trivial center of the Heisenberg algebra, becomes now a non-trivial operator. These are however the minimal changes that seem to be required if stability of the algebra of observables (in the tangent space) is a good guiding principle. Two constants define this deformation. One is ℓ, the *fundamental length*, the other the sign of ϵ_5. The "tangent space" algebra (33–41) is the kinematical algebra appropriate for microphysics. However, for physics in the large, it should be the full stable algebra (16–18, 28–32) to play a role. In the last part of this section, I will discuss two important roles that the non-vanishing of $\frac{1}{R}$ may play for the physical construction. However, for the most part, the emphasis here will be in the tangent space limit $R \to \infty$.

The stabilization of the Heisenberg–Poincaré algebra has been further studied and extended in [30–32]. The idea of modifying the algebra of the space-time components x_μ in such a way that they become non-commuting operators had already appeared several times in the physical literature. Rather than being motivated (or forced) by stability considerations, the aim of those proposals was to endow space-time with a discrete structure, to be able, for example, to construct quantum fields free of ultraviolet divergences. Sometimes they simply postulated a non-zero commutator, others they were guided by the formulation of field theory in curved spaces. Although the algebra arrived at in [28], Eqs. (33–41), is so simple and appears in such a natural way in the context of deformation theory, it seems that, strangely, it differed in some way or another from the past proposals. In one scheme, for example, the coordinates were assumed to be the generators of rotations in a five-dimensional space with constant negative curvature. This possibility was proposed long ago by Snyder [33, 34] and the consequences of formulating field theories in such spaces have been extensively studied by Kadishevsky and collaborators [35, 36]. The coordinate commutation relations $[x_\mu, x_\nu]$ are identical to (37), however, because of the representation chosen for the momentum operators, the Heisenberg algebra is different and, in particular, $[P_\mu, x_\nu]$ has non-diagonal terms. Banai [37] also proposed a specific non-zero commutator which only operates between time and space coordinates, breaking Lorentz invariance. Many other discussions exist concerning the emergence and the role of discrete or quantum space-time, which however, in general, do not specify a complete operator algebra [38–51].

Notice that there other ways to deform the algebra $\Re_0$ to the simple algebra of the pseudo-orthogonal group in six dimensions. They correspond to different physical identifications of the generators $M_{\mu 4}, M_{\mu 5}$, and M_{45}.

For example, putting

$$P_\mu = \frac{1}{R'}(M_{\mu 4} + M_{\mu 5}), \tag{42}$$

$$x_\mu = \frac{\ell'}{2}(M_{\mu 4} - M_{\mu 5}), \tag{43}$$

$$\Im = \frac{\ell'}{R'} M_{45} \tag{44}$$

and $\epsilon_4 = -\epsilon_5 = 1$, the coordinates and momenta are now commuting variables and the changes occur only in the Heisenberg algebra and the nature of $\Im$, namely

$$[P_\mu, x_\nu] = i\left(\frac{\ell'}{R'} M_{\mu\nu} + \eta_{\mu\nu}\Im\right), \tag{45}$$

$$[P_\mu, \Im] = -i\frac{\ell'}{R'} P_\mu, \tag{46}$$

$$[x_\mu, \Im] = i\frac{\ell'}{R'} x_\mu. \tag{47}$$

However this identification of the physical observables in the deformed algebra does not seem so natural as the previous one. In particular Eq. (45) implies a radical departure from the Heisenberg algebra and the fundamental length scale is tied up to the large scale of the manifold curvature radius, in the sense that, if we take $R' \to \infty$, the whole deformation vanishes.

The $\Re_{\ell,\infty}$ algebra (33–41) has a simple representation by differential operators in a five-dimensional space with coordinates $(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4)$

$$P_\mu = i\frac{\partial}{\partial \xi^\mu} + iD_{P_\mu}, \tag{48}$$

$$M_{\mu\nu} = i\left(\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}\right) + \Sigma_{\mu\nu}, \tag{49}$$

$$x_\mu = \xi_\mu + i\ell\left(\xi_\mu \frac{\partial}{\partial \xi^4} - \epsilon_5 \xi^4 \frac{\partial}{\partial \xi^\mu}\right) + \ell\Sigma_{\mu 4}, \tag{50}$$

$$\Im = 1 + i\ell\frac{\partial}{\partial \xi^4} + i\ell D_{\xi^4}. \tag{51}$$

The set $(\Sigma_{\mu\nu}, \Sigma_{\mu 4})$ is an internal spin operator for the groups $O(4,1)$ (if $\epsilon_5 = -1$) or $O(3,2)$ (if $\epsilon_5 = +1$) and D_{P_μ} and D_{ξ^4} are derivations operating in the space where $(\Sigma_{\mu\nu}, \Sigma_{\mu 4})$ acts. In this representation the deformation has a simple interpretation. The space-time coordinates x_μ, in addition to the usual (continuous spectrum) component have a small angular momentum component corresponding to a rotation (or hyperbolic rotation) in the extra dimension. And the center of the Heisenberg algebra picks up a small momentum in the extra dimension.

The algebra (33–41) is seen to be the algebra of the pseudo-Euclidean groups $E(1,4)$ or $E(2,3)$, depending on whether ϵ_5 is -1 or $+1$. For the construction of quantum fields it might be convenient to use this representation. Notice however that only the Poincaré part of $E(1,4)$ or $E(2,3)$ corresponds to symmetry operations and only this part has to be implemented by unitary operators.

Physical consequences of the non-commutative space-time structure implied by the $\Re_{\ell,\infty}$ algebra have been explored in a series of publications [52–57]. Depending on the sign of ϵ_5 the time ($\epsilon_5 = +1$) or one space variable ($\epsilon_5 = -1$) will have discrete spectrum. In any case ℓ, a new fundamental constant, sets a natural scale for time and length. If ℓ is of the order of Planck's length, observation of most of the effects worked out in the cited references will be beyond present experimental capabilities. However, if ℓ is much larger than Planck´s length (for example, of order 10^{-27}–10^{-26} seconds) the effects might already be observable in the laboratory or in astrophysical observations. I refer the reader to the references above for a detailed analysis of the experimental predictions and just add here a few remarks. Some of the most noteworthy effects arise from the modification of the phase space volume and from interference effects. In addition, the simple fact that the space-time coordinates do not commute already implies that many notions currently used in the analysis of laboratory experiments become ill-defined. For example, because the space and the time coordinates cannot be simultaneously diagonalized, speed can only be defined in terms of expectation values,

$$v_\psi^i = \frac{1}{\langle \psi_t, \psi_t \rangle} \frac{d}{dt} \left\langle \psi_t, x^i \psi_t \right\rangle, \tag{52}$$

ψ being a state with a small dispersion of momentum around a central value p. This would imply a deviation from $c\,(=1)$ of the "effective speed" of massless particles of order [55]

$$\Delta v_\psi = -3\epsilon_5 \ell^2 \left(p^0\right)^2. \tag{53}$$

The deviation would be negative for $\epsilon_5 = +1$ (ℓ a fundamental time) or positive for $\epsilon_5 = -1$ (ℓ a fundamental length). In any case such deviation should not be confused with a modification of the value of the fundamental constant c.

Most of the consequences worked out in the references [52–56] are rather conservative, in the sense that they simply explore the non-vanishing of the right-hand side of the commutators of previously commuting variables. Deeper consequences are to be expected from the radical change from a commutative to a non-commutative space-time geometry. The new geometry was studied in [58].

