ON A STOCHASTIC PROCESS ASSOCIATED TO NON-ABELIAN GAUGE FIELDS

R. VILELA MENDES

CFMC – Instituto Nacional de Investigação Científica, Av. Gama Pinto, 2, P-1699 Lisbon Codex, Portugal

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A stochastic process is constructed from a ground state measure that generalizes to non-abelian fields the ground state of abelian (free) gauge fields without fermions. Using a latticized version one shows how the process leads to a well-defined quantum theory in the Schrödinger representation. An analysis of the qualitative behaviour of the theory seems to imply a quasi-free behaviour at short distances and a maximally disordered field strength configuration for the low-momentum component of the ground state. Scaling relations for the mass gap are inferred from the theory of small random perturbations of dynamical systems.

Euclidean fields are defined by a probability measure $d\mu$ on the space of vector-valued distributions $\prod_{\otimes\alpha}\mathscr{D}'(\mathbb{R}^d)$, d being the dimension of space–time and α denoting the tensor and internal degrees of freedom of the fields. The densities S_n of the nth order moments of $d\mu$ are the Schwinger functions. From the path space $L_2(\prod_{\otimes\alpha}\mathscr{D}'(\mathbb{R}^d), d\mu)$ for quantum operators, reflection positivity yields [1] the positivity of the inner product in the Hilbert space where Minkowski fields act, and thus the existence of the Schrödinger representation

$$\mathscr{H}=L_2\left(\prod_{\otimes\alpha}\mathscr{D}'(\mathbb{R}^{d-1}), d\nu\right),$$

where $d\nu$, the measure defined by the ground state of the hamiltonian operator, is the restriction of $d\mu$ to the time $t=0$ subspace.

Conversely, given a measure $d\nu$ on the space of fixed time field configurations one defines the form

$$\mathscr{E}(u, v)=\int Du\cdot Dv\, d\nu$$

where Du denotes the generalized gradient in the space E of $t=0$ field configurations. If $\mathscr{E}$ is a closable form, its closure $\bar{\mathscr{E}}$ is a Dirichlet form [2]. Then, there is a positive definite self-adjoint operator H in $L_2(E, d\nu)$ associated to $\bar{\mathscr{E}}$ by $\bar{\mathscr{E}}(u, v)=(\sqrt{H}u, \sqrt{H}v)$. $T_t=\exp(-tH)$ is a symmetric contraction semigroup and a Markov process $((X_t)_{t\geqslant 0}, (P_z)_{z\in E})$ with state space E will be associated to $\bar{\mathscr{E}}$ if for any bounded measurable function u on E

$$\int u(X_t)dP_z=T_t u(z) \quad \text{for } \mu-\text{a.e. } z\in E .$$

Then $d\nu$ is the invariant measure for $(X_t)_{t\geqslant 0}$. Therefore closability of the form defined by the measure $d\nu$ allows the reconstruction of a Hilbert space and a hamiltonian operator. Schwinger functions are associated to the stochastic correlations of the Markov process X_t.

The construction of the theory through ground state measures and the corresponding Dirichlet forms met a remarkable success in the quantum theory of systems with finitely many degrees of freedom [3,4]. Not only was one able to deal with singular interactions that cannot be consistently described by potentials, but also the stochastic process provides a natural framework for the evaluation of non-perturbative effects [5,6]. The singular nature of quantum fields and the fact that the formal definition an infinite-dimensional theory through hamiltonian or lagrangian densities is an ambiguous specification [#1], makes quantum fields natural candidates for a description through ground state measures. Here one faces however the difficult mathematical problems of closability of infinite dimensional Dirichlet forms and the

[#1] See for example the discussion in ref. [7].

associated existence problem for the infinite-dimensional Markov process. It is only very recently that reasonable progress has been made towards a satisfactory formulation of the necessary tools [8–10].

The existence problem of the Markov process associated to $\bar{\mathscr{E}}$ may be turned around by defining the theory directly in terms of a stochastic differential equation. The hamiltonian is the generator of the process and the problem is then one of proving the existence of solutions and existence of an ergodic invariant measure, i.e., a measure for a unique ground state. This point of view is perhaps the most appropriate one for quantum fields [11].

