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Asymptotic dynamics for gauge theories

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Rigorous methods are used to analyze the asymptotic (large time) behavior of gauge-theory Hamiltonians in the interaction picture. A simple "asymptotic dynamics" for four-dimensional gauge theories is obtained allowing an explicit construction of asymptotic spaces. The structure of these spaces is studied; and in particular one proves that for Abelian theories they contain states with an arbitrary number of particles and antiparticles, whereas for non-Abelian theories either the symmetry is spontaneously broken or the asymptotic space contains no states with observable charges.

I. INTRODUCTION

In gauge theories, asymptotic spaces and infrared phenomena are closely related problems. Even in QED, where the infrared asymptotic behavior exponentiates in a simple manner, the infrared phenomena still cause some difficulties. The conventional asymptotic in and out fields belonging to charge-carrying fields do not exist, and it was only in recent years that appropriate substitutes for them were proposed.[1-3]

In non-Abelian gauge theories the problem of infrared behavior and asymptotic states is of central importance, in particular because of the possibility that the infrared structure of quantum chromodynamics (QCD) may provide a dynamical mechanism for the confinement of quarks. Contrary to earlier speculations, however, there is no evidence for quark confinement from perturbation theory calculations, although these results are not very significant because the infrared differential equations, suggested by perturbation theory itself, point to the nonperturbative nature of the problem.[4]

One possible approach would be to start from gauge theories at small distances where, because of asymptotic freedom, things are supposedly simple and by successive (nonperturbative) steps try to evaluate the behavior at large distances. This is the philosophy behind the use of instantons, merons, etc.[5] for this purpose.

Although simple at short distances non-Abelian gauge theories are tremendously complicated at intermediate distances and the construction of, for example, a good approximation to the vacuum of the exact theory may be a hopeless task. However, it is quite possible that, if one goes to the other extremity of the distance range and is concerned with approximations that are good for large distances only, the problem might also be manageable. This reasoning leads us to attempt to study directly the asymptotic (large time) dynamics of gauge theories.

As is well known, in S-matrix theory, one deals with two spaces[6]: the space $\mathcal{H}$ of actual states of the system and an asymptotic space $\mathcal{H}_a$ with the same asymptotic behavior for large $|t|$. In general $\mathcal{H}_a$ may be very different from $\mathcal{H}$, the only requirement being that given a state ψ in $\mathcal{H}$ there should exist states $\phi_+(\psi)$ and $\phi_-(\psi)$ in $\mathcal{H}_a$ such that

$$\lim_{t\to\pm\infty} |\langle\psi|Q(t)|\psi\rangle - \langle\phi_\pm|Q'(t)|\phi_\pm\rangle| = 0\,, \tag{1.1}$$

with $Q(t)$ and $Q'(t)$ being corresponding observables in $\mathcal{H}$ and $\mathcal{H}_a$.

It is also desirable that for each $\phi\in\mathcal{H}_a$, there exist unique $\psi_\pm(\phi)$ such that $\phi_+(\psi_+(\phi))=\phi$ and $\phi_-(\psi_-(\phi))=\phi$.

From the mappings $\phi_\pm$ and $\psi_\pm$ one defines the S matrix as an operator in $\mathcal{H}_a$,

$$S_a=\phi_+\psi_-\,, \tag{1.2}$$

or as an operator in $\mathcal{H}$,

$$S=\psi_-\phi_+\,. \tag{1.3}$$

Because in general the space $\mathcal{H}$ of actual states cannot be constructed, it is the definition (1.2) that has practical interest. As long as Eq. (1.1) is satisfied the asymptotic space may be chosen to be much simpler than the exact theory and, as we know from the nonrelativistic Coulomb problem, still be nontrivially different from a free-particle space.

Our approach will consist in writing the interaction Hamiltonians in the interaction picture (which is defined for all times), to use rigorous methods to isolate the nonintegrable contributions in the asymptotic large-time expansion and finally construct the asymptotic space from the truncated Hamiltonian.

This approach is therefore very similar in spirit to the one used by Kulish and Faddeev[2] to deal with the infrared problem in QED. However, instead of merely isolating the presumably dominant contributions in the integrals of the interaction Hamiltonian in the interaction picture, by going further and deriving exact asymptotic expansions of such integrals it has been possible to obtain a great simplification in the asymptotic dynamics which makes the problem tractable even for non-Abelian gauge theories.

In Sec. II one analyzes the asymptotic behavior of the boson-fermion term which leads to the asymptotic dynamics discussed in Sec. III. In particular, one finds a very clear difference between Abelian and non-Abelian theories, with Abelian theories allowing spaces with arbitrary numbers of particles and antiparticles and non-Abelian gauge theories being either spontaneously broken or, if unbroken, restricted to asymptotic spaces with no observable charges. In Sec. IV a similar analysis is carried out for the boson sector of non-Abelian gauge theories.

II. ASYMPTOTIC FERMION-BOSON INTERACTION

Let L be the Lagrangian of a gauge theory of spinor $\underline{\Psi}(x)$ fields interacting with vector (gauge) $B^a_\mu(x)$ fields,

$$L(x)=\underline{\overline{\Psi}}(i\gamma^\mu D_\mu - M)\underline{\Psi} - \tfrac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu}\,, \tag{2.1}$$

where

$$D_\mu=\partial_\mu + igB^a_\mu \frac{\chi^a}{2}\,,\quad \left[\frac{\chi^a}{2},\frac{\chi^b}{2}\right]=if_{abc}\frac{\chi^c}{2}\,, \tag{2.2}$$

$$F^a_{\mu\nu}=\partial_\mu B^a_\nu - \partial_\nu B^a_\mu - gf_{abc}B^b_\mu B^c_\nu\,. \tag{2.3}$$

We will split L as follows:

$$L=L_0+L_{I\Psi}+L_{IB}\,.$$

L_0 is the free Lagrangian

$$L_0=\underline{\overline{\Psi}}(i\gamma^\mu\partial_\mu - M)\underline{\Psi} - \tfrac{1}{4}(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu)^2 \tag{2.4}$$

and $L_{I\Psi}$ and L_{IB} are the interacting spinor and self-interacting gauge field terms, respectively, namely

$$L_{I\Psi}=-g\underline{\overline{\Psi}}\gamma^\mu B^a_\mu \frac{\chi^a}{2}\underline{\Psi}\,, \tag{2.5}$$

and L_{IB} contains the cross terms arising from (2.3). In this section we will focus our attention on the $L_{I\Psi}$ interaction term.

For simplicity, let the spinors have equal masses, $M=m\,1$. In the interaction picture explicit momentum expansions can be written that hold for all times:

$$B^a_\mu(t,x)=\frac{1}{(2\pi)^{3/2}}\sum_\lambda\int\frac{d^3k}{(2\omega_k)^2}\epsilon_\mu(k\lambda)[a_a(k\lambda)e^{-ik\cdot x}+a^\dagger_a(k\lambda)e^{ik\cdot x}]\,, \tag{2.6a}$$

$$\underline{\Psi}(t,x)=\sum_{\pm s}\int\frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m}{E_p}\right)^{1/2}[\underline{b}(ps)u(ps)e^{-ip\cdot x}+\underline{d}^\dagger(ps)v(ps)e^{ip\cdot x}]\,, \tag{2.6b}$$

where the momentum-space operators $a^\dagger_a$, $\underline{b}$, and $\underline{d}$ obey free field equations of motion. Their normalization is chosen such that their (anti) commutation relations are

$$[b_a(ps),b^\dagger_c(p's')]=\delta_{ss'}\delta^3(p-p')\delta_{ac}\,, \tag{2.7a}$$

$$[a_a(k\lambda),a^\dagger_b(k'\lambda')]=(2\omega_k)^3\delta^3(k-k')\delta_{\lambda\lambda'}\delta_{ab}\,. \tag{2.7b}$$

The unusual normalization for the boson operators $a(k\lambda)$ was chosen to simplify the form of the asymptotic expansion of the interaction Hamiltonian. Of course the choice of normalization is arbitrary. Completely equivalent results would be obtained with, for instance, pure $\delta^3(k-k')$ normalization provided one specifies that when applied to the relevant states $\lim_{k\to 0}\omega_k^{3/2}a(k\lambda)$ is finite.