For this non-commutative geometry, the differential algebra may be defined either by duality from the derivations of the algebra or from the triple $(H, \pi(U_{\Re}), D)$, where $U_{\Re}$ is the enveloping algebra of $\Re_{\ell,\infty}$, to which a unit and, for later convenience, the inverse of $\Im$, are added.

$$U_{\Re} = \{x_\mu, M_{\mu\nu}, p_\mu, \Im, \Im^{-1}, 1\}, \tag{54}$$

$\pi(U_{\Re})$ is a representation of the $U_{\Re}$ algebra in the Hilbert space H and D is the Dirac operator, the commutator with the Dirac operator being used to generate the one-forms. In a general non-commutative framework [59, 60] it is not always possible to use the derivations of the algebra to construct by duality the differential forms. Many algebras have no derivations at all. However when the algebra has enough derivations it is useful to consider them [61, 62] because the correspondence of the non-commutative geometry notions to the classical ones becomes very clear. One considers the set V of derivations with basis $\{\partial_\mu, \partial_4\}$ defined as follows[b]

$$\begin{aligned}
\partial_\mu(x_\nu) \quad &= \eta_{\mu\nu}\Im, \\
\partial_4(x_\mu) \quad &= -\epsilon_5 \ell p_\mu \Im, \\
\partial_\sigma(M_{\mu\nu}) &= \eta_{\sigma\mu}p_\nu - \eta_{\sigma\nu}p_\mu, \\
\partial_\mu(p_\nu) \quad &= \partial_\mu(\Im) = \partial_\mu(1) = 0, \\
\partial_4(M_{\mu\nu}) &= \partial_4(p_\mu) = \partial_4(\Im) = \partial_4(1) = 0.
\end{aligned} \tag{55}$$

In the commutative ($\ell = 0$) case a basis for one-forms is obtained, by duality, from the set $\{\partial_\mu\}$. In the $\ell \neq 0$ case the set of derivations $\{\partial_\mu, \partial_4\}$ is the minimal set that contains the usual ∂_μ's, is maximal abelian and is action closed on the coordinate operators, in the sense that the action of ∂_μ on x_ν leads to the operator $\Im$ associated to ∂_4 and conversely.

The operators that are associated to the physical coordinates are just the four x_μ, $\mu \in (0, 1, 2, 3)$. However, an additional degree of freedom appears

[b]Notice that the definition of ∂_4 here is slightly different from the one in [58].

in the set of derivations. This is not a conjectured extra dimension but simply a mathematical consequence of the algebraic structure of $\Re_{\ell,\infty}$ which, in turn, was a consequence of the stabilizing deformation of relativistic quantum mechanics. No extra dimension appears in the set of physical coordinates, because it does not correspond to any operator in $\Re_{\ell,\infty}$. However the derivations in V introduce, by duality, an additional degree of freedom in the exterior algebra. Therefore all quantum fields that are Lie algebra-valued connections will pick up some additional components. These additional components, in quantum fields that are connections, are a consequence of the length parameter ℓ which does not depend on its magnitude, but only on ℓ being $\neq 0$.

The Dirac operator [58] is

$$D = i\gamma^a \partial_a \tag{56}$$

with $\partial_a = (\partial_\mu, \partial_4)$ and the γ's being a basis for the Clifford algebras $C(3,2)$ or $C(4,1)$

$$\begin{array}{ll} \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = \gamma^5\right) & \epsilon_5 = +1 \\ \left(\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^4 = i\gamma^5\right) & \epsilon_5 = -1 \end{array}, \tag{57}$$

How to construct quantum, scalar, spinor and gauge fields, as operators in $U_\Re$, has been described in [58]. In particular the role of the additional dimension in the exterior algebra on gauge interactions has been emphasized (see also [56]). Here, another potential interesting consequence for spinor fields will be described. Because

$$\left[p_\mu, e^{\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+}\right] = -k_\mu e^{\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} \tag{58}$$

a spinor field is written

$$\psi = \int d^4k\delta(k^2 - m^2)\left\{b_k u_k e^{-\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} + d_k^* v_k e^{\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+}\right\}, \tag{59}$$

$$\psi \in U_\Re : D\psi - m\psi = 0. \tag{60}$$

For a massless field, the (extended) Dirac equation is

$$D\psi = i\gamma^a\partial_a\psi = \left(i\gamma^\mu\partial_\mu + i\gamma^4\partial_4\right)\psi = 0. \tag{61}$$

Write

$$\psi = e^{\frac{i}{2}k_\nu\left\{x^\nu, \Im^{-1}\right\}_+} u(k).$$

From

$$\begin{aligned} \partial_\mu e^{\frac{i}{2}k_\nu\{x^\nu,\Im^{-1}\}_+} &= ik_\mu e^{\frac{i}{2}k_\nu\{x^\nu,\Im^{-1}\}_+} \\ \partial_4 e^{\frac{i}{2}k_\nu\{x^\nu,\Im^{-1}\}_+} &= -i\epsilon_5\ell\left(k^\mu p_\mu + \frac{1}{2}k^2\right)e^{\frac{i}{2}k_\nu\{x^\nu,\Im^{-1}\}_+}, \end{aligned} \tag{62}$$

one obtains, using (62) and (58)

$$\begin{aligned} \left(-\gamma^\mu k_\mu - \gamma^5\ell\frac{1}{2}k^2\right)u(k) = 0 \quad \epsilon_5 = +1, \\ \left(-\gamma^\mu k_\mu + i\gamma^5\ell\frac{1}{2}k^2\right)u(k) = 0 \quad \epsilon_5 = -1. \end{aligned} \tag{63}$$

Let $\epsilon_5 = -1$. Iterating (63)

$$\left(k^2 - \frac{\ell^2}{4}\left(k^2\right)^2\right)u(k) = 0. \tag{64}$$

This equation has two solutions, the massless solution ($k^2 = 0$) and another one, of large mass (ℓ being small)

$$k^2 = \frac{4}{\ell^2}. \tag{65}$$

For $\epsilon_5 = +1$ the large $|k^2|$ solution is tachyonic. The solutions of the extended Dirac equation for $k^2 = 0$ are the usual ones and for $k^2 = \frac{4}{\ell^2}$, in the rest frame and the Weyl (chiral) basis

$$\begin{aligned} \begin{pmatrix} a \\ -ia \end{pmatrix} \ \text{Positive energy} \ \left(m_0 = \frac{2}{\ell}\right), \\ \begin{pmatrix} a \\ ia \end{pmatrix} \ \text{Negative energy} \ \left(m_0 = -\frac{2}{\ell}\right) \end{aligned} \tag{66}$$

the solutions of non-zero momentum being obtained by the application of a proper Lorentz transformation. a is an arbitrary two-vector.

So far and in [52–56] consequences were explored of the ($\ell \neq 0, 1/R \to 0$) case. However, as pointed out by several authors [63–66], even a very small non-vanishing of the right-hand side of the commutator $[P_\mu, P_\nu]$ may have striking consequences on the nature of the representations of the algebra, which instead of Poincaré, becomes de Sitter or anti-de Sitter.

Another interesting possibility, still unexplored, would be to promote the right-hand side of the commutator $[P_\mu, P_\nu]$, which in (28) is written $-i\frac{\epsilon_4}{R^2}M_{\mu\nu}$, to a space-time dependent field $C_{\mu\nu}(x)$, from which a theory of gravity as a deformation might be constructed.

Finally, notice that when using algebraic stability to study the kinematical algebras, the primary results so far have concerned the nature of one-particle states. If, instead, one is concerned with two-particle effects (or aggregates) it is probably the deformation theory of bialgebras that comes into play. The suggestion is that the stability theory of bialgebras might provide useful information on the nature of the *stable interactions*.

4. Stability, Universality and Critical Phenomena

4.1. *Bifurcations and universality*

Many families of differential equations and discrete-time mappings depending on one parameter μ, exhibit, when μ varies, a cascade of successive period-doubling bifurcations of stable periodic orbits [67, 68]. A typical example is the quadratic map $x \to 1 - \mu x^2$. As μ approaches the value $\mu_\infty = 1.40155$ from below, the ratio

$$\frac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n},$$

tends to $\delta = 4.669\ldots$, μ_n being the value at which the 2^n-cycle is born. Similarly the size of the domains in phase space associated to the successive cycles (for example the separation of two points in the orbits that contain the critical point at $x = 0$) also scales to a constant $\lambda = 0.399\ldots$. [68].

The universality of these constants is associated to the existence of a fixed point for the Feigenbaum functional equation [69–72].

$$-\frac{1}{\lambda}\psi \circ \psi(-\lambda x) = \psi(x).$$

The values $\delta = 4.669\ldots$ and $\lambda = 0.399\ldots$ depend on the quadratic nature of the critical point. Other critical points also lead to scaling behavior but with different constants [73]. However the fact that the above constants are the ones that are actually found in so many one-parameter systems and also on experimental results [74] clearly seems to be a manifestation of the fact that, as discussed in Appendix A, the quadratic map is the only stable one-dimensional map.