In this paper, keeping to a minimum the formal mathematical constructions, one discusses questions related to the *ground state measure formulation* of theories of gauge fields.

Consider continuum QED (without fermions) in the Schrödinger picture and in the temporal gauge, $A_0(x)=0$ [12]. The hamiltonian operator is

$$H=\tfrac{1}{2}\int d^3x\left(-\frac{\delta^2}{\delta A_i(x)^2}+B_i(x)^2\right), \tag{1}$$

with the magnetic field $B_i(x)=\epsilon_{ijn}\partial_j A_n(x)$. The Gauss law $\partial_i F^{0i}=0$ is imposed as a constraint, $\partial_i(\delta/\delta A_i(x))\psi[A]=0$, and is equivalent to requiring gauge invariance of the wave functionals.

It is convenient to use both the configuration space fields and their Fourier transforms

$$A_i(k)=\left(\frac{1}{2\pi}\right)^{3/2}\int d^3x\,\exp(-ik\cdot x)A_i(x)\,.$$

In these coordinates the hamiltonian becomes

$$H=\tfrac{1}{2}\int d^3k\left(-\frac{\delta}{\delta A_i(k)}\frac{\delta}{\delta A_i(-k)}+B_i(k)\,B_i(-k)\right), \tag{2}$$

with $B_j(k)=-i\epsilon_{jmn}k_m A_n(k)$.

Up to a constant the hamiltonian of eq. (2) admits two decompositions in the form

$$H+\text{const.}=\tfrac{1}{2}\int d^3k\left(-\frac{\delta}{\delta A_i(k)}+L_i(-k)\right)\times\left(\frac{\delta}{\delta A_i(-k)}+L_i(k)\right), \tag{3}$$

where the $L_i(k)$ are gradients of a functional of the fields, $L_i^{(\alpha)}(k)=-(\delta/\delta A_i(-k))\sigma^{(\alpha)}[\mathrm{A}]$. They are [12–15]

$$L_i^{(1)}(k)=-i\epsilon_{imn}k_m A_n(k)\,, \tag{4a}$$

$$L_i^{(2)}(k)=|k|\,A_i(k)-\frac{k_i k_j}{|k|}A_j(k)\,, \tag{4b}$$

with

$$\sigma^{(1)}[A]=\frac{i}{2}\int d^3k\,\epsilon_{imn}A_i(-k)\,k_m\,A_n(k)\,, \tag{5a}$$

$$\sigma^{(2)}[A]=-\tfrac{1}{2}\int d^3k\left(A_i(-k)\,|k|\,A_i(k)-A_i(-k)\frac{k_i k_j}{|k|}A_j(k)\right). \tag{5b}$$

A decomposition of the type of eq. (3) means that $\phi_{(\alpha)}=\exp(\sigma^{(\alpha)}[A])$ are eigenstates of H. Making the transformation $H\to H'_{(\alpha)}=\phi^{-1}_{(\alpha)}H\phi_{(\alpha)}$ one obtains

$$H'_{(\alpha)}=\int d^3k\left(-\frac{1}{2}\frac{\delta}{\delta A_i(k)}\frac{\delta}{\delta A_i(-k)}+L_i{}^{(\alpha)}(k)\frac{\delta}{\delta A_i(k)}\right), \tag{6}$$

implying that $-H'_{(\alpha)}$ is the generator of a diffusion process

$$dA_i^{(\alpha)}(k)=-L_i^{(\alpha)}(k)\,dt+dW_i(t) \tag{7}$$

with drift $-L_i^{(\alpha)}(k)$ and invariant density $\exp(2\sigma^{(\alpha)}[A])$ (in the space of fixed-time field configurations).

Of course, the correctness of the above statements presumes that one can carry to the infinite dimensional setting of fields the corresponding manipulations of finite dimensional analysis. The justification of these constructions may be carried out using a lattice and approaching the continuum limit through scaling relations [11], or directly by infinite-dimensional techniques [16,17] which, at least in the (free-field) abelian gauge field case, are expected to succeed.

The wave functional $\phi_{(2)}$ is the usual perturbative ground state corresponding to an infinite number of harmonic modes. The corresponding stochastic process describes fluctuations around the $\{A_i(k)=0\}$ configuration, which is an attractive fixed point for the classical (noiseless) motion. By contrast for the process associated to $\phi_{(1)}$ the $\{A_i(k)=0\}$ configuration is an hyperbolic (unstable) fixed point.