By substitution of the expansions (2.6) and (2.7) in Eq. (2.5) one obtains a long expression for the corresponding interaction Hamiltonian term which, for the benefit of the reader, we reproduce here because it is from the analysis of its asymptotic behavior that all our results will follow:

$$H_{I\Psi}(t)=\frac{g}{(2\pi)^{3/2}}\sum_{\lambda}\sum_{\pm s,s'}\int d^3p\,d^3p'\frac{m}{(E_pE_{p'})^{1/2}}$$
$$\times\Bigg\{\bar{u}(ps)\gamma^{\mu}u(p's')\underline{b}^{\dagger}_{ps}\frac{\chi^a}{2}\underline{b}_{p's'}\left[\epsilon_\mu(\vec{p}-\vec{p}'\lambda)\frac{a_a(p-p'\lambda)}{(2\omega_{p-p'})^2}e^{i(p^0-p'^0-\omega_{p-p'})t}+\epsilon_\mu(\vec{p}'-\vec{p},\lambda)\frac{a_a^{\dagger}(p'-p,\lambda)}{(2\omega_{p'-p})^2}e^{i(p^0-p'^0+\omega_{p-p'})t}\right]$$
$$+\bar{u}(ps)\gamma^{\mu}v(p's')\underline{b}^{\dagger}_{ps}\frac{\chi^a}{2}\underline{d}^{\dagger}_{p's'}\left[\epsilon_\mu(\vec{p}+\vec{p}'\lambda)\frac{a_a(p+p'\lambda)}{(2\omega_{p+p'})^2}e^{i(p^0+p'^0-\omega_{p+p'})t}+\epsilon_\mu(-\vec{p}-\vec{p}'\lambda)\frac{a_a^{\dagger}(-p-p'\lambda)}{(2\omega_{-p-p'})^2}e^{i(p^0+p'^0+\omega_{p+p'})t}\right]$$
$$+\bar{v}(ps)\gamma^{\mu}u(p's')\underline{d}_{ps}\frac{\chi^a}{2}\underline{b}_{p's'}\left[\epsilon_\mu(-\vec{p}-\vec{p}'\lambda)\frac{a_a(-p-p'\lambda)}{(2\omega_{-p-p'})^2}e^{-i(p^0+p'^0+\omega_{-p-p'})t}+\epsilon_\mu(\vec{p}+\vec{p}'\lambda)\frac{a_a^{\dagger}(p+p'\lambda)}{(2\omega_{p+p'})^2}e^{-i(p^0+p'^0-\omega_{p+p'})t}\right]$$
$$+\bar{v}(ps)\gamma^{\mu}v(p's')\underline{d}_{ps}\frac{\chi^a}{2}\underline{d}^{\dagger}_{p's'}\left[\epsilon_\mu(\vec{p}'-\vec{p},\lambda)\frac{a_a(p'-p\lambda)}{(2\omega_{p'-p})^2}e^{-i(p^0-p'^0+\omega_{p'-p})t}+\epsilon_\mu(\vec{p}-\vec{p}'\lambda)\frac{a_a^{\dagger}(p-p'\lambda)}{(2\omega_{p-p'})^2}e^{-i(p^0-p'^0-\omega_{p-p'})t}\right]\Bigg\}. \quad (2.8)$$

Our purpose is to derive the leading term in an asymptotic expansion of $H_{I\Psi}(t)$ as $t\to\pm\infty$. Let us analyze the first term in the right-hand side of Eq. (2.8). For fixed $\vec{p}$ it involves the following integration over $\vec{p}'$:

$$\int p'^2dp'd(\cos\theta)d\phi\, e^{ih(p,p',\cos\theta)t}f(p,p')/(2\omega_{p-p'})^2, \quad (2.9a)$$

where

$$f(p,p')=\sum_{\lambda}\sum_{ss'}\frac{m}{(E_pE_{p'})^{1/2}}\bar{u}(ps)\gamma^{\mu}u(p's')\underline{b}^{\dagger}_{ps}\frac{\chi^a}{2}\times\underline{b}_{p's'}\epsilon_\mu(\vec{p}-\vec{p}'\lambda)a_a(\vec{p}-\vec{p}'\lambda), \quad (2.9b)$$

$$h(p,p',\cos\theta)=(p^2+m^2)^{1/2}-(p'^2+m^2)^{1/2}-(p^2+p'^2-2pp'\cos\theta+\mu^2)^{1/2}. \quad (2.9c)$$

It is understood that the physical results are obtained in the limit $\mu^2\to 0$. The exact nature of this limit is discussed below.

In Eq. (2.9) we perform first the integration in $p'=|\vec{p}'|$ at fixed $\cos\theta$ and ϕ. The phase $h(p,p',\cos\theta)$ has a nondegenerate critical point ($\partial h/\partial p'=0$, $\partial^2h/\partial p'^2\neq 0$) at

$$p'=p'_c=\frac{mp\cos\theta}{m+\mu'},$$

with

$$\mu'=[\mu^2+p^2(1-\cos^2\theta)]^{1/2},$$

$$\frac{d^2h}{dp'^2}\bigg|_{p'=p'_c}=-\left(\frac{1}{m}+\frac{1}{\mu'}\right)\left[\frac{p^2\cos^2\theta}{(m+\mu')^2}+1\right]^{-3/2}<0.$$

Using the stationary-phase method (see the Appendix) for the integration in p', the leading term, when $|t|\to\infty$, is found to be

$$\frac{\sqrt{\pi}}{|t|^{1/2}}e^{-i\epsilon(t)\pi/4}\int d(\cos\theta)d\phi\, p'^2_cf(p,p'_c)\left(\frac{m\mu'}{m+\mu'}\right)^{1/2}\times\left[\frac{p^2\cos^2\theta}{(m+\mu')^2}+1\right]^{3/4}\frac{e^{ih(p,p'_c,\cos\theta)t}}{(2\omega_{p-p'_c})^2}. \quad (2.10)$$

The remaining phase $h(p,p'_c,\cos\theta)$ has a critical point $\partial h/d\cos\theta=0$ at $\cos\theta=0$. If one applies again the method of stationary phase to the integration in $\cos\theta$ one obtains a vanishing contribution for the leading term because the integrand contains the factor $p'^2_c\sim\cos^2\theta$. This means that the overall contribution of this critical point vanishes at least as fast as $|t|^{-3/2}$ when $|t|\to\infty$.

One should notice, however, that the method of stationary phase is applicable only for regular integrand functions. If there are singularities, then other methods, namely the methods of the theory of distributions, must be applied to obtain the contributions of the regions near the singularities. If $\mu^2=0$ the integral (2.10) has indeed a singular point at $\cos\theta=1$. In the neighborhood of $\theta=0$ the integral in $\cos\theta$ becomes

$$\frac{1}{4}\int_0^{\epsilon} d\theta \frac{1}{\sqrt{\theta}} \exp\left[-i\frac{mp}{(p^2+m^2)^{1/2}}\theta t\right] \times \sqrt{p}\left(\frac{p^2}{m^2}+1\right)^{-1/4} f(p,p_c') = \frac{1}{4}\int_0^{\epsilon'} dx \frac{1}{\sqrt{x}} e^{-ixt} f(p,p_c'),$$

where the last step is obtained by the change of variables

$$x = \frac{mp}{(p^2+m^2)^{1/2}}\theta .$$

Applying Eq. (A6) this integral is seen to have a leading asymptotic contribution equal to

$$\frac{1}{|t|^{1/2}} \frac{\pi}{\Gamma(\frac{1}{2})\cos(\frac{1}{4}\pi)} \frac{1}{4} f(p,p). \qquad (2.11)$$

Substituting in (2.10) it leads to a contribution of order $|t|^{-1}$ which cannot be integrated over t and therefore implies a nontrivial asymptotic behavior.