For higher dimensions, however, we might have stable sequences of higher order bifurcations corresponding to fixed point solutions of the functional equation

$$-\frac{1}{\lambda}\underbrace{\psi \circ \cdots \circ \psi}_{n}(-\lambda x) = \psi(x).$$

Bifurcation sequences of period-tripling, period-quadrupling, etc. have been studied for complex mappings [75–77]. Consider a family $f(z, \mu)$ of quadratic mappings of C^1 into C^1 depending on a complex parameter μ. In the complex μ plan there is a domain U_0 of parameter values for which there is a stable fixed point. The boundary of the U_0 domain consists of the parameter values for which the map derivative at the fixed point lies on the unit circle. Touching U_0 there are two smaller domains $U_3^{(1)}$ and $U_3^{(2)}$ corresponding to the values of μ for which there is a stable period-3 orbit. The contact points of the domain U_0 with $U_3^{(1)}$ and $U_3^{(2)}$ are the cubic roots of the unit $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. Then, adjoining each of the domains $U_3^{(1)}$ and $U_3^{(2)}$, there are two domains corresponding to stable period-9 orbits and so on. Choosing parameter values μ to follow the successive contact points of all these domains one obtains a period-tripling bifurcation sequence. The corresponding (complex) universal constant is $\delta^{(1,2)}(3) = 4.600 \cdots \pm i8.981...$. A similar scheme operates for other n-tuplings for which the complex universal constants have also been computed [76].

A complex $C^1 \to C^1$ mapping may be regarded as a real $R^2 \to R^2$ mapping and sequences of n-tuplings might therefore also be expected in real mappings as a two-parameter effect. Structural stability imposes however some restrictions on the observability of this phenomenon. Let us write the quadratic $z \to z^{'} = 1 - \mu z^2$ complex mapping as a real $C^2 \to C^2$ mapping. With $\mu = \alpha + i\beta$ and $z = x + iy$ one obtains

$$x^{'} = 1 - \alpha(x^2 - y^2) + 2\beta xy,$$
$$y^{'} = -\beta(x^2 - y^2) - 2\alpha xy.$$

This map however has at $x = y = 0$ a singularity of the Σ^2-type which is stable only for real maps of dimension four and above. Therefore, on the basis of the stability principle, for physical systems described by real maps, *one should expect the n-tupling effect (with $n > 2$) to be generic only for phenomena which are not reducible to an effective dynamics below four dimensions.* Conversely the observation of such higher n-tuplings in actual complex physical systems may be a guide for the dimensional requirements of their mathematical models.

4.2. *Universality in phase transitions*

The renormalization group analysis [78, 79] provides a great deal of information on continuous phase transitions. At a continuous phase transition point the correlation length diverges, the dynamics is dominated by long-range collective effects and one expects the physics of the problem to be insensitive to scale transformations.

In configuration (or real)-space renormalization, for a system defined on a lattice, one replaces, at each step, all the degrees of freedom contained in a block by a single block variable. Therefore the block variable ($\sigma_i^{(n+1)}$) at step n+1 is a function of the block variables of the preceding step ($\sigma_i^{(n)}$).

$$\sigma_i^{(n+1)} = f(\sigma_k^{(n)}). \tag{67}$$

The function f may be a smooth function and is normalized in such a way that the mean-square value of the block variables is preserved at all renormalization steps. Each time a blocking is performed, the lattice parameter changes from a to ba. Therefore to keep the same nominal lattice spacing, lengths are at each step scaled down by a factor b^{-1}. The effective Hamiltonian $H(\sigma^{(n+1)})$ of the renormalized system is obtained by summing over the variables of the preceding step, namely

$$\frac{1}{Z^{(n+1)}} e^{-\mathcal{H}^{(n+1)}(\sigma_i^{(n+1)})} = \sum_{f(\sigma_k^{(n)})=\sigma_i^{(n+1)}} \frac{1}{Z^{(n)}} e^{-\mathcal{H}^{(n)}(\sigma_k^{(n)})}, \tag{68}$$

where the sum in the right-hand side is over all the configurations of the $\sigma_k^{(n)}$ variables that lead to the specified $\sigma_i^{(n+1)}$. The temperature dependence is included in the effective Hamiltonian. In the first step we have

$$\mathcal{H}^{(0)}(\sigma_i^{(0)}) = \frac{1}{kT} H(\sigma_i^{(0)}), \tag{69}$$

H being the temperature-independent usual Hamiltonian. However, after the renormalization, the effective Hamiltonians obtained from Eq. (68) will in general have a much more complicated dependence on the temperature and on the other variables. However, they will be functions of the same variables as $\mathcal{H}^{(0)}(\sigma_i^{(0)})$ and furthermore assumed to be smooth functions.

Here, I will be mostly concerned with the dependence on temperature and on a parameter which, for definiteness, is assumed to play the same role as an external magnetic field coupled by a term $\frac{B}{kT}\sum_i \sigma_i^{(0)}$ in $\mathcal{H}^{(0)}$. Hence

$$\mathcal{H}^{(0)}(\sigma_i^{(0)}) = \mathcal{H}^{(n)}\left(T, \frac{B}{T}, \ldots\right). \tag{70}$$

The dots stand for other variables like the spin–spin coupling strengths, etc.

At high temperatures, variables become independent and the correlation length vanishes. On the other hand, using a sufficiently high-temperature for the starting point of the renormalization, the correlation length at step n is $\xi^{(n)} = \frac{\xi}{b^n}$, it tends to zero as $n \to \infty$ and one expects it to be driven to the high-temperature fixed point.

On the other hand, close enough to $T = 0$ all variables are near their ground-state values and the block averaging, resulting from the renormalization, will make the block variables increasingly more uniform. Therefore starting from a sufficiently small temperature the system is driven by renormalization towards the low-temperature fixed point.

Consider now a system that has only one phase transition. Then, between the functions that are attracted to the high-temperature fixed point and those that are attracted to the low-temperature fixed point, there is, in the space of smooth functions, those that are attracted to neither one. These functions are said to lie in the *critical surface* and, at least some of them, correspond to effective Hamiltonians for phase transition points at distinct values of the physical parameters.

To make the connection with the *structural stability scenario*, notice that this is typically a codimension-one framework (Appendix A). Therefore the critical surface may be taken to be a codimension-one subset $\mathcal{S}_c$ in the space of all smooth functions. The missing dimension is precisely the direction taken by the renormalization transformation when it drives nearby functions either to the low or the high-temperature limits. This is the precise physical meaning of the codimension of the critical surface, as defined here. It should not be confused with the number of *relevant directions*, because if there is a renormalization group fixed point in the critical surface, some of the directions associated to eigenvalues greater than one may point along the critical surface.

Of course, not every function in $\mathcal{S}_c$ may be reached from any other by a renormalization transformation. This is understandable because the finite codimension subsets in the space of all smooth functions are defined by R-equivalence, that is by arbitrary diffeomorphisms and the renormalization transformation is just a particular type of change of variables. Also, as defined, the critical surface may contain the effective Hamiltonians of many different physical systems. For each particular system the renormalization group generates a (not necessarily dense) orbit in the critical surface. Notice also that, instead of the critical surface containing the effective Hamiltonians, we may consider a space of (Helmholtz) free energy functions.

So far this a very general framework which depends only on the existence of the low and high-temperature limits and one phase transition. A further assumption of the renormalization group analysis is the existence, in the critical surface, of quasi-homogeneous functions. A function is quasi-homogeneous [80] of degree d with indices $y_1, \ldots, y_n$ if for any $b > 0$ we have

$$f(b^{y_1} x_1, \ldots, b^{y_n} x_n) = b^d f(x_1, \ldots, x_n). \tag{71}$$

For the effective Hamiltonians the assumption is that there is a fixed point in the critical surface and the corresponding result for the free energy per unit mass is a relation of the type of Eq. (71). Actually, even at the fixed point, the transformation of the free energy is slightly more complicated, namely

$$f(b^{y_1} x_1, \ldots, b^{y_n} x_n) = b^d \{f(x_1, \ldots, x_n) - g(x_1, \ldots, x_n)\},$$

where the function g is needed to satisfy the normalization conditions. However for the purpose of computation of the critical exponents the first term is considered to be sufficient (for a discussion see [79]).