These two functionals have an interesting geometrical relation. The drift $-L_i=\delta\sigma^{(\alpha)}/\delta A_i$ associated to any wave functional $\exp(\sigma^{(\alpha)})$, satisfying the Gauss law constraint, must be orthogonal to an arbitrary gauge transformation

$$\delta A_i(k)=\mathrm{i}k_i\,\delta\alpha(k)\ . \tag{8a}$$

One finds that $L_i^{(2)}(k)$ is the exterior product

$$\boldsymbol{L}^{(2)}(k)=\delta\boldsymbol{A}(k)\times\boldsymbol{L}^{(1)}(k)$$

for $\delta\alpha(k)=-1/|k|$.

For non-abelian gauge fields the Gauss law constraint is also equivalent to imposing gauge invariance on the wave functionals, i.e., invariance for

$$\begin{aligned}\delta A_i^\alpha(x)&=(\mathrm{D}_i)_{\alpha\beta}\delta\alpha^\beta\\&=[\partial_i\delta_{\alpha\beta}-gf_{\alpha\gamma\beta}A_i^\gamma(x)]\delta\alpha^\beta(x)\ .\end{aligned} \tag{8b}$$

A drift vector $-L_i^\beta(x)=\delta\sigma/\delta A_i^\beta(x)$ will be orthogonal to an arbitrary $\delta A_i^c(x)$ if $\int\mathrm{d}^3x\, L_i^\beta(\mathrm{D}_i\delta\alpha)^\beta=0$, i.e.

$$\begin{aligned}&(\mathrm{D}_iL_i(x))^\alpha\\&\quad=[\partial_i\delta_{\alpha\beta}-gf_{\alpha\gamma\beta}A_i^\gamma(x)]L_i^\beta(x)=0.\end{aligned} \tag{9}$$

The general solution to eq. (9) is

$$L_i^\alpha(x)=\left(\frac{1}{1-M}\right)_{i\alpha,k\beta}\Gamma_k^\beta\ , \tag{10}$$

where $M_{i\alpha,k\beta}=gf_{\alpha\gamma\beta}(\partial_i/\Delta)A_k^\gamma(x)$ and Γ_k^β is any solution of $\partial_k\Gamma_k^\beta=0$.

Eq. (10) may be used to construct general gauge invariant diffusion processes for non-abelian gauge fields. Here we will limit ourselves to the non-abelian generalizations of $L_i^{(1)}$ and $L_i^{(2)}$ of eqs. (4). For this purpose we rewrite the exponents $\sigma^{(\alpha)}[A]$ of the corresponding wave functionals $(\exp(\sigma^{(\alpha)}))$ in terms of the configuration space fields and exhibiting their gauge transformation properties.

$$\sigma^{(1)}[A]=\tfrac{1}{2}\int\mathrm{d}^3x\,\epsilon_{0ijk}A_i(x)\,\partial_jA_k(x)\ , \tag{5c}$$

$$\begin{aligned}\sigma^{(2)}[A]&=-\frac{1}{4\pi^2}\int\mathrm{d}^3x\,\mathrm{d}^3y\,B_i(x)\,\frac{1}{|x-y|^2}\,B_i(y)\\&=-\tfrac{1}{2}\int\mathrm{d}^3x\,B_i(x)\,\frac{1}{\sqrt{-\Delta}}\,B_i(x)\ ,\end{aligned} \tag{5d}$$

$\sigma^{(1)}[A]$ generalizes easily to the non-abelian case as

$$\sigma^{(1)}[A]=\int\mathrm{d}^3x\,\epsilon_{0ijk}\,\mathrm{Tr}\{\underset{\sim}{A}_i(x)[\partial_j\underset{\sim}{A}_k(x)+\mathrm{i}\tfrac{2}{3}\underset{\sim}{A}_j(x)\underset{\sim}{A}_k(x)]\}\ . \tag{11}$$