A remark is in order concerning the massless limits $\mu^2 \to 0$. The denominator that originates the singular points is

$$(\omega_{\vec{p}} - _{\vec{p}_c'})^2 = \frac{p^2\cos^2\theta\,\mu'^2}{(m+\mu')^2} + \mu'^2 . \qquad (2.12)$$

For $\mu^2 > 0$ this denominator never vanishes, therefore, if the (massless) gauge theory is considered as the $\mu^2 \to 0^+$ limit of a massive theory the interaction vanishes always as fast as $|t|^{-3/2}$ and asymptotically it would be a free field theory.

A nontrivial asymptotic behavior exists only at the point $\mu^2 = 0$. Therefore, gauge theories are qualitatively different from massless limits of massive theories. If one were to insist on treating the massless theory as a $\mu^2 \to 0$ limit, then the only consistent possibility seems to be to have $\mu^2 < 0$ and a $\mu^2 \to 0^-$ limit [notice that for $\mu^2 < 0$ (2.12) has zeros at $\cos^2\theta = 1 + \mu^2/p^2$].

To complete the calculation of the contribution of the first term of Eq. (2.8) to the asymptotic expansion of the interaction Hamiltonian, the only thing remaining now is the computation of

$$f(p,p) = \lim_{\vec{p}' \to \vec{p}} f(p,p_c') .$$

From Eq. (2.9b),

$$\lim_{\vec{p}' \to \vec{p}} f(p,p') = \sum_s \frac{1}{E_p} \underline{b}^{\dagger}_{ps} \frac{\chi^a}{2} \underline{b}_{ps} \times \lim_{p' \to p} \sum_{\lambda} p^{\mu} \epsilon_{\mu}(\vec{p}-\vec{p}'\lambda) a_a(\vec{p}-\vec{p}'\lambda),$$

where we have used $\bar{u}(ps)\gamma^{\mu}u(ps') = (p^{\mu}/m)\delta_{ss'}$. On the other hand,

$$\lim_{\substack{\vec{p}\to\vec{p}'\\ \mu^2\to 0}} \left(\sum_{\lambda=1}^{3} p^{\mu}\epsilon_{\mu}(\vec{p}-\vec{p}'\lambda)a_a(\vec{p}-\vec{p}'\lambda)\right) = -p^k \lim_{p\to p'} a_a(p-p'k) = -p^k a_a(0k)_{\epsilon(t)}\, e^{+i\epsilon(t)\pi/4},$$

where we defined $a(0\lambda)_{\epsilon(t)} = \lim_{p \to p'} a(p-p'\lambda)e^{-i\epsilon(t)\pi/4}$, the "infrared boson operator." Notice that because the phase contains $\epsilon(t) \equiv \operatorname{sgn} t$ we are in fact working with two different infrared operators for $t \to +\infty$ and $t \to -\infty$ [$a(0\lambda)_- = ia(0\lambda)_+$]. This choice is made to simplify the form of the asymptotic interaction.

To have the infrared boson operators completely determined one has to specify their commutation relations with their adjoints. From Eq. (2.7b) one sees that the unusual normalization chosen for the $a(k\lambda)$ operators is, in fact, the one appropriate to have a consistent limit when $\vec{k}$ and $\vec{k}'$ approach zero. Because $(2\omega_k)^3\delta^3(k-k')$ is a dimensionless quantity we can consistently set

$$[a_a(0\lambda), a_b^{\dagger}(0\lambda')] = \text{const} \times \delta_{\lambda\lambda'}\delta_{ab},$$

or considering the constant absorbed in the definition of the operators, we can obtain simply

$$[a_a(0\lambda), a_b^{\dagger}(0\lambda')] = \delta_{\lambda\lambda'}\delta_{ab}.$$

Putting together the results of all calculations one finally obtains, for the leading asymptotic behavior of the first term of Eq. (2.8),

$$-\frac{1}{|t|} g \frac{\sqrt{\pi}}{4} \sum_s \int d^3p \frac{p^k}{(p^2+m^2)^{1/2}} \underline{b}^{\dagger}_{ps} \frac{\chi^a}{2} \underline{b}_{ps} a_a(0k)_{\epsilon(t)} .$$

The computation of the contributions of the other terms of Eq. (2.8) follows similar steps, and we merely make a few remarks about them. The calculation for the second, seventh, and eighth terms is absolutely similar to the computation of the first with simple replacements of $\underline{b}^{\dagger}\frac{1}{2}\chi\underline{b}$ by $\underline{d}\frac{1}{2}\chi\underline{d}^{\dagger}$ and a by $a^{\dagger}$.

For the computation of the fifth term we change the integral from $\vec{p}$ to $-\vec{p}$. An overall $\exp(-i2p^0t)$ phase is obtained multiplying the same phase $\exp[ih(p,p',\cos\theta)]$ as in the first term. The computation proceeds similarly and the only other difference lies in the spinors where now the relevant bilinear is

$$(\bar{v}(-ps)\gamma^{\mu}u(ps'))^{\dagger} = -(\sigma^i)_{s',-s}\left[\left(\frac{p^0}{m}-1\right)\frac{p^{\mu}p^i}{p^2} + \delta^{\mu i}\right].$$

The computation of the third, fourth, and sixth terms is similar to the computation of the fifth and putting together all contributions one finally obtains, for the leading asymptotic behavior of the interaction Hamiltonian,

$$H_I^{\text{asym}}(t) = -\frac{1}{|t|}g\frac{\sqrt{\pi}}{4}\sum_s \int d^3p \frac{p^k}{(p^2+m^2)^{1/2}}\left(\underline{b}^\dagger_{ps}\frac{\chi^a}{2}\underline{b}_{ps}+\underline{d}_{ps}\frac{\chi^a}{2}\underline{d}^\dagger_{ps}\right)[a_a(0k)_{\epsilon(t)}+a_a^\dagger(0k)_{\epsilon(t)}]$$
$$+\frac{1}{|t|}g\frac{\sqrt{\pi}}{4}\sum_{s,s'}\int d^3p \frac{m}{(p^2+m^2)^{1/2}}e^{-i2p^0t}f^k_{s,s'}\underline{d}_{ps}\frac{\chi^a}{2}\underline{b}_{ps'}[a_a(0k)_{\epsilon(t)}+a_a^\dagger(0k)_{\epsilon(t)}]+\text{c.c.}, \tag{2.13}$$

where

$$f^k_{s,s'} = (\sigma^i)^*_{s',-s}\left[\left(\frac{p^0}{m}-1\right)\frac{p^kp^i}{p^2}+\delta^{ki}\right].$$

On the integration in $\vec{p}$ required to compute matrix elements of this Hamiltonian one should yet consider the effects of the oscillating phase $\exp(-i2p^0t)$. Noticing that $dp^0/dp = p/(p^2+m^2)^{1/2}$, the only critical point at $p=0$ is seen not to contribute in the application of the stationary-phase method because the volume element d^3p contains p^2dp and $p^2\to 0$ at the critical point. Therefore, the matrix elements coming from the second term behave at least as $1/|t|^2$ when $|t|\to\infty$. Therefore, the asymptotic dynamics is dominated by the first part of Eq. (2.13) alone.

Obtained by rigorous asymptotic methods, this asymptotic interaction has a form similar to the Faddeev and Kulish ansatz[2] for the asymptotic dynamics of QED. Although rigorous our result is actually simpler because it only involves an integration over one spinor momentum and the infrared boson operators are perfectly determined. We now study the asymptotic dynamics implied by this interaction for Abelian and non-Abelian gauge theories.