For a continuous phase transition, physical intuition, derived from the divergence of the correlation length, indeed suggests the existence of a scale-independent point. Nevertheless the actual existence of a renormalization fixed point in the critical surface is an assumption and more complex (periodic or chaotic) behaviors are possible. Notice also that, at the fixed point, the effective Hamiltonian that is obtained by the action of the renormalization group may not correspond to any particularly interesting set of parameters. The physical phase transition points are all over the critical surface. However because the critical exponents are preserved along renormalization group orbits, they may be computed at the fixed point.

Let, in Eq. (71), x_1 be the reduced temperature

$$x_1 = t = \frac{|T - T_c|}{T_c}$$

and x_2 the magnetic field

$$x_2 = B.$$

Then y_1 and y_2 are the *temperature* and *magnetic indices* (or *eigenvalues*) and Eq. (71) becomes Widom's [81, 82] scaling hypothesis

$$f(b^{y_t} t, b^{y_B} B) = b^d f(t, B). \tag{72}$$

All critical exponents may be computed from the two numbers y_t and y_B [79]. $\xi \sim |T - T_c|^{-\nu}$

$$c_B \sim \alpha^{-1}\left(\left(\frac{|T - T_c|}{T_c}\right)^{-\alpha} - 1\right);\ \alpha = 2 - \frac{d}{y_t},$$

$$m \sim (T_c - T)^{\beta};\ B = 0;\ \beta = \frac{d - y_B}{y_t},$$

$$c_B \sim \alpha^{-1}\left(\left(\frac{|T - T_c|}{T_c}\right)^{-\alpha} - 1\right);\ \alpha = 2 - \frac{d}{y_t},$$

$$m \sim B^{\frac{1}{\delta}};\ T = T_c;\ \delta = \frac{y_B}{d - y_B},$$

$$G^{(2)}(r) \sim \frac{1}{\gamma^{d-2+\eta}};\ T = T_c;\ B = 0;\ \eta = d + 2 - 2y_B,$$

$$\xi \sim |T - T_c|^{-\nu};\ B = 0;\ \nu = \frac{1}{y_t}.$$

For each pair (y_t, y_B) of renormalization group eigenvalues one has a set of critical exponents, which apply to a class of different physical systems. Each set of values (y_t, y_B) defines a *universality class*. This provides an appreciable unification in our knowledge of critical phenomena and understanding the mechanism, through which very different physical systems may have the same critical exponents, was the great achievement of the renormalization group analysis. However, the renormalization group is powerless in determining the pair (y_t, y_B) or in finding out how many universality classes there is.

We now turn to structural stability considerations. One imposes, as an hypothesis, that *the critical surface is a structurally stable codimension-one family of functions*. From the table in Appendix A one knows that there is only one stable family of codimension-one. This family contains all the functions that are R-equivalent to the canonical form A_2. The canonical forms listed in the table of Appendix A are defined up to a Morse function in the other variables. Hence, for two variables, one has

$$f_\alpha(x_1, x_2) = x_1^2 + x_2^3 + \alpha x_2. \tag{73}$$

The last term is the unfolding that vanishes ($\alpha = 0$) on the critical surface. By R-equivalence one generates all kinds of complex functions in the critical surface. However the canonical form is already all one needs because it is a quasi-homogeneous function. Hence for the stable codimension-one family, the existence of a quasi-homogeneous point is not a separate assumption.

Notice that the canonical form in Eq. (73) is only appropriate at the fixed point. Because the other functions in the critical surface are obtained from this one by arbitrary diffeomorphisms, there is no simple relation between the canonical form at the quasi-homogeneous point and the free energy at other physical phase transition points. Therefore the canonical form is only appropriate to derive renormalization group invariants like the critical exponents and nothing else. Notice also that it is only at the fixed point that the extra functional dimension, pointing towards the high and low temperature limits, is generated by αx_2.

To apply the canonical form to derive the critical exponents we still have to identify the variables x_1 and x_2. Referring back to Eq. (70) we conclude that the natural identification of even and odd variables is not t and B, but t and $\frac{B}{t}$. Then

$$x_1 = t \text{ and } x_2 = \frac{B}{t}.$$

Therefore from

$$f_0\left(t, \frac{B}{t}\right) = b^{-d} f_0\left(b^{y_t} t, b^{y_B - y_t}\frac{B}{t}\right)$$

and Eq. (73) one obtains

$$\begin{aligned} 2y_t &= d, \\ 3(y_B - y_t) &= d, \end{aligned}$$

that is, $y_t = \frac{d}{2}$ and $y_B = \frac{5}{6}d$. Then,

$$\alpha = 0;\ \beta = \frac{1}{3};\ \gamma = \frac{4}{3};\ \delta = 5;\ \eta = 2 - \frac{2}{3}d;\ \nu = \frac{2}{d}. \tag{74}$$

These values, obtained from the structural stability of the critical surface, are indeed close to the experimental values for three-dimensional physical systems undergoing continuous phase transitions.

The similarity of the measured critical exponents for many different experimental systems and in particular the proximity of their values to simple rational numbers has intrigued many authors. Cardy [83], for example, uses the fact that, by letting the length rescaling factor depend continuously on position, scale invariance is generalized to conformal invariance. Then the critical exponents are restricted to rational numbers which, by trial, may be identified with particular models. However no unique or strongly preferred result is obtained.

Here using a structural stability hypothesis, in the codimension-one setting, a unique result is obtained. To require structural stability, that is, to require that the physical laws are not too sensitive to the precise values of the couplings, is perhaps a natural requirement, at least for phenomena that do not seem to depend on the detailed properties of the system but only on a general scaling behavior. In obtaining the result (74) an important role is also played by the identification of t and $\frac{B}{t}$ as the variables in the quasi-homogeneous free energy at the fixed point. This however seems a natural choice in view of Eq. (70). Is the result (74) an accident, or is it appropriate to use structural stability in this context? I leave to the reader to decide.

5. Appendix A: Structural Stability in Dynamical Systems Theory

5.1. *Structural stability of phase portraits*

Let (M, U_t) be a *dynamical system*. M is the *state space* and U_t (with $t \in K{=}R, Z, R^+$ or Z^+) the *time evolution operator*. For each initial condition $x_0 \in M$, the set $\{U_t\ x_0 : t \in K\}$ is an *orbit* of the dynamical system. The set of all orbits is called the *phase portrait* P of the system.

The problem of *structural stability* in the theory of dynamical systems is, in qualitative terms, the following: "*If the dynamical system* (M, U_t) *with phase portrait* P *is perturbed to a slightly different system* $(M, U_t)'$, *is the new phase portrait* P' *also a small perturbation of* P? *That is, is the new system equivalent to the first?* (equivalent in a sense to be specified later)". The perturbation of the dynamical system may be, for example, a small change in the numerical parameters of the evolution operator.

Structural stability is a question of great physical importance because, even if (M, U_t) is an accurate model for a physical system, the results obtained by the study of this model are, in practice, never applied to the actual (M, U_t) model of the real world but to a nearby system because the parameters of the system, being obtained experimentally, are only known approximately. Therefore, underlying all attempts to describe natural phenomena is the assumption that the structures in Nature enjoy some stability, otherwise we could hardly think of the possibility to describe them in an experimentally reproducible way. Hence, *the only qualitative properties of a family of dynamical systems which are physically relevant are those that are preserved under perturbations.*

To discuss structural stability we have to specify what are the allowed deformations of the systems we are concerned with (that is, what is the topology given to the set of dynamical systems) and what is the equivalence relation that decides when the perturbed system is equivalent to the unperturbed one. In the classical theory of dynamical systems, the evolution operator U_t is either a discrete power of a mapping $T : M \to M$ (discrete time) or the flow induced by a vector field X in M (continuous time). The topology is in both cases the C^r-topology. Two maps are C^r-close when their values and the values of their derivatives up to order r are close at every point. An ϵ-*neighborhood* of a map f in the C^r-topology is the set of all C^r-maps which together with their derivatives up to order r differ from f less than ϵ.