$\sigma^{(1)}[A]$ is proportional to the Pontryagin index of the field configuration and leads to a gauge invariant drift even for topologically non-trivial transformations. This "winding number functional" first mentioned by Loos [15], has been discussed by several authors [12,18]. As in the abelian case the $\{A_i^\alpha=0\}$ configuration is an unstable fixed point, and in the continuum formulation with unbounded variables it is not possible to use the corresponding drift

$$-L_{i,\alpha}^{(1)}=\frac{\delta\sigma^{(1)}}{\delta A_i^\alpha}$$

to define a recurrent process with an ergodic ground state measure. However, if the variables are compactified in a lattice version [11], existence (and uniqueness) of the solutions to the stochastic differential equation may be proved, as well as existence of a unique invariant measure and a well defined scaling limit at weak coupling. Although this process is unrelated to the perturbative ground state it nevertheless defines a well-behaved quantum theory.

Here we will concern ourselves with the non-abelian generalization [#2] of $\sigma^{(2)}[A]$ to

$$\sigma^{(2)}[A]=-\tfrac{1}{2}\int\mathrm{d}^3x\,B_k^\alpha(x)\left(\frac{1}{\sqrt{-\mathrm{D}_i\mathrm{D}_i}}\right)_{\alpha\beta}B_k^\beta(x)\ . \tag{12}$$

For eq. (12) to make sense a meaning has to be as-

[#2] Previous attempts [12,19] to generalize the abelian ground state functional $\exp(\sigma^{(2)}[A])$ start from $\sigma^{(2)}[A]=-(1/4\pi^2)\int\mathrm{d}^3x\,\mathrm{d}^3y\,B_k(x)\,|x-y|^{-2}B_k(y)$, and attempt to generalize the kernel $|x-y|^{-2}$. The problems encountered by these authors are probably related to the difficulty in finding a well-defined explicit form for the (symbolic) kernel $(1/\sqrt{-\mathrm{D}_i\mathrm{D}_i})\delta^3(x-y)$.

signed to $(-\mathrm{D}_i\mathrm{D}_i)^{-1/2}$. The abelian version $(-\Delta)^{-1/2}$ is well defined through the Fourier transform. A similar transform does not exist for the nonabelian case because the operator set $\{\mathrm{D}_i, i=1, 2, 3\}$ has a common system of eigenvalues only if A_i^α is a gauge transform of $A_i^\alpha(x)=0$.

Due to the antisymmetry of the structure constants, in the space of vector-valued functions of compact support in $\mathbb{R}^3$ with norm

$$(u, v)=\int \mathrm{d}^3x\, u^\alpha(x)v^\alpha(x)$$

one has $(u, \mathrm{D}_i v)=-(\mathrm{D}_i u, v)$. Therefore one may use the integral representation for fractional powers of positive operators [20] and define

$$\frac{1}{\sqrt{-\mathrm{D}_i\mathrm{D}_i}}=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\frac{1}{\lambda-\mathrm{D}_i\mathrm{D}_i}\,\mathrm{d}\lambda\,. \tag{13}$$

Using (13) in eq. (12) and a formal series expansion of $1/(\lambda-\mathrm{D}_i\mathrm{D}_i)$ to compute the functional derivative one obtains

$$\frac{\delta\sigma^{(2)}}{\delta A_i^\alpha(x)}=-\epsilon_{ijk}(\mathrm{D}_j)_{\alpha\beta}\left(\frac{1}{\sqrt{-\mathrm{D}_s\mathrm{D}_s}}B_k\right)^\beta$$
$$+\frac{g}{\pi}\int_0^\infty \lambda^{-1/2}\left(\frac{1}{\lambda-D_mD_m}B_k\right)^\beta$$
$$\times\left(F_\alpha\mathrm{D}_i\frac{1}{\lambda-\mathrm{D}_s\mathrm{D}_s}B_k\right)^\beta\mathrm{d}\lambda \tag{14}$$

for the drift of the process. The quantities involved in eq. (14) are

$$(\mathrm{D}_i)_{\alpha\beta}=\partial_i\delta_{\alpha\beta}-gf_{\alpha\gamma\beta}A_i^\gamma(x)\,,$$
$$B_k^\alpha=\epsilon_{kmn}[\partial_mA_n^\alpha(x)-\tfrac{1}{2}gf_{\alpha\beta\gamma}A_m^\beta(x)A_n^\gamma(x)]\,,$$
$$(F_\alpha)_{\beta\gamma}=f_{\beta\alpha\gamma}\,.$$