III. ASYMPTOTIC DYNAMICS

In this section one studies the structure of the asymptotic space generated by the leading term of the interaction Hamiltonian, Eq. (2.13),

$$V_{as}^D(t) = -\frac{g\sqrt{\pi}}{4|t|}\sum_s\int d^3p\left(\underline{b}^\dagger_{ps}\frac{\chi^a}{2}\underline{b}_{ps}+\underline{d}_{ps}\frac{\chi^a}{2}\underline{d}^\dagger_{ps}\right)\times\frac{p^k}{p^0}[a_a(0k)_{\epsilon(t)}+a_a^\dagger(0k)_{\epsilon(t)}]. \tag{3.1}$$

All results in this section pertain to the fermion sector. The asymptotic space for the boson sector of non-Abelian gauge theories will be analyzed in Sec. IV.

The asymptotic states (in the interaction picture) are the solutions $|ta\rangle^D$ of the equation

$$i\frac{\partial}{\partial t}|ta\rangle^D = :V_{as}^D(t):|ta\rangle^D. \tag{3.2}$$

To solve this equation one has to choose an appropriate basis. For this purpose one first finds the eigenvalues of $a(0\lambda)+a^\dagger(0\lambda)$.

In the usual Fock space representation the eigenvectors of boson operators are $|0\rangle, a^+(k\lambda)|0\rangle,\ldots,[a^\dagger(k\lambda)]^n|0\rangle$, with the lowest-energy state $|0\rangle$ defined by $a(k\lambda)|0\rangle=0$. If it is also $a(0\lambda)|0\rangle=0$, then

$$[a(0\lambda)+a^\dagger(0\lambda)]|0\rangle = a^\dagger(0\lambda)|0\rangle \neq \text{const}|0\rangle,$$

i.e., $|0\rangle$ is not an eigenstate of $a(0\lambda)+a^\dagger(0\lambda)$. In the same way $[a^\dagger(0\lambda)]^n|0\rangle$ is not an eigenstate of $a(0\lambda)+a^\dagger(0\lambda)$. Therefore, in the spectrum we could maintain $a(k\lambda)|0\rangle=0$ for $k\neq 0$ but not for $k=0$. Therefore, we have to find a different basis of vectors that diagonalizes an operator of the form $a+a^\dagger$ (with $[a,a^\dagger]=1$).

Although this is not the most efficient and mathematically correct way of doing it there is some interest in doing this calculation in the framework of the Fock space representation, i.e., in a space with a basis $\{|n\rangle=(1/\sqrt{n!})(a^\dagger)^n|0\rangle, n=0,1,2,\ldots\}$. If $a+a^\dagger$ has an eigenvector

$$(a+a^\dagger)|\lambda\rangle = \lambda|\lambda\rangle,$$

its coefficients in a Fock space basis $|\lambda\rangle=\sum_n c_n|n\rangle$ must satisfy

$$\lambda c_n = c_{n-1}\sqrt{n}+c_{n+1}(n+1)^{1/2}.$$

The solution of this equation is

$$c_n = c_0\frac{H_n(\lambda/\sqrt{2})}{(2^nn!)^{1/2}}$$

where $H_n(\;)$ are Hermite polynomials. Therefore

$$|\lambda\rangle = c_0\sum_{n=0}^\infty\frac{H_n(\lambda/\sqrt{2})}{n!\,2^n}(a^\dagger)^n|0\rangle$$
$$= c_0e^{\lambda a^\dagger-(a^\dagger/\sqrt{2})^2}|0\rangle, \tag{3.3}$$

where in the second step we used the definition of the generating function for Hermite polynomials.

Defining $|\Omega\rangle$ as the state corresponding to the $\lambda=0$ eigenvector,

$$|\Omega\rangle = |\lambda=0\rangle = c_0e^{-(a^\dagger/\sqrt{2})^2}|0\rangle, \tag{3.4}$$

the space of eigenvectors of $a+a^\dagger$ contains $|\Omega\rangle$ and coherent states of the form

$$|\lambda\rangle = e^{\lambda a^\dagger}|\Omega\rangle. \tag{3.5}$$

The fundamentally nonperturbative nature of the spectrum of the Hermitian infrared boson operator $a(0\lambda)+a^\dagger(0\lambda)$ is clearly seen by attempting for example to compute the Fock space norm of $|\Omega\rangle$,

$$\langle\Omega|\Omega\rangle = |c_0|^2 \sum_n \frac{[H_n(0)]^2}{2^n n!},$$

which, of course, is found to be divergent. This is really the interest of doing this calculation in the Fock space basis because one then sees clearly that the eigenstates of $a+a^\dagger$ lie outside the Fock space, in which case it would be quite hopeless to obtain a correct evaluation of the infrared effects in a perturbative way.

Leaving aside the Fock space considerations we may, in the space of eigenstates of $a+a^\dagger$, normalize $|\Omega\rangle$ by definition, $\langle\Omega|\Omega\rangle=1$, and have a set (3.5) of well-defined eigenvectors,

$$(a+a^\dagger)\, e^{\lambda a^\dagger}|\Omega\rangle = \lambda\, e^{\lambda a^\dagger}|\Omega\rangle .$$

Reintroducing the polarization indices

$$|\Omega\rangle = \exp\{-[\underline{a}^\dagger(0k)\cdot\underline{a}^\dagger(0k)]/2\}|0\rangle , \tag{3.6}$$

where a sum over k and a dot product on the internal indices are implied.

A general eigenvalue of $a_a(0k)+a_a^\dagger(0k)$ will be

$$|\lambda\rangle = e^{\lambda_k^a a_a^\dagger(0k)}|\Omega\rangle , \tag{3.7}$$

where in particular the coefficient λ_k^a in this coherent state is chosen of the form

$$\lambda_k^a = v_k l^a$$

to have independence between internal and space-time degrees of freedom.

Let us now treat separately the Abelian and the non-Abelian cases.

Abelian theory

In this case choose a basis

$$\{|\lambda\rangle,\ b_{p_1s_1}^\dagger \cdots b_{p_ns_n}^\dagger d_{q_1s_1}^\dagger \cdots d_{q_ms_m}^\dagger|\lambda\rangle\}, \tag{3.8}$$

of fermion states obtained by application of arbitrary finite numbers of fermion and antifermion creation operators to a particular $|\lambda\rangle$ state. In this basis Eq. (3.2) becomes $[\langle nm|ta\rangle^D = \psi_{nm}^\lambda(t)]$,

$$i\frac{\partial}{\partial t}\psi_{nm}^\lambda(t) = \langle nm|:V_{as}^D(t):|ta\rangle^D$$
$$= -\frac{g\sqrt{\pi}}{4|t|}\left(\sum_n \frac{\vec{p}_n\cdot\vec{\lambda}}{p_n^0} - \sum_m \frac{\vec{q}_m\cdot\vec{\lambda}}{q_m^0}\right)\psi_{nm}^\lambda(t). \tag{3.9}$$

The normal ordering of $:V_{as}^D(t):$ can be carried out without ambiguity in the interaction picture. The solution of (3.9) is

$$\psi_{nm}^\lambda(t) = \psi_{nm}(ps|qr) \times \exp\left[i\epsilon(t)\ln\frac{|t|}{t_0}\left(\sum_n \frac{\vec{p}_n\cdot\vec{\lambda}}{p_n^0} - \sum_m \frac{\vec{q}_m\cdot\vec{\lambda}}{q_m^0}\right)\frac{g\sqrt{\pi}}{4}\right]. \tag{3.10}$$

To compute the energy of the asymptotic states,

$$\langle nm|:H_{as}^D:|ta\rangle^D = \left[\sum_n\left(p_n^0 - \frac{g\sqrt{\pi}}{4|t|}\frac{\vec{p}_n\cdot\vec{\lambda}}{p_n^0}\right) + \sum_m\left(q_m^0 - \frac{g\sqrt{\pi}}{4|t|}\frac{\vec{q}_m\cdot\vec{\lambda}}{q_m^0}\right)\right]\langle nm|ta\rangle^D . \tag{3.11}$$

When $|t|\to\infty$, which is the situation where $H^D \sim H_{as}^D$ and the asymptotic states become identical to the states of the actual theory, the energy becomes

$$\sum_n p_n^0 + \sum_m q_m^0 ,$$

i.e., identical to the energy of a state of n free particles and m free antiparticles.