The equivalence relation is *topological conjugacy* for maps and *topological equivalence* for flows. Two maps T_1 and T_2 are C^0-conjugate if there is a homeomorphism h such that $h \circ T_1 = T_2 \circ h$. Two vector fields X_1 and X_2 are C^0-equivalent if there is a homeomorphism h which takes the orbits of X_1 to orbits of X_2, preserving senses but not necessarily the time parametrization. This is because, for example, we allow the periods of closed orbits to be different. Notice also that the most relevant notion of equivalence is topological (C^0) equivalence, not for example C^1-equivalence or C^1-conjugacy. This latter equivalence would be too restrictive because it would impose invariance of the eigenvalues of the linear part of the dynamics at periodic points.

A map f (or vector field X) is *structurally stable* if it has an ϵ-neighborhood topologically conjugate (or topologically equivalent) to f (to X). We may however not be concerned with the transients of the dynamics. Therefore we may consider stability restricted to the main part of the orbit structure, that is to the *non-wandering set* Ω. A point is non-wandering if, for any neighborhood U of x, there is an integer n such that $f^n U \cap U \neq \emptyset$. Then, $\Omega-$*structural stability* is structural stability restricted to the non-wandering set. That is, given a small perturbation the perturbed system has a non-wandering set $\widetilde{\Omega}$ and there is a surjective map $\Omega \to \widetilde{\Omega}$ sending orbits to orbits.

For general dynamical systems the notions of structural stability, hyperbolicity and transversality are closely related. Some of the strongest results proved so far are:

Theorem: (Mañé [84]) A C^1-diffeomorphism is C^1-structurally stable if and only if it satisfies Axiom A and the strong transversality condition.

Theorem: (Palis [85]) A C^1-diffeomorphism is C^1-Ω-structurally stable if and only if it satisfies Axiom A and the no-cycle condition.

The meaning of the terms used in these theorems is the following:

An *Axiom A dynamical system* is a map (or flow) such that

(1) The non-wandering set Ω is compact and hyperbolic.
(2) The fixed points and periodic orbits are dense in Ω.

A set is *hyperbolic* when there is a continuous splitting $TM\mid_\Omega = V^+ + V^-$ of the tangent bundle restricted to Ω such that $(Df)V^{\pm} \subset V^{\pm}$ and $\|Df^{\pm n}\mid_{V^{\mp}}\| \leq c\theta^{-n}$, $n \geq 0$ for some $c > 0$, $\theta > 1$.

The *stable* and *unstable manifolds* of a point are

$$W_x^s = \{y \in M : \lim_{n\to\infty} d(f^n x, f^n y) = 0\},$$
$$W_x^u = \{y \in M : \lim_{n\to-\infty} d(f^n x, f^n y) = 0\}.$$

A dynamical system f satisfies the *strong transversality* condition if, for each $y \in M$ there are stable and unstable manifolds through y such that

$$T_y M = T_y W_x^u + T_y W_{x'}^s (+X),$$

where X is added in the flow case if y is not a fixed point.

For Ω-stability, strong transversality is replaced by the no-cycle condition. The non-wandering set Ω of an Axiom A system f is a finite union $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ of disjoint closed invariant sets called *basic sets*, such that f is topologically transitive on each Ω_i. Topological transitivity means that there is an x with dense orbit in Ω_i. One writes

$$\Omega_i \gg \Omega_j \quad \text{if} \quad W_{\Omega_i}^s \cap W_{\Omega_j}^u \neq \emptyset.$$

The *no-cycle condition* means that one cannot find distinct $\Omega_{i_1}, ..., \Omega_{i_p}$ ($p > 1$) such that

$$\Omega_{i_1} \gg \Omega_{i_2} \gg \cdots \gg \Omega_{i_p} \gg \Omega_{i_1}.$$

The stability results quoted above are difficult mathematical theorems. However, the relation between structural stability, hyperbolicity and transversality is fairly intuitive and was the object of an old conjecture (Palis, Smale [86]). Structural stability means that the nature of the system does not change for small perturbations and, for example, a periodic point must be hyperbolic if it remains of the same nature for small perturbations. On the other hand, transversality means that the stable and unstable manifolds, that are the organizers of the dynamics, must be in general position.

The generic nature of hyperbolicity and transversality might suggest that almost all systems are structurally stable in the sense that structurally stable systems are dense in the set of all smooth systems. Actually this has been proven not to be true [87, 88] except for low-dimensional cases [89].

Structural stability as it has been defined is a property that concerns the topological properties of the dynamical system. Another notion of stability was proposed (Zeeman [90]) that deals with the invariant measure ρ of the system under a small random perturbation. A small random perturbation is added to the system because for a large class of noisy systems the invariant measure is unique whereas in general a deterministic system has many invariant measures. The measure is the solution of the Fokker–Planck equation

$$\partial_t \rho = -\nabla(\rho X) + \epsilon \triangle \rho,$$

where X is the deterministic vector field and the diffusion coefficient ϵ is a small quantity. Two functions ρ and $\rho^{'}$ are *equivalent* if there are diffeomorphisms α and β of M and R such that $\rho^{'} \circ \alpha = \beta \circ \rho$. Then two vector fields X and $X^{'}$ are $\epsilon-$*equivalent* in this sense when the corresponding solutions $\rho^{X,\epsilon}$ and $\rho^{X^{'},\epsilon}$ are equivalent. A vector field X is ϵ-*stable* if it has an ϵ-equivalent neighborhood. It is called *stable* if it is ϵ-stable for arbitrarily small $\epsilon > 0$.

Both structural stability and (measure) stability in Zeeman's sense are designed for general dynamical systems and leave out whole classes of physical interest. For example they are not suitable for Hamiltonian systems which are all structurally unstable. This is because the perturbations allowed in the C^r-topology do not preserve any constants of motion or symmetries that the dynamical system may have.

No system with regular first integrals may be structurally stable, in the general sense, because the property of having no regular first integrals is C^1 generic [91, 92]. To define a structural stability concept for Hamiltonian systems we must exclude non-Hamiltonian perturbations. Restricting the perturbations to the space χ_H of Hamiltonian vector fields, and using the C^r-topology in this space, we may define stability of the phase portrait in the same way as before. This notion of stability is very strong and it seems more appropriate to require only stability of the phase portrait on a single energy surface under small perturbations of the Hamiltonian and the energy. Because generically an Hamiltonian system restricted to an energy surface has no other first integrals, the conflict with general structural stability would seem to be avoided. However the problem with the several definitions that have been proposed so far [93] is that they do not apply to generic Hamiltonian systems [94].

In addition to stability of the phase portraits, there are two other notions of dynamical stability which are reviewed in next two subsections. They are of importance for the applications described in Section 4.

5.2. *Stability of smooth mappings and stable dynamical families*

The preceding subsection was concerned with the stability of the phase portrait of a dynamical system, that is, the stability of the realization in phase space of a dynamical law. Given two equivalent phase portraits, one may in fact say that one is dealing with the same dynamics as seen in two reference frames, related by a continuous change of coordinates. This subsection deals not with stability of the phase portrait but with stability of the type of dynamical law. This will be clear after the definition of equivalence and stability of smooth mappings.

Being mostly concerned with local properties of maps between smooth manifolds M and N one may, by a choice of local charts, reduce the problem to $R^n \rightarrow R^p$ maps. Two smooth maps $f_1, f_2 : R^n \rightarrow R^p$ are *equivalent* when there are diffeomorphisms $g : R^n \rightarrow R^n$ and $h : R^p \rightarrow R^p$ such that $f_1 = h^{-1} \circ f_2 \circ g$. A mapping f is *stable* when there is a neighborhood where all mappings are equivalent to f. *Neighborhoods* in the space of mappings are defined by

$$U_f(k,\epsilon) = \left\{ g : \max_{\alpha \leq k} \|\partial^\alpha (f-g) \partial x^{\alpha_1} \cdots \partial x^{\alpha_n}\| < \epsilon, \alpha = \alpha_1 + \cdots + \alpha_n \right\}$$

the derivatives being taken up to order r for the C^r-topology.