The task now is to obtain a precise meaning for the stochastic process with drift given by (14). As in ref. [11] we use a lattice formulation, define the variables $\theta^\alpha(n, n+\hat{\imath})$ corresponding to the scaled fields $A_i^\alpha(n)$ and make the replacements

$$gaA_i^\alpha(n)\to\theta^\alpha(n, n+\hat{\imath})\,, \tag{15a}$$
$$ga^2B_i^\alpha(n)\to\beta_i^\alpha(n)$$
$$=\tfrac{1}{4}\gamma_{ijk}[\theta^\alpha(n+\hat{\jmath}, n+\hat{\jmath}+\hat{k})$$
$$-\tfrac{1}{2}f_{\alpha\beta\gamma}\theta^\beta(n, n+\hat{\jmath})\theta^\gamma(n, n+\hat{k})]\,, \tag{15b}$$

where $\gamma_{ijk}=(\operatorname{sign} i)(\operatorname{sign} j)(\operatorname{sign} k)\epsilon_{|i|\,|j|\,|k|}$; $i, j, k\in\{1, 2, 3\}$, $\hat{\imath}$ denotes the unit lattice vector along the i-direction and a is the lattice spacing.

On the other hand the covariant derivative of a quantity $v^\alpha(n)$ transforming under the adjoint representation of the internal symmetry group, becomes

$$(\mathrm{D}_i)_{\alpha\beta}v^\beta(n)\to\frac{1}{a}\{\tfrac{1}{2}[v^\alpha(n+\hat{\imath})-v^\alpha(n-\hat{\imath})]$$
$$-f_{\alpha\gamma\beta}\theta^\gamma(n, n+\hat{\imath})v^\beta(n)\}$$
$$=\frac{1}{a}(\mathcal{D}_i)_{\alpha\beta}v^\beta(n)\,. \tag{16}$$

With these definitions and using eq. (13) the lattice version of $\sigma^{(2)}[A]$ is

$$\sigma^{(2)}[\theta]=-\frac{1}{2\pi g^2}\sum_n\int_0^\infty \mathrm{d}\lambda\,\lambda^{-1/2}\beta_k^\alpha(n)$$
$$\times\left(\frac{1}{\lambda-\mathcal{D}_i\mathcal{D}_i}\right)_{\alpha\gamma}\beta_k^\gamma(n)\,, \tag{17}$$

and for the drift of the lattice process we obtain

$$-L(\theta^\alpha(n, n+\hat{\imath}))$$
$$=\frac{1}{\pi a}\int_0^\infty \mathrm{d}\lambda\lambda^{-1/2}\left[-\epsilon_{ijk}(\mathcal{D}_j)_{\alpha\gamma}\left(\frac{1}{\lambda-\mathcal{D}_s\mathcal{D}_s}\beta_k(n)\right)^\gamma\right.$$
$$\left.+\left(\frac{1}{\lambda-\mathcal{D}_s\mathcal{D}_s}\beta_k(n)\right)^\gamma\left(F_\alpha\mathcal{D}_i\frac{1}{\lambda-\mathcal{D}_m\mathcal{D}_m}\beta_k(n)\right)^\gamma\right]. \tag{18}$$

Notice that, for simplicity, we have written for $-L(\theta^\alpha(n, n+\hat{\imath}))$ the lattice version of eq. (14) rather than recomputing $(g^2/a)\delta\sigma^{(2)}[\theta]/\delta\theta^\alpha(n, n+\hat{\imath})$. The two quantities should however coincide in the continuum limit.

We have now two ways to study the lattice model. The first takes the energy form associated to the density $\rho(\theta)=\exp(2\sigma^{(2)}[\theta])$

$$E(u, v)=\int\prod_{n,i,\alpha}\mathrm{d}\theta^\alpha(n, n+\hat{\imath})\,\rho(\theta)\,\mathrm{D}u(\theta)\,\mathrm{D}v(\theta), \tag{19}$$

and proceeds, through the proof of closability, to the existence of a positive self-adjoint generator of time translations and a Markov process.