The states $\psi_{nm}^\lambda(t)$ of Eq. (3.10) define therefore a consistent set of finite-energy asymptotic states for the Abelian theory.

For S-matrix calculations the Dyson operator $S_D(t_1,t_2)$ should operate between the true asymptotic states, i.e.,

$$\lim_{|t|\to\infty}\psi_{nm}^\lambda(t).$$

However, notice that, at least formally, one can define an operator

$$U_\lambda(t) = \exp[-\tfrac{1}{2}a^\dagger(0k)a^\dagger(0k)]\exp\left[i\epsilon(t)\ln\frac{|t|}{t_0}\sum_s\int d^3p\,(b_{ps}^\dagger b_{ps} - d_{ps}d_{ps}^\dagger)\frac{g\sqrt{\pi}}{4}\frac{\vec{p}\cdot\vec{\lambda}}{p^0}\right]\exp[\vec{\lambda}\cdot\vec{a}(0)],$$

which maps the usual Fock space into the true asymptotic space. Therefore the S matrix can be formally redefined,

$$S = \lim_{\substack{t_2\to-\infty\\ t_1\to\infty}} U_\lambda^\dagger(t_1)S_D(t_1,t_2)U_\lambda(t_2),$$

so that it operates in the usual Fock space.

From Eq. (3.11) it is clear that for $|t|\to\infty$, the $|\lambda\rangle$ states are zero-energy states; they form therefore a degenerate set of vacuums. For each $|\lambda\rangle$ one obtains a different space $\mathcal{H}_a^\lambda$ of asymptotic states.

The physical results, however, do not depend on the choice of λ. In particular each $\mathcal{H}_a^\lambda$ is invariant for global U(1) gauge transformations $U(\alpha) = \exp(i\alpha Q)$ as can be seen from

$$\langle nm|U(\alpha)|ta\rangle^D = \exp[i\alpha(n-m)]\langle nm|ta\rangle^D .$$

Hence, the vacuum degeneracy does not imply spontaneous breaking of U(1) invariance. Because of the superselection rule that is known to operate for gauge charges[7] the space $\mathcal{H}_a^\lambda$ should be considered decomposed in disjoint coherent subspaces of well-defined charge,

$$\mathcal{H}_a^\lambda = \bigoplus_q \mathcal{H}_{aq}^\lambda ,$$

with no superposition of states in different sectors being allowed as a physical state. U(1) global transformations also map each $\mathcal{H}_{aq}^\lambda$ into itself.

In conclusion, the states $\psi_{nm}^\lambda(t)$ of Eq. (3.10) supplemented by the choice of basis (3.8) define a completely consistent set of asymptotic states for Abelian gauge theory containing arbitrary numbers of fermions and antifermions.

Non-Abelian theory

Denoting by $\langle\,|$ a vector of the basis to be chosen in this case, Eq. (3.2) becomes

$$i\frac{\partial}{\partial t}\langle\,|ta\rangle^D = -\left\langle\left|\sum_s \int d^3p \frac{g\sqrt{\pi}}{4|t|}\left(\underline{b}_{ps}^\dagger \frac{\chi^a}{2}\underline{b}_{ps} + :\underline{d}_{ps}\frac{\chi^a}{2}\underline{d}_{ps}^\dagger :\right) l^a \frac{\vec{p}\cdot\vec{v}}{p^0}\right| ta\right\rangle^D .$$

To solve this equation one should choose a basis diagonalizing $l^a\chi^a$.

To be specific let us analyze in detail the case of an SU(2) gauge group. As we will see the treatment is completely analogous for higher gauge groups. For the SU(2) gauge group,

$$\frac{\chi^a}{2} l^a = \tfrac{1}{2}\begin{pmatrix} l^3 & l^1 - il^2 \\ l^1 + il^2 & -l^3 \end{pmatrix}. \tag{3.12}$$

In this case $\underline{b}_{ps}^\dagger, \underline{b}_{ps}, \underline{d}_{ps}^\dagger, \underline{d}_{ps}$ are two spinors, and in writing (3.12) the basis chosen is such that it diagonalizes the Q_3 charge.

Let us diagonalize the matrix. The eigenvalues are $\pm\frac{1}{2}|\underline{l}|$. Denote by $U(\lambda)$ the unitary matrix that diagonalizes $\frac{1}{2}\chi^a l^a$,

$$\tfrac{1}{2}\chi^a l^a = U(\lambda)\Gamma(\lambda)U(\lambda), \quad \Gamma(\lambda) = |\underline{l}|\tfrac{1}{2}\sigma^3 .$$

For this particular fixed $\lambda_k^a = v_k l^a$, we therefore choose for the fermion sector of the basis a set of spinors created by operators $\underline{\beta}_{ps}(\lambda) = U^\dagger(\lambda)\underline{b}_{ps}$ and $\underline{\delta}_{ps}^\dagger(\lambda) = U^\dagger(\lambda)\underline{d}_{ps}^\dagger$, i.e.,

$$\{|\lambda\rangle,\ |nm\rangle = \underline{\beta}_{p_1s_1}^\dagger(\lambda)\cdots\underline{\beta}_{p_ns_n}^\dagger(\lambda)\underline{\delta}_{p_1s_1}^\dagger(\lambda)\cdots\underline{\delta}_{p_ms_m}^\dagger(\lambda)|\lambda\rangle\},\quad n,m = 1,2,\ldots .$$

Therefore, because

$$(\underline{b}_{ps}^\dagger \tfrac{1}{2}\chi^a \underline{b}_{ps} + :\underline{d}_{ps}\tfrac{1}{2}\chi^a\underline{d}_{ps}^\dagger:)l^a = \underline{\beta}_{ps}^\dagger(\lambda)\Gamma(\lambda)\underline{\beta}_{ps}(\lambda) - \underline{\delta}_{ps}^\dagger(\lambda)\Gamma(\lambda)\underline{\delta}_{ps}(\lambda)$$

in this basis, the equation for $\langle nm|ta\rangle^D = \psi_{nm}^\lambda(t)$ is

$$i\frac{\partial}{\partial t}\psi_{nm}^\lambda(t) = -\frac{g\sqrt{\pi}}{4|t|}\left(\sum^n \frac{p\cdot v}{p^0}q(\lambda) - \sum^m \frac{p'\cdot v}{p'^0}q'(\lambda)\right)\tfrac{1}{2}|\underline{l}|\psi_{nm}^\lambda(t), \tag{3.13}$$

where $q(\lambda)$ and $q'(\lambda)$ (with values $+1$ or -1) are the charge projections of particles and antiparticles defined in relation to the λ direction in the internal space. The solution of Eq. (3.13) is

$$\psi_{nm}^\lambda(t) = \psi(psq(\lambda)|p's'q'(\lambda))\exp\left[i\epsilon(t)\ln\frac{|t|}{t_0}\left(\sum^n\frac{p\cdot v}{p^0}q(\lambda) - \sum^m \frac{p'\cdot v}{p'^0}q'(\lambda)\right)\right]. \tag{3.14}$$

The functions $\psi(psq(\lambda)|p's'q'(\lambda))$ define the occupation of states with a certain number n of particles and m antiparticles with well-defined charges quantized along the internal direction.