When dealing with maps between different spaces, R^n and R^p, the equivalence relation means that different choices of coordinate systems in the source and the target spaces are allowed. If however one identifies the source and the target space, as in a map $f : R^n \rightarrow R^n$ defining a discrete time dynamical system and the diffeomorphisms h and g are distinct, different dynamics are in fact obtained. Two equivalent maps in the above sense may generate very different phase portraits. The set

$$\{f^{'} : f^{'} = h^{-1} \circ f \circ g\} \tag{75}$$

for all possible difeomorphisms h and g represents not a single dynamical system but a *family* of related systems. We know that in Nature we sometimes have to deal with phenomena that depend on a certain number of control parameters, which may indeed induce very different dynamical behavior, phase transitions, etc., but which nevertheless we want to identify with different conditions of the same physical system. The action of the diffeomorphisms in (75) gives then a precise and general meaning to the notion of *change of parameters* in a *stable family of dynamical systems*. This contains the usual notion of change of parameters in many classical examples. For example for mappings of the unit interval $x \to f_\mu(x) = 1 - \mu x^2$, $f_{\mu'}$ and $f_\mu$are related by $h(x) = x$ and $g(x) = (\mu^{'}/\mu)^{\frac{1}{2}} x$.

If the phase portrait is not preserved, what are the features of the dynamics that are preserved under this equivalence relation? That is, what are the invariant properties that characterize the dynamical systems family defined by (75). The most significant ones are the singularities of the mappings. f is said to have a *singularity* or *critical point* at x if the rank of the derivative map Df at x is less than the maximum possible value (n for $R^n \to R^n$ mappings). The kind of dynamical properties that are controlled by the critical points are universality in the approach to bifurcation accumulation points [73] and bifurcation patterns.

For the singular points of smooth mappings one uses Boardman's notation $\Sigma^{i_1,\cdots,i_k}$. A point is said to belong to Σ^{i_1} if the dimension of the kernel of Df is i_1. The full notation is defined recursively by considering the kernels of the restriction of Df to Σ^{i_1}, etc. That is, $\Sigma^{i_1,\ldots,i_k} = \Sigma^{i_k}(Df \mid \Sigma^{i_1,\ldots,i_{k-1}})$. Actually this characterization of the Boardman symbol $\Sigma^{i_1,\ldots,i_k}$ is correct only if these sets are submanifolds, which is the case for the stable maps that concern us here. That is, for stable maps the Boardman sets coincide with Thom's singularity sets.

The stable maps for low dimensions have been fully classified [80, 95, 96]. They are characterized in terms of germs and unfoldings. A *smooth germ* at the point x is an equivalence class of maps which coincide when restricted to some neighborhood of x. Given a germ $f_0 : (R^n, 0) \to (R^n, 0)$ in the neighborhood of zero, an *r-parameter unfolding* of f_0 is the germ $F : (R^r \times R^n, 0) \to (R^r \times R^n, 0)$ given by $F(u, x) = (u, f(u, x))$ with $f(0, x) = f_0(x)$. Therefore an unfolding is a $(r + n)$-dimensional map, the first r components being the identity map and the other n a *deformation* of the original f_0 map.

A classification of stable $R^n \to R^n$ maps for $n \leq 4$ is listed below, in terms of equivalence of its germ at any point to a standard form. Let f be a stable map; then its germ at any point is equivalent to one of the following:

$n = 1$

$$\Sigma^0(x_1' = x_1),$$
$$\Sigma^{1,0}(x_1' = x_1^2).$$

$n = 2$

$$\Sigma^0(x_1' = x_1; x_2' = x_2),$$
$$\Sigma^{1,0}(x_1' = x_1; x_2' = x_2^2),$$
$$\Sigma^{1,1,0}(x_1' = x_1; x_2' = x_2^3 + x_1x_2).$$

$n = 3$

$$\Sigma^0(x_1' = x_1; x_2' = x_2; x_3' = x_3),$$
$$\Sigma^{1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3^2),$$
$$\Sigma^{1,1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3^3 + x_1x_3),$$
$$\Sigma^{1,1,1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3^4 + x_1x_3 + x_2x_3^2).$$

$n = 4$

$$\Sigma^0(x_1' = x_1; x_2' = x_2; x_3' = x_3; x_4' = x_4),$$
$$\Sigma^{1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3; x_4' = x_4^2),$$
$$\Sigma^{1,1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3; x_4' = x_4^3 + x_1x_4),$$
$$\Sigma^{1,1,1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3; x_4' = x_4^4 + x_1x_4 + x_2x_4^2),$$
$$\Sigma^{1,1,1,1,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3; x_4' = x_4^5 + x_1x_4 + x_2x_4^2 + x_3x_4^3),$$
$$\Sigma^{2,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3x_4; x_4' = x_3^2 + x_4^2 + x_1x_3 + x_2x_4),$$
$$\Sigma^{2,0}(x_1' = x_1; x_2' = x_2; x_3' = x_3x_4; x_4' = x_3^2 - x_4^2 + x_1x_3 + x_2x_4).$$

On the left of each standard form is the Boardman symbol corresponding to the singularity set to which the singular point belongs. Notice that in all cases the standard forms for singularities of type Σ^i are written as $(n-i)$-parameter unfoldings of i-dimensional maps.

For a stable $R^n \to R^p$ map the singularity set $\Sigma^{i_1,\ldots,i_k}$ is a smooth submanifold of codimension

$$(p - n + i_1)\mu(i_1, \ldots, i_k) - (i_1 - i_2)\mu(i_2, \ldots, i_k) - \cdots - (i_k - i_{k-1})\mu(i_k),$$

where $\mu(i_s, \ldots, i_k)$ denotes the number of sequences of integers $(j_s, \ldots, j_k)$ satisfying $j_s \geq j_{s+1} \geq \cdots \geq j_k \geq 0$ with $i_r \geq j_r$ for all $s \leq r \leq k$ and $j_s > 0$.

In particular, for the equidimensional case ($n = p$), Σ^i has codimension i^2. That is why, in the list above, singularities of the type Σ^2 only appear for $n \geq 4$.

The (Boardman) singularity symbols $\Sigma^{i_1,\ldots,i_k}$ are equivalence invariants, that is, they are invariant under a change of parameters (in the sense defined above), and therefore, they are a robust characterization of the dynamical system families. Notice however that, for example, the last two stable $R^4 \to R^4$ maps listed above have the same $\Sigma^{2,0}$ symbol but are not equivalent. Hence the classification of singularities in Σ^I classes is not complete.

For low dimensions, stable maps are dense in the space of all $R^n \to R^n$ maps. However for $n \geq 9$ this is no longer true.

In discussing the stability of critical properties of dynamical system families through the stability of smooth mappings one is directly concerned with discrete time dynamics. This is not a serious limitation because in a continuous time system one may always consider the intersections of the orbits with some transversal surface in phase space. Conversely for a discrete dynamical system defined in $K \subset R^n$ there is [97] a continuous time system in R^{2n+1} for which K is a global section.

5.3. *Stable dynamical families with degeneracies*

Here we are concerned with properties of smooth functions $f : R^n \to R$. The structural stability conditions for functions is given by Morse theory:

(i) f is stable if and only if the critical points are non-degenerate (non-vanishing Hessian) and distinct.
(ii) If f is stable, local coordinates $(x_1, \ldots, x_n)$ may be defined in such a way that in the neighborhood of each point $\overrightarrow{x}$ the function may be written either as

$$f(\overrightarrow{x}) = x_1$$

or

$$f(\overrightarrow{x}) = x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$$

(iii) Stable functions on a compact manifold are everywhere dense in the space of all smooth functions.

Hence, in the space of all functions, stable functions are generic and the non-stable functions form a codimension-one hypersurface, that is, a submanifold defined by one equation. This hypersurface is called the *bifurcation set*. The bifurcation set is the union of the hypersurface of functions having degenerate critical points and the hypersurface of functions with coinciding critical values. The bifurcation set divides the function space into components. When in the previous subsection we spoke of the notion of *change of parameters in a stable family of dynamics*, as induced by the diffeomorphisms h and g, this operates solely inside one of the components of the space of functions. However we may have a more general situation. Consider for example a one-parameter family. This is represented by a curve in function space. If the intersection of this curve with the bifurcation hypersurface is transversal then the intersection is stable in the sense that it cannot be destroyed by a small variation of the one-parameter family. For a neighboring family the intersection will occur for a slightly different value of the parameter and the point of intersection itself is slightly different. However the intersection cannot be removed by small perturbations and the situation is qualitatively the same for all the neighboring families. An example is

$$f_t(x) = x^3 - tx$$

which has a degenerate critical point at $t = 0$ which cannot be removed from the family by small perturbations. We therefore reach the notion of *stable dynamical family with degeneracies*. Such families represent the stable ways to connect two non-equivalent classes of functions. This is the reason why they might be relevant to the theory of phase transitions as illustrated in Section 4.