The second studies directly the stochastic differential equation

$$d\theta^\alpha(n, n+\hat{i})$$
$$= -L(\theta^\alpha(n, n+\hat{i}))\, dt + \frac{g}{\sqrt{a}} dW_i^\alpha(n, n+\hat{i}) , \quad (20)$$

trying to insure existence of solutions and an invariant measure.

In both cases to insure the existence of a well defined Markov process is an essential step to be able to use the (non-perturbative) Wentzell and Freidlin technique [21,7] to infer the continuum behavior through scaling relations for the mass gap.

Before showing that $\exp(2\sigma^{(2)}[\theta])$ as the density of a Dirichlet form (or (18) as the drift of a stochastic differential equation) lead to a well-defined quantum theory, we examine the relevant features of physical significance associated to the ground state wave functional $\exp(\sigma^{(2)}[A])$ (with $\sigma^{(2)}[A]$ as in eq. (12)).

For field configurations for which $A_i^\alpha(x)$ varies slowly in space, the effect of the derivative terms in the denominator of $1/(\lambda - D_i D_j)$ is small and the operator becomes a local one. I.e. in the neighbourhood of these slowly varying field configurations the ground state wave functional factors out into a product $\Pi_x \exp[\Gamma(x)]$ where each $\Gamma(x)$ depends only on the fields at the point x.

$$\Gamma(x) \sim -\tfrac{1}{2} \int_0^\infty d\lambda\, \lambda^{-1/2}$$
$$\times B_k^\alpha(x) \left(\frac{1}{\lambda - g^2 F_\gamma F_\xi A_i^\gamma(x) A_i^\xi(x)} \right)_{\alpha\beta} B_k^\beta(x) .$$

Considering the Fourier transform of $A_i^\alpha(x)$ it is clear that the slow varying configurations are the low-momentum components that control the long-distance behaviour. Hence the main correlations between field configurations at different points in space, imposed by the low-momentum component of the ground state wave functional, are only those that follow from the Bianchi identity. I.e. the long-distance behaviour is associated to a maximally disordered field strength configuration.

Conversely for high-momentum (short distance) the derivative terms in $1/(\lambda - D_i D_i)$ are dominant and the kernel will approach the $1/|x-y|^2$ of the U(1) free theory. From the qualitative limiting behaviour of the operators one already expects the $\exp(2\sigma^{(2)}[A])$ ground state measure to describe both asymptotic freedom at short distances and a maximally disordered situation at low momentum.

The exponent $\sigma^{(2)}[A]$ is negative semi-definite. Therefore the maximum of the (real) ground state functional is reached for the $B_k^\alpha(x) = 0$ configurations, the functional being peaked at all homotopically non-equivalent classical vacua.

For the quantum theory defined by the ground state measure $\exp\{2\sigma[\theta]\}$ (i.e. the lattice version) we now make a few exact statements and sketch their proofs. Whenever consideration of a specific group is called for, SU(2) has, for simplicity, been considered.

Statement 1. For a finite lattice with periodic boundary conditions the ground state measure $\rho[\theta] = \exp\{2\sigma[\theta]\}$ with $\sigma[\theta]$ of eq. (17) characterizes a well-defined quantum theory.

In a finite lattice with periodic boundary conditions

$$(u, \mathscr{D}_i v) = \sum_{n,\alpha\beta} u_k^\alpha(n) (\mathscr{D}_i)_{\alpha\beta} v_k^\beta(n)$$
$$= - \sum_{n,\alpha,\beta} (\mathscr{D}_i)_{\alpha\beta} u_k^\beta(n) v_k^\alpha(n) ,$$

and $-\mathscr{D}_i \mathscr{D}_i$ is a positive operator.

$\sigma^{(2)}[\theta]$ being $\leqslant 0$, $\rho(\theta)$ is bounded.