The asymptotic space $\mathcal{H}_a^{\underline{\lambda}}$ is the space of the solutions (3.14). Let us examine the transformation properties of $\mathcal{H}_a^{\underline{\lambda}}$ under an SU(2) gauge transformation. One obtains from

$$\begin{aligned}\langle nm|U^{-1}(\underline{\alpha})U(\underline{\alpha})|ta\rangle^D &= \langle\underline{\lambda}|\beta_{p_1s_1q_1}\cdots\delta_{p_1's_1'q_1'}\cdots U^{-1}(\underline{\alpha})U(\underline{\alpha})|ta\rangle^D\\ &= \langle\underline{\lambda}|[U^\dagger(\lambda)\underline{b}_{p_1s_1}]_{q_1}\cdots[\underline{d}_{p_1's_1'}U(\lambda)]_{q_1'}\cdots U^{-1}(\underline{\alpha})U(\underline{\alpha})|ta\rangle^D\\ &= \langle D^{(1)}(\underline{\alpha})\underline{\lambda}|[U^\dagger(\lambda)D^{(1/2)\dagger}(\alpha)\underline{b}_{p_1s_1}]_{q_1}\cdots[\underline{d}_{p_1's_1'}D^{(1/2)}(\alpha)U(\lambda)]_{q_1'}U(\alpha)|ta\rangle^D ,\end{aligned}$$

that by applying an SU(2) gauge transformation to a state containing n particles and m antiparticles, one obtains a new state containing the same number of particles and antiparticles but lying in a different space and having charges quantized along a different internal direction $D^{(1)}(\underline{\alpha})\underline{\lambda}$.

The global SU(2) transformations therefore map $\mathcal{H}_a^{\underline{\lambda}}$ into a different space $\mathcal{H}_a^{D(\underline{\alpha})\underline{\lambda}}$ of states $\psi_{nm}^{D(\underline{\alpha})\underline{\lambda}}(t)$

defined in relation to a basis generated by application to $|D(\underline{\alpha})\underline{\lambda}\rangle$ of arbitrary numbers of the rotated operators $\underline{\beta}^{\dagger\underline{\alpha}}_{ps} = b^{\dagger}_{ps}D^{1/2}(\underline{\alpha})U(\underline{\lambda})$ and $\underline{\delta}^{\dagger\underline{\alpha}}_{ps} = U^{\dagger}(\underline{\lambda})D^{1/2\dagger}(\underline{\alpha})d^{\dagger}_{ps}$. That is, if $\mathcal{H}^{\underline{\lambda}}_{\underline{a}}$ is the space of physical states the global gauge symmetry is spontaneously broken.

The space $\mathcal{H}^{D(\underline{\alpha})\underline{\lambda}}_a$ is physically equivalent to $\mathcal{H}^{\underline{\lambda}}_{\underline{a}}$ the only difference is that the eigenstates of the Hamiltonian are eigenstates of different linear combinations of the internal generators (Q_i). If one wants to avoid the spontaneous symmetry breaking and construct a manifestly [SU(2)] invariant theory one may use a suitable restriction in the algebra of observables or explicitly integrate over the manifold of $\mathcal{H}^{D(\underline{\alpha})\underline{\lambda}}_a$ spaces. In this latter case the basis states are obtained by direct integrals, namely

$$|0\rangle = \int_{\oplus} |D^{(1)}(\underline{\alpha})\lambda\rangle d\mu(\underline{\alpha}),$$

with $\mu(\underline{\alpha})$ an invariant group measure, is the invariant vacuum and

$$\int_{\oplus} \underline{\delta}^{\dagger\underline{\alpha}}_{ps} |D^{(1)}(\underline{\alpha})\lambda\rangle d\mu(\underline{\alpha})$$

and

$$\int_{\oplus} \underline{\beta}^{\dagger\underline{\alpha}}_{ps} |D^{(1)}(\underline{\alpha})\lambda\rangle d\mu(\underline{\alpha})$$

form a basis for one-particle states.

The asymptotic space is in this case the direct integral space $\mathcal{H}_a = \int_{\oplus} \mathcal{H}^{D(\underline{\alpha})\underline{\lambda}}_a d\mu(\underline{\alpha})$ which is the space of states $\psi^{D(\underline{\alpha})\underline{\lambda}}_{nm}(t)$, each state being defined as a continuous set of functions of the form of Eq. (3.14), i.e., each state corresponds to a function over n, m, t and also the group space $\underline{\alpha}$.

As shown by Strocchi and Wightman[7] there is a superselection rule also for non-Abelian gauge theories if the symmetry is not broken. One therefore divides the asymptotic space $\mathcal{H}_a$ in sectors labeled by the irreducible representations of the global gauge group [in the SU(2) case these are states with a fixed eigenvalue of Q_iQ_i]:

$$\mathcal{H}_a = \mathcal{H}^{(0)}_a + \mathcal{H}^{(1/2)}_a + \mathcal{H}^{(1)}_a + \cdots .$$

In the singlet (zero charge) sector things work nicely because the singlet operators formed from $\underline{\beta}^{\dagger}, \underline{\beta}, \underline{\delta}^{\dagger}, \underline{\delta}$ are, by definition, invariant for all global transformations and therefore are the same for all component spaces and factorize out of the direct integral. The states in $\mathcal{H}^{(0)}_a$ are therefore eigenvectors of the charge Q_i.

For the nonsinglet sectors, however, it turns out that (contrary to what one should expect from a nontrivially charged space) because of the direct integral operation the expectation value of any Q_i charge is zero. For example,

$$\begin{aligned}\langle \beta_j | Q_i | \beta_j^{\dagger}\rangle &= \int d\mu(\underline{\alpha})\langle D(\underline{\alpha})\lambda|\beta_j^{\alpha} Q_i \beta_j^{\alpha\dagger}|D(\underline{\alpha})\lambda\rangle \\ &= \int \sum_i q_i |D_{ij}(\underline{\alpha})|^2 d\mu(\underline{\alpha}) \\ &= \text{const} \sum_i q_i = 0 .\end{aligned}$$

The origin of this result is traced to the fact that in each component space $\mathcal{H}^{D(\underline{\alpha})\underline{\lambda}}_a$ the direction of quantization of charges is a different one. This comes about because in the asymptotic space the eigenstates of the Hamiltonian are nondegenerate and correspond to well-defined linear combinations of the fermion operators. It is only in the limit $|t| \to \infty$ that the up and down states, say, become of the same energy but in the limiting procedure the whole structure is preserved.

With vanishing expectation value everywhere the charge is a trivial unobservable quantity in any sector of the direct integral space. Therefore nothing is gained by using nonsinglet sectors because the expectation value of the charges is zero anyway. Furthermore, in the nonsinglet sectors of the direct integral space the integrated states are not eigenstates of the charges. Hence, to obtain a space where we can proceed with a consistent construction of eigenvectors and operators one should restrict oneself to the singlet sector.

All considerations above apply to the case $\underline{\lambda} \neq 0$. If $\underline{\lambda} = 0$ no spontaneous symmetry breaking would occur. However, the point $\underline{\lambda} = 0$ is unstable in the sense that any small displacement $\underline{\lambda} + \delta\underline{\lambda}$ leads to a qualtitatively different solution.

I have worked out in detail an SU(2) example. From the general structure that emerges from this example it is, however, clear that the same results apply in general to other non-Abelian gauge theories.

In conclusion, non-Abelian gauge theories (except at the isolated point $\underline{\lambda} = 0$) are either spontaneously broken or the asymptotic space of physical fermion states contains only singlet states.

I have some comments on this "confinement" result: Some of the phenomenological views on confinement in non-Abelian theories make it depend on some sort of potential growing to infinity at large distances. Such a potential which is known to occur in $(1+1)$-dimensional QED would then require the elementary spinor states to carry infinite energy and therefore to be unobservable. This is what we may call an "energetic confinement."