To classify the possible classes of stable parametrized families, the notion of universal unfolding plays an essential role. For the space of function germs E_n one uses in general a notion of equivalence finer than the one defined for general maps. Two function germs $f, g : R^n \to R$ are said to be right-equivalent if there is a diffeomorphism germ h and a constant c such that

$$g(x) = f \circ h(x) + c.$$

The action of all possible diffeomorphisms h acting on a function f defines an orbit of a smooth action. The *codimension of the orbit* is the number of independent functional directions missing from the orbit. The codimension is obtained by finding the quotient of the functional space with the tangent space to the orbit. The functional space to consider is M_n, the ideal of germs vanishing at the origin, and the tangent space is the Jacobian ideal

$\Delta(f) = \left\{ g_1 \frac{\partial f}{\partial x_1} + \cdots + g_n \frac{\partial f}{\partial x_n} : g_1, \ldots, g_n \in E_n \right\}$. Then

$$\operatorname{cod}(f) = \dim \frac{M_n}{\Delta(f)}.$$

Whenever the codimension of f is finite the construction of a stable family of dynamics $f_\alpha(x)$ based on f is straightforward. A basis $\{u_1, \ldots, u_l\}$ is found for $\frac{M_n}{\Delta(f)}$ and

$$f_\alpha(x) = f(x) + \alpha_1 u_1(x) + \cdots + \alpha_l u_l(x). \tag{76}$$

This *unfolding* of the function f is called *universal* because any other unfolding may be *induced* from it by a smooth change of parameters and the number l of unfolding directions is as small as possible. If the function f is stable the unfolding coincides with the function itself. A family of function germs is *structurally stable* if any small perturbation is equivalent to it, as an unfolding. (Equivalence for two unfoldings means that they may be obtained from each other by a smooth change of parameters). Hence a universal unfolding of a germ of finite codimension is structurally stable.

The unfolding (76) is linear in the parameters α and for finite codimension this construction characterizes all possible parametrized functional families. A useful result is the *splitting lemma* which states that if the rank of the second differential (the Hessian) of f at a singularity is r then f is right equivalent to

$$g(x_1, \ldots, x_{n-r}) \pm x_{n-r+1}^2 \pm \cdots \pm x_n^2.$$

The splitting lemma reduces the effective number of variables to $n - r$ and the classification of possible classes for f depends only on the classification of g. $n - r$ is called the *corank*. A list all the classes of universal unfoldings for codimension ≤ 5 is included here. By the splitting lemma, in each case, we may add an arbitrary quadratic (Morse) function on the other variables. A more extensive list may be found in [80].

The symbols A_k, D_k, and E_6 are used because of the relation of these singularities to the crystallographic groups with the same symbols. A_k and D_k correspond to two infinite series with germs $g(x_1, \ldots, x_{n-r})$ equivalent to x^{k+1} and $x^2y + y^{k-1}$.

When using the stable unfoldings to model natural phenomena the first and most important number to be concerned with is the codimension (of the germ g), because degenerate singularities are irremovable only in the case of a family depending on parameters. In particular a singularity of codimension c is irremovable only if the number of parameters is $\geq c$.

Conversely if, for some process, there are l relevant parameters then all classes up to codimension l should be considered.

Symbol		Corank, Codimension
A_2	$x^3 + \alpha x$	1,1
A_3	$\pm x^4 + \alpha_1 x^2 + \alpha_2 x$	1,2
A_4	$x^5 + \alpha_1 x^3 + \alpha_2 x^2 + \alpha_3 x$	1,3
A_5	$\pm x^6 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x$	1,4
A_6	$x^7 + \alpha_1 x^5 + \alpha_2 x^4 + \alpha_3 x^3 + \alpha_4 x^2 + \alpha_5 x$	1,5
D_4	$x^3 - xy^2 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 y$	2,3
D_4	$x^3 + xy^2 + \alpha_1 x^2 + \alpha_2 x + \alpha_3 y$	2,3
D_5	$\pm(x^2 y + y^4) + \alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 x + \alpha_4 y$	2,4
D_6	$x^5 - xy^2 + \alpha_1 y^3 + \alpha_2 x^2 + \alpha_3 y^2 + \alpha_4 x + \alpha_5 y$	2,5
D_6	$x^5 + xy^2 + \alpha_1 y^3 + \alpha_2 x^2 + \alpha_3 y^2 + \alpha_4 x + \alpha_5 y$	2,5
E_6	$\pm(x^3 + y^4) + \alpha_1 xy^2 + \alpha_2 y^2 + \alpha_3 xy + \alpha_4 x + \alpha_5 y$	2,5

6. Appendix B: Algebraic Deformation Theory. Basic Notions

Deformation theory, as the study of continuous families of mathematical structures, already implicit in the work of Riemann [98], traces its modern origins to the work of Fröhlicher–Nijenhuis [99] and Kodaira–Spencer [100] on deformations of complex manifolds and of Gerstenhaber [103] and Nijenhuis–Richardson [102] on the deformations of associative and Lie algebras. So far, it is the deformation theory of algebras that seems to play the main role on physical applications.

6.1. *Deformation of Lie algebras*

For physics it is useful to have an explicit representation of the deformation parameters, because they may play the role of fundamental constants in the deformed stable theories. I will therefore focus in the *theory of formal deformations of Lie algebras* [104]. A formal deformation of a Lie algebra L_0 defined on a vector space V over a field K is an algebra L_t on the space $V \otimes K[t]$ (where $K[t]$ is the field of formal power series), defined by

$$[A, B]_t = [A, B]_0 + \sum_{i=1}^{\infty} \phi_i(A, B) t^i \tag{77}$$

with A, B, $\phi_i(A, B) \in V$ and $t \in K$. The adjoint representation of L_0 is

$$\rho(A)(B) = [A, B]_0. \tag{78}$$

The (Chevalley) cohomology groups play a key role in characterizing the stability of the Lie algebra. An *n-cochain* (relative to the adjoint representation) is a multilinear, skew-symmetric mapping

$$V \times \cdots \times V \to V$$

and the n-cochains form a vector space $C^n(\rho, V)$. In particular $\phi_i(A, B)$ in Eq. (77) must be a 2-cochain. One also has:

The *coboundary operator*

$$\begin{array}{l} d(A_1, \ldots, A_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} \rho(A_i) \phi(A_1, \ldots, \widehat{A_i}, \ldots, A_{n+1}) \\ + \sum_{1 \le i < j \le n+1} (-1)^{i+j} \phi([A_i, A_j], A_1, \ldots, \widehat{A_i}, \ldots, \widehat{A_j}, \ldots, A_{n+1}) \end{array}. \tag{79}$$

A *cocycle* $\phi \in C^n(\rho, V)$ whenever $d\phi = 0$. The set of all n-cocycles is a vector space denoted $Z^n(\rho)$.

A *coboundary* if $\phi \in d(C^{n-1}(\rho, V))$. The set of all coboundaries is a vector space denoted $B^n(\rho)$.

The quotient space

$$H^n(\rho) = \frac{Z^n(\rho)}{B^n(\rho)}$$

is the *n-cohomology group* (relative to the ρ-representation). From (79) it follows that $d^2\phi = 0$. However not all cocycles need to be coboundaries and the n-cohomology groups may be non-trivial.

To illustrate the relevance of these concepts to the deformation problem use the deformed commutation relations (77) and differentiate the Jacobi identity

$$[A, [B, C]_t]_t + [B, [C, A]_t]_t + [C, [A, B]_t]_t = 0 \tag{80a}$$

in the variable t. Then, setting $t = 0$ one obtains

$$d\phi_1(A, B, C) = 0,$$

that is, for the deformation in (77) to be a Lie algebra, ϕ_1 must be a 2-cocycle.