If μ_0 is the lowest positive eigenvalue of $-\mathscr{D}_i\mathscr{D}_i$ the norm of the operator $(1/\pi)\int_0^\infty \lambda^{-1/2}(\lambda - \mathscr{D}_i\mathscr{D}_i)^{-1} d\lambda$ is bounded by $(\mu_0)^{-1/2}$. However zero eigenvalues of $-\mathscr{D}_i\mathscr{D}_i$ may exist and one worries about possible zeros of $\rho(\theta)$, more precisely about the measure of the singular set $S(\rho)$

$$S(\rho) = \left\{ \{\theta\}: \int \rho^{-1} \sum_{n,\alpha,\hat{i}} d\theta^\alpha(n, n+\hat{i}) = \infty \right.$$
$$\left. \text{for any neighbourhood of } \{\theta\} \right\} ,$$

i.e., the set of field configurations $\{\theta\}$ for which ρ^{-1} is not locally integrable.

Let $v_k^\beta(n)$ be a zero eigenvalue of $-\mathscr{D}_i\mathscr{D}_i$. Then from $(\mathscr{D}_i v, \mathscr{D}_i v) = 0$ it follows that

$$\tfrac{1}{2}[v_k^\alpha(n+\hat{i}) - v_k^\alpha(n-\hat{i})] - f_{\alpha\beta\gamma}\theta^\gamma(n, n+\hat{i}) v_k^\beta(n) = 0$$
$$\forall n, \alpha, k, i . \quad (21)$$

Let the lattice have N^3 sites and the Lie algebra dimensionality be d_L. Eq. (21) is a system of $9N^3d_L$ equations for $3N^3d_L$ variables. Consider the Nd_L-dimensional subsystems associated to N sites along one coordinate direction and fixed k. For generic θ the equations in these subsystems are linearly independent. If (21) has a non-trivial solution then, at least one of the subsystems should have a non-zero solution. This imposes a determinant relation on the θ coordinates that reduces the dimensionality by one unit, at least. I.e. the subset in θ-space associated to a zero eigenvalue of $-\mathscr{D}_i\mathscr{D}_i$ has dimensionality less than $3N^3d_L$.

Therefore, the singular set $S(\rho)$ has zero measure and the density $\rho(\theta)$ satisfies the conditions of theorem 1 in ref. [22]. The form $E(u, v)$ of eq. (19) is closable, there is associated to it a diffusion process and, in the Hilbert space of square-integrable field configurations, its generator defines a positive self-adjoint hamiltonian operator. I.e. a well defined quantum theory is associated to $\rho(\theta)$.

Statement 2. The "constant field configurations" contribution to the Wilson loop of the theory defined in statement 1 has area law decay.

By constant field contribution one means precisely the contribution of the amplitudes

$$\psi_0(\theta) = \exp\Bigg[-\frac{1}{128\pi g^2} \sum_n \int_0^\infty d\lambda\, \lambda^{-1/2} \times \gamma_{ijk}\gamma_{ij'k'} f_{\alpha\beta\gamma}\theta^\beta(n, n+\hat{j})\theta^\gamma(n, n+\hat{k}) \times \left(\frac{1}{\lambda - F_\xi F_{\xi'}\theta^\xi(n, n+\hat{r})\theta^{\xi'}(n, n+\hat{r})}\right)_{\alpha\alpha'} \times f_{\alpha'\beta'\gamma'}\theta^{\beta'}(n, n+\hat{j}')\theta^{\gamma'}(n, n+\hat{k}')\Bigg], \tag{22}$$

which are obtained from eq. (17) in the constant field limit.

One can now either consider spatial Wilson loops and use ψ_0^2 as a dimensionally reduced partition function, or compute the time correlation of two strings, which is the form taken by the space–time Wilson loop in the temporal gauge.

$$\begin{aligned} W(N, T) &= \langle \mathrm{Tr}\{U^\dagger(N, T) \ldots U^\dagger(1, T) U(1, 0) \ldots U(N, 0)\}\rangle \\ &= \langle \mathrm{Tr}\{U^\dagger(N, 0) \ldots U^\dagger(1, 0) \\ &\quad \times \exp(-HT) U(1, 0) \ldots U(N, 0)\}\rangle . \end{aligned} \tag{23}$$