The situation we have found here for $(3+1)$-dimensional non-Abelian gaugh theory seems rather different. It is related to a situation where spontaneous breaking is avoided by a restriction on the algebra of observables, corresponding to a "confinement by symmetry restoration," a possibility I have suggested in very general terms elsewhere.[8]

For those used to thinking of confinement in terms of interactions and cancelation of channels it may seem strange that such a general mechanism might lead to such drastic results on the observation of the "color" quantum number. Notice, however, that a complicated physical mechanism may very well exist at the level of the exact theory. What was studied here was the structure of the asymptotic space which presumably leads to the same matrix elements as the exact theory for infinite times. It is therefore possible that an intuitive physically appealing mechanism might exist at the level of the exact theory which, for lack of more detailed interaction structure, the asymptotic space can merely mimic in this very general geometrical way.

IV. BOSON ASYMPTOTIC SPACE

In Abelian theories there are no boson self-interactions but in non-Abelian theories one has third- and fourth-order terms,

$$L_{IB}=L_B^{(3)}+L_B^{(4)},$$

$$L_B^{(3)}(x)=(g/4)f_{abc}[(\partial_\mu B_\nu^a-\partial_\nu B_\mu^a)B^{b\mu}B^{c\nu}+B_\mu^b B_\nu^c(\partial^\mu B^{a\nu}-\partial^\nu B^{a\mu})], \quad (4.1)$$

$$L_B^{(4)}(x)=-(g^2/4)f_{abc}f_{ab'c'}B_\mu^b B_\nu^c B^{b'\mu}B^{c'\nu}. \quad (4.2)$$

The analysis is very similar to the one for the boson-fermion interaction term. Using the momentum expansions and integrating over space the contribution of the third-order terms to the Hamiltonian is

$$\begin{aligned}H_B^{(3)}(t)=\frac{g}{(2\pi)^{3/2}}f_{abc}\sum_{\lambda\lambda'\lambda''}\int\frac{d^3k\,d^3k'}{(4\omega_k\omega_{k'})^2}[k_\mu\epsilon_\nu(k\lambda)-k_\nu\epsilon_\mu(k\lambda)]\epsilon^\mu(k'\lambda')\\ \times\mathrm{Re}\Bigg\{ia_a(k\lambda)\Bigg[\frac{\epsilon^\nu(-k-k'\lambda'')}{(2\omega_{-k-k'})^2}a_b(k'\lambda')a_c(-k-k'\lambda'')\,e^{-i(\omega_k+\omega_{k'}+\omega_{-k-k'})t}\\ +\frac{\epsilon^\nu(k-k'\lambda'')}{(2\omega_{k-k'})^2}a_b^\dagger(k'\lambda')a_c^\dagger(k-k'\lambda'')\,e^{-i(\omega_k-\omega_{k'}-\omega_{k-k'})t}\\ +\frac{\epsilon^\nu(k+k'\lambda'')}{(2\omega_{k+k'})^2}a_b(k'\lambda)a_c^\dagger(k+k'\lambda'')\,e^{-i(\omega_k+\omega_{k'}-\omega_{k+k'})t}\\ +\frac{\epsilon^\nu(k'-k\lambda'')}{(2\omega_{k'-k})^2}a_b^\dagger(k'\lambda')a_c(k'-k\lambda'')\,e^{-i(\omega_k-\omega_{k'}+\omega_{k'-k})t}\Bigg]\Bigg\}. \quad (4.3)\end{aligned}$$

The phases and denominator singularities in these integrals are similar to those in Eq. (2.8) and a similar analysis leads to a leading contribution,

$$H_B^{(3)}(t)\simeq\frac{1}{|t|}g\frac{\sqrt{\pi}}{8}\sum_\lambda\int\frac{d^3k}{(4k^2)^2}k^i f_{abc}\,i[a_a^\dagger(k\lambda)a_b(k\lambda)-a_a(k\lambda)a_b^\dagger(k\lambda)][a_c(0i)_{\epsilon(t)}+a_c^\dagger(0i)_{\epsilon(t)}]. \quad (4.4)$$

For the fourth-order term,

$$\begin{aligned}H_B^{(4)}(t)&=\frac{g^2}{4}\int d^3x\,f_{abc}B_\mu^b B_\nu^c f_{ab'c'}B^{b'\mu}B^{c'\nu}\\ &=\frac{g^2}{4}f_{abc}f_{abc}\frac{1}{(2\pi)^3}\int\frac{d^3k_1d^3k_2d^3k_3}{(8\omega_1\omega_2\omega_3)^2}\sum_{\lambda_1\lambda_2\lambda_3\lambda_4}\epsilon_\mu(k_1\lambda_1)\epsilon_\nu(k_2\lambda_2)\epsilon^\mu(k_3\lambda_3)\epsilon^\nu(k_4\lambda_4)\\ &\quad\times2\,\mathrm{Re}\bigg(a_b^\dagger(k_1\lambda_1)a_c^\dagger(k_2\lambda_2)a_{b'}(k_3\lambda_3)a_{c'}(k_1+k_2-k_3\lambda_4)\\ &\quad\times\frac{\exp[i(\omega_1+\omega_2-\omega_3-\omega_{1+2-3})t]}{(2\omega_{1+2-3})^2}+\cdots\bigg). \quad (4.5)\end{aligned}$$

Let us analyze for example the integral

$$\int d^3k_3 f(k_1,k_2,k_3)\frac{\exp[i(\omega_1+\omega_2-\omega_3-\omega_{1+2-3})t]}{(2\omega_{1+2-3})^2}.$$

Choosing θ and ϕ as the angles of $\vec{k}_3$ with the direction of $\vec{k}_1+\vec{k}_2$, the analysis of the asymptotic behavior proceeds as in Sec. II, leading to

$$\frac{1}{|t|}\frac{\pi^2}{\sqrt{2}}e^{-i\epsilon(t)\pi/4}f(k_1,k_2,k_1+k_2)\,e^{i(\omega_1+\omega_2-\omega_{1+2})t}.$$

The integration in d^3k_2 then contributes at least another $1/|t|$ factor, hence $H_B^{(4)}(t)$ is at most of order $1/|t|^2$ and therefore does not contribute to the asymptotic dynamics.

The leading $1/|t|$ contribution of the third-order term, Eq. (4.4), defines (in the interaction picture) the simplified Hamiltonian V'^D_{as} that one should use to build the boson sector of the asymptotic space in non-Abelian theories. As before one uses the eigenvectors of the infrared operators,

$$|\lambda\rangle = e^{\lambda_k^a a_a^\dagger(0k)}|\Omega\rangle\,, \quad \lambda_k^a = v_k l^a\,.$$

Operating on these states $V'^D_{as}(t)$ becomes

$$V'^D_{as}(t) = \frac{1}{|t|} g\frac{\sqrt{\pi}}{8}\sum_\lambda \int \frac{d^3k}{(4k^2)^2}\vec{k}\cdot\vec{v}\, l^c i f_{abc} \times [a_a^\dagger(k\lambda)a_b(k\lambda) - a_a(k\lambda)a_b^\dagger(k\lambda)]\,. \tag{4.6}$$

To proceed, let us specialize, as we have done in the discussion of the fermion sector, to an SU(2) gauge group. As before the general structure of the results obtained in this particular group can be carried over in a straightforward manner to higher gauge groups.