A deformation of L_0 is said to be *trivial* if the algebra L_t is isomorphic to L_0. This means that there is an invertible linear transformation $T_t : V \to V$ such that

$$[A, B]_t = T_t^{-1}[T_t A, T_t B]_0. \tag{81}$$

If all deformations L_t are isomorphic to L_0 then L_0 is said to be *stable* or *rigid*. Suppose now that the second cohomology group $H^2(\rho)$ is trivial.

This means that all 2-cocycles are 2-coboundaries. Then, there must be a 1-cochain γ such that $\phi_1 = d\gamma$. Applying the linear transformation $M_t^{'} = \exp\{-t\gamma\}$ to the algebra L_t

$$[A, B]_t^{'} = M_t^{'-1}([M_t^{'}A, M_t^{'}B]_0).$$

From $\phi_1 = d\gamma$ one now obtains, by a simple calculation

$$\phi_1^{'}(A, B) = \phi_1(A, B) - [\gamma(A), B] - [A, \gamma(B)] + \gamma([A, B]) = 0.$$

Therefore, the power series expansion for $[A, B]_t^{'}$ begins with terms of second order in t

$$[A, B]_t^{'} = [A, B]_0 + \phi_2^{'}(A, B)t^2 + \cdots$$

and from the Jacobi identity, as above, it follows $d\phi_2^{'}(A, B) = 0$. Iterating the whole process all powers of t are successively eliminated. It means that the limit

$$T_t^{-1} = M_t^{'-1} M_t^{''-1} \ldots$$

is the transformation that establishes the equivalence of L_t and L_0. In conclusion, *the vanishing of the second cohomology group is a sufficient condition for non-existence of non-trivial deformations, that is, it is a sufficient condition for stability* (*or rigidity*) *of the Lie algebra*. This is the content of the "rigidity theorem" of Nijenhuis and Richardson [102]. However the condition is not necessary and there are rigid Lie algebras for which the second cohomology group is non-vanishing [8, 105, 106].

Cocycles and coboundaries have a nice geometrical interpretation. The set L^n of all possible complex n-dimensional Lie algebras is an algebraic variety embedded in the linear space of alternating bilinear mappings in C^N (isomorphic to $C^{(n^3-n^2)/2}$), the defining algebraic relations being the Jacobi identity relations between the structure constants. In L^n one has two natural topologies. One is the topology induced on L^n by the open sets in C^N. the other is the Zariski topology defined by taking closed sets to be zeros of polynomials on L^n.

The isomorphism relation (81) is an action of the linear group $GL(n, C)$

$$L^n \times GL(n, C) \to L^n : (l, T) \to T^{-1} \circ l \circ T \times T, \tag{82}$$

where $l \in L^n$ denotes the Lie algebra law. Denoting $l_0(A, B) \doteq [A, B]_0$, l_0 will be a *rigid algebra* if its *orbit* $O(l_0)$ under the action of $GL(n, C)$ is

(Zariski) *open*. Considering an infinitesimal transformation

$$T = Id + \varepsilon\phi,$$

acting on l_0 ($T : l_0 \to l_0'$), a simple computation using (81) leads to

$$\lim_{\varepsilon\to 0} \frac{l_0' - l_0}{\varepsilon}\,(X,Y) = d\phi(X,Y).$$

Therefore the tangent space to the orbit $O(l_0)$ at l_0 coincides with $B^2(l_0)$. On the other hand, considering a tangent line to L^n at l_0

$$l_t = l_0 + \varepsilon\psi,$$

l_t will satisfy Jacobi identity if and only if

$$d^2\psi = 0,$$

that is, if $\psi \in Z^2(l_0)$, the 2-cocycle space $Z^2(l_0)$ of l_0. Then we understand why the vanishing of the second cohomology group is a sufficient condition for rigidity of the algebra. The correspondence does not work both ways because, in general, the algebraic variety L^n has singular points.

Semisimple Lie algebras have a vanishing second cohomology group [107] and are stable. More generally, for any subalgebra l of a semisimple Lie algebra that contains a maximal solvable algebra one has $H^p(l) = 0$ for all $p \geq 0$ [108].

In the general case, the construction of the cohomology groups is not a simple matter. This led to the development of different, non-cohomological, techniques to classify the rigid Lie algebras [27, 106, 109–112]. Here an important role is played by the techniques of non-standard analysis. In this context a Lie algebra law l_0 is said to be rigid if any perturbation is isomorphic to l_0. A perturbation of l_0 is an algebra such that

$$l(A,B) \sim l_0(A,B) \tag{83}$$

for A, B standard or limited. The symbol $\sim$ means *infinitesimally close*. There is a decomposition of any perturbation of l_0 as follows

$$l = l_0 + \epsilon_1\phi_1 + \epsilon_1\epsilon_2\phi_2 + \cdots + \epsilon_1\epsilon_2\ldots\epsilon_k\phi_k \tag{84}$$

which is unique up to equivalence. The ϕ's are standard antisymmetric bilinear mappings, the ϵ's are non-zero infinitesimals and $k \leq N$.

The most useful result for the characterization of the rigid Lie algebras is the theorem that states that if l_0 is rigid there is a standard non-zero vector X such that $ad_{l_0}X$ is diagonalizable ($ad_{l_0}X(Y) = [X, Y]$). The converse result is not true and to classify the rigid algebras in dimension n one still has to exclude the non-rigid ones with a diagonalizable vector. A large number is simply excluded by checking the rank of the root system and for the rest (which is a finite number) one has to check explicitly the isomorphism of the perturbation. For details refer to [27, 109, 111, 112].

A related question is the *strong rigidity.* A finite-dimensional complex Lie algebra is called strongly rigid if its universal enveloping algebra is rigid as an associative algebra. Results on the strong rigidity question may be found in [113].

6.2. *Bialgebras*

A *bialgebra* over the field K is an algebra $\mathcal{A}$ which, in addition to the *product* m, is equipped with a *coproduct* $\triangle : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and a *counit* $\varepsilon : \mathcal{A} \to K$ satisfying

$$\begin{array}{c} \triangle \circ m = (m \otimes m) \circ \tau \circ (\triangle \otimes \triangle) \; ; \varepsilon \circ m = m \circ (\varepsilon \otimes \varepsilon) \\ \triangle \circ i = i \otimes i \qquad\qquad ; \qquad\quad \varepsilon \circ i = I \end{array}, \tag{85}$$

where i is the *unit* of the algebra, I the *identity* map and τ a permutation on the nearby indices. With an additional operation called the *antipode* $S : \mathcal{A} \to \mathcal{A}$ and

$$m \circ (I \otimes S) \circ \triangle = m \circ (S \otimes I) \circ \triangle = i \circ \varepsilon \tag{86}$$

the bialgebra becomes an *Hopf algebra*. These properties have a natural realization on (and were abstracted from) the algebra of a group G where

$$\begin{array}{cc} m\,(g \otimes h) = gh & \varepsilon\,(g) = 1 \\ \triangle\,(g) = g \otimes g & S\,(g) = g^{-1} \end{array}. \tag{87}$$

Then, for example, the first equation in (85) reads

$$\begin{aligned} \triangle \circ m\,(g,h) &= gh \otimes gh = (m \otimes m) \circ \tau \circ (\triangle \otimes \triangle)\,(g,h) \\ &= (m \otimes m) \circ \tau \circ (g \otimes g, h \otimes h) = gh \otimes gh, \end{aligned}$$

where the last step follows from exchanging the second and third argument (that is why the permutation τ is sometimes denoted $(2,3)$).

Other common realizations are:

For the algebra of functions on a group

$$\begin{array}{cc} m\,(f \otimes g)\,(x) = f\,(x)\,g\,(x) & \varepsilon\,(f) = f\,(e) \\ \triangle\,(f)\,(x,y) = f\,(xy) & S\,(f)\,(x) = f\left(x^{-1}\right) \end{array}. \tag{88}$$

For the tensor algebra on a vector space

$$\begin{array}{cc} m\,(x,y) = x \otimes y & \varepsilon\,(x) = 0 \\ \triangle\,(x) = x \otimes 1 + 1 \otimes x & S\,(x) = -x \end{array}. \tag{89}$$

A deformation theory of bialgebras has been developed [114] and a partial classification of rigid bialgebras has also been obtained [115].

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