Transforming the hamiltonian

$$H = \frac{g^2}{2a}\sum_{n,i,\alpha}\left(-\frac{\partial}{\partial\theta^\alpha(n, n+\hat{i})} - \frac{\partial\sigma^{(2)}}{\partial\theta^\alpha(n, n+\hat{i})}\right) \times \left(\frac{\partial}{\partial\theta^\alpha(n, n+\hat{i})} - \frac{\partial\sigma^{(2)}}{\partial\theta^\alpha(n, n+\hat{i})}\right) \tag{24}$$

by the string operator one obtains

$$W(n, T) = \langle \mathrm{Tr}\{\exp(-\hat{H}T)\}\rangle , \tag{25}$$

where $\hat{H}$ differs from H by the replacement

$$\frac{\partial}{\partial\theta^\alpha(n, n+\hat{i})} \to U^\dagger(s, 0)\frac{\partial}{\partial\theta^\alpha(n, n+\hat{i})}U(s, 0) ,$$

whenever the link $s \doteq (n, n+\hat{i})$ belongs to the string, and remains unchanged otherwise. To compute eq. (25) one uses (22) with

$$\theta^\alpha(n, n-\hat{i}) = -\theta^\alpha(n, n+\hat{i}) + a\frac{\partial\theta^\alpha}{\partial\hat{i}} ,$$

the contribution of the second term vanishing in the constant field limit. One then finds that the operator in eq. (25) factors out in a product of independent site operators leading to a $\exp(-cNT)$ area law.

Remark. the reason why this is not a confinement result follows from the fact that constant field configurations have zero measure in the space of all configurations. The crucial step missing is, for example, proof that a positive measure neighbourhood of the constant field configurations has the same area law behaviour.

That "constant field configurations" produce a non-trivial contribution to the quantum ground state measure is seen from the following explicit calculation in the SU(2) group:

Consider the matrix $M_{\alpha\beta} = \sum_i \theta_i^\alpha \theta_i^\beta$, $\theta_i^\alpha \doteq \theta^\alpha(n, n+\hat{i})$ $\forall n, \hat{i}$ being the constant field. M may be diagonalized by a global gauge transformation. In the new gauge the three (for SU(2)) three-vectors $\boldsymbol{\theta}^\alpha$ are orthogonal. Without loss of generality, space axis may be chosen such that

$$\theta_i^1=(a_1,0,0),\quad \theta_i^2=(0,a_2,0),\quad \theta_i^3=(0,0,a_3),$$

to which correspond the chromomagnetic fields

$$\beta_i^1=(-a_2a_3,0,0),\quad \beta_i^2=(0,-a_3a_1,0),$$

$$\beta_i^3=(0,0,-a_1a_2)\,.$$

Then

$$\rho(\theta)=\exp\left[-\sum_n\frac{1}{g^2}\left(\frac{a_2^2a_3^2}{\sqrt{a_2^2+a_3^2}}+\frac{a_1^2a_3^2}{\sqrt{a_1^2+a_3^2}}+\frac{a_2^2a_1^2}{\sqrt{a_2^2+a_1^2}}\right)\right]. \tag{26}$$

The last statement pertains to the approach to the continuum limit through scaling relations:

Statement 3. The mass gap of the theory defined in statement 1 scales as $am\sim\exp(-c/g^2)$ when $a\to0$, $g(a)\to0$.

The drift $-L$ (eq. (18)) of the lattice process is independent of g, therefore the model satisfies the necessary condition for the mass gap to have the stated behaviour [7] when $g\to0$. The proof of statement 3 is based on the asymptotics of the lowest eigenvalue of elliptic operators, as obtained from the theory of small random perturbations of dynamical systems [21,7]. The application of the small random perturbations theory is relevant to the continuum limit only if $a\to0$ implies also $g(a)\to0$. We refer to ref. [7] for details on the application of this technique to lattice problems.

A sufficient condition for $am\sim\exp(-c/g^2)$ when $g\to0$ is that the ω-limit set of the deterministic problem

$$\frac{d\theta^\alpha(n,n+\hat{i})}{dt}=-L(\theta^\alpha(n,n+\hat{i}))$$

be attractive in the domain of the Dirichlet problem. This is fulfilled because $\sigma^{(2)}$ is negative definite with maxima in the $\beta=0$ field configurations.

From $am\sim\exp(-c/g^2)$ one sees that for the physical mass gap to remain fixed when $a\to0$, one should require $g^2(a)\sim|c/\log a|$. Therefore when $a\to0$ $g(a)\to0$, consistent with the use of the theory of small random perturbations.

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