The equation obeyed by the asymptotic boson states $\phi_n^\lambda(t)$ is

$$i\frac{\partial}{\partial t}\phi_n^\lambda(t) = \langle n|:V'^D_{as}(t):|\phi(t)\rangle = \frac{1}{|t|}\frac{g\sqrt{\pi}}{4}\sum_\mu \int \frac{d^3k}{(4k^2)^2}k\cdot v i f_{abc} \times l^c\langle n|a_a^\dagger(k\mu)a_b(k\mu)|\phi\rangle\,. \tag{4.7}$$

The combination $if_{abc}l^c$ is a nondiagonal Hermitian matrix in the internal space

$$M = \begin{pmatrix} 0 & il_3 & -il_2 \\ -il_3 & 0 & il_1 \\ il_2 & -il_1 & 0 \end{pmatrix},$$

which can be diagonalized by a unitary transformation

$$U^\dagger(\lambda)MU(\lambda) = |\underline{l}| \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}.$$

Defining the new boson operators $\underline{\alpha}_\lambda(k\mu) = U^\dagger(\lambda)\underline{a}(k\mu)$ and a basis

$$\{|\lambda\rangle\,,\ \underline{\alpha}_\lambda^\dagger(k_1\mu_1)\cdots\underline{\alpha}_\lambda^\dagger(k_n\mu_n)|\lambda\rangle\}\,, \tag{4.8}$$

the solution of Eq. (4.7) is

$$\phi_n^\lambda(t) = \phi(k\mu q(\lambda)) \times \exp\left[-i\epsilon(t)\ln\frac{|t|}{t_0} g\frac{\sqrt{\pi}}{4}\sum_{i=1}^n \frac{k_i\cdot v}{2k}|\underline{l}|q_i(\lambda)\right], \tag{4.9}$$

where the $q_i(\lambda)$ are the boson charges (±1 or 0) defined in relation to the λ direction in the internal space.

The states $\phi_n^\lambda(t)$ define the boson sector $\mathcal{H}_{aB}^\lambda$ of the asymptotic space for the non-Abelian theory. The structure is quite similar to the one found for the fermion sector with $\mathcal{H}_{aB}^\lambda$ being noninvariant for global gauge transformations (except at the isolated $\underline{\lambda}=0$ point). We may therefore skip the detailed analysis of Sec. III and state, as before, that for non-Abelian theories either there is spontaneous symmetry breaking or if the breaking is avoided by symmetry restoration the charges are not observable and a consistent boson asymptotic space should be restricted to the singlet sector.

As concerns the relevance of the present results to the quark-gluon theories of strong interactions, one should expect that in a situation of unbroken color symmetry, gluons are, like fermions, only observable in color-singlet combinations. If gluons are flavorless, "truly neutral gluon bound states" would be as probable as color-singlet fermion states.

APPENDIX: ASYMPTOTIC METHODS

In this appendix we summarize the mathematical results used in the text to find the asymptotic behavior of functions defined by integrals.

1. Stationary-phase method

Let

$$F(t) = \int_a^b e^{ith(x)}f(x)dx \tag{A1}$$

and let the phase $h(x)$ have a certain number N of nondegenerate critical points

$$\left.\frac{dh}{dx}\right|_{x=c_i} = 0\,, \quad \left.\frac{d^2h}{dx^2}\right|_{x=c_i} \neq 0\,,$$

in the interval (a, b). Consider

$$\sqrt{|t|}\,F(t) = \sum_{i=i}^N \int_{a_i}^{a_{i+1}} \sqrt{|t|}\,e^{ith(x)}f(x)dx = \sum_i I_i\,,$$

where the decomposition of the interval (a, b) in subintervals is such that $a_1 \equiv a$, $a_{N+1}\equiv b$, and $c_i \in (a_i, a_{i+1})$. Define

$$K_i(t)=\int_{a_i}^{a_{i+1}}\sqrt{|t|}\,e^{ith(x)}dx \tag{A2}$$

and $\epsilon(t)=\text{sgn}t$. Then

$$\begin{aligned}I_i=&K_i(t)f(c_i)\\&+\int_{a_i}^{a_{i+1}}\sqrt{|t|}\,e^{ith(x)}[f(x)-f(c_i)]dx\\=&k_i(t)f(c_i)+\frac{\epsilon(t)}{i\sqrt{|t|}}[e^{ith(a_{i+1})}f_1(a_{i+1})-e^{ith(a_i)}f_1(a_i)]\\&-\frac{\epsilon(t)}{i\sqrt{|t|}}\int_{a_i}^{a_{i+1}}e^{ith(x)}f_1'(x),\end{aligned} \tag{A3}$$

where

$$f_1(x)=\frac{f(x)-f(c_i)}{h'(x)}.$$

If all the functions involved in Eq. (A3) are regular,

$$\lim_{t\to\pm\infty}\sqrt{|t|}\,F(t)=\sum_{i=1}^{N}k_i(\pm\infty)f(c_i) \tag{A4}$$

and the problem is reduced to the computation of the asymptotic limit of $K_i(t)$.

Making in Eq. (A2) the change of variables

$$y^2=\gamma_i[h(x)-h(c_i)],\quad \gamma_i=\text{sgn}\left.\frac{d^2h}{dx^2}\right|_{x=c_i}$$

and applying the same reasoning as in (A3) and (A4) for the nondegenerate critical point at $y=0$, one obtains

$$\begin{aligned}\lim k_i(t)=&e^{ith(c_i)}\left[\sqrt{2}\Big/\left(\gamma_i\frac{d^2h}{dx^2}\right)^{1/2}\Big|_{x=c_i}\right]\\&\times\lim_{t\to\pm\infty}\int_{y(a_i)}^{y(a_{i+1})}\sqrt{|t|}\,e^{it\gamma_i y^2}dy\\=&\left[\sqrt{\pi}\Big/\left(\gamma_i\frac{d^2h}{dx^2}\right)^{1/2}\Big|_{x=c_i}\right]\\&\times\exp\{i[th(c_i)+\epsilon(t)\gamma_i\pi/4]\},\end{aligned}$$

and the final result is

$$\begin{aligned}\lim_{t\to\pm\infty}\sqrt{|t|}\,F(t)=&\sum_{i=1}^{N}f(c_i)\left[\sqrt{\pi}\Big/\left(\gamma_i\frac{d^2h}{dx^2}\right)^{1/2}\Big|_{x=c_i}\right]\\&\times\exp\{i[th(c_i)+\epsilon(t)\gamma_i\pi/4]\}.\end{aligned} \tag{A5}$$

2. Some singular integrands

As we have remarked the method of stationary phase applies only to regular functions, for if there are singularities Eq. (A4) does not necessarily follow from Eq. (A3).

To find the asymptotic behavior of functions such as $(\sin xt)/|x|^p$ and $(\cos xt)/|x|^p$ that are singular at $x=0$, one uses the methods of the sequential approach to the theory of distributions.[9] Namely, given a sequence of functions one constructs primitives of successively higher orders until one finds a sequence whose limit belongs to the class of equivalence of a known continuous function. After this is found the limit of the original sequence is identified with the distribution that is the appropriate generalized derivative of the continuous function.

Denoting by P the primitive, one finds, for $0<p<1$,

$$P\left(\lim_{t\to\infty}t^{1-p}\frac{\sin xt}{|x|^p}\right)=\begin{Bmatrix}0, & x<0\\0, & x>0\\-c', & x=0\end{Bmatrix}\equiv\{0\},$$

$$P\left(\lim_{t\to\infty}t^{1-p}\frac{\cos xt}{|x|^p}\right)=\begin{Bmatrix}0, & x<0\\2c, & x>0\\c, & x=0\end{Bmatrix}\equiv\{2c\,\theta(x)\},$$

$$c=\frac{\pi}{2\Gamma(p)\cos(p\pi/2)},\quad c'=\frac{\pi}{2\Gamma(p)\sin(p\pi/2)},$$

i.e., the sequence of first-order primitives tends in the first case to a function in the class of equivalence of the zero function and in the second case to an element of the equivalent class of the piecewise continuous $2c\,\theta(x)$ (whose primitive is continuous). Therefore,

$$\lim_{t\to\pm\infty}|t|^{1-p}\frac{\sin xt}{|x|^p}=0, \tag{A6a}$$

$$\lim_{t\to\pm\infty}|t|^{1-p}\frac{\cos xt}{|x|^p}=\frac{\pi}{\Gamma(p)\cos(p\pi/2)}\delta(x). \tag{A6b}$$

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