Physics Letters A 180 (1993) 39–42
North-Holland

PHYSICS LETTERS A

q-deformed Brownian motion

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Received 23 March 1993; accepted for publication 17 June 1993
Communicated by J.P. Vigier

Brownian motion may be embedded in the Fock space of bosonic free fields in one dimension. Extending this correspondence to a family of creation and annihilation operators satisfying a q-deformed algebra, the notion of q-deformation is carried from the algebra to the domain of stochastic processes. The properties of q-deformed Brownian motion, in particular its non-Gaussian nature and cumulant structure, are established.

The concept of symmetry plays an essential role in the description of physical phenomena. In most cases this symmetry is related to covariance under the transformations induced by a Lie algebra. A generalization of this mathematical structure, the q-deformed (or quantum) algebras, has recently emerged [1–7]. q-deformed algebras, first discovered in the context of integrable lattice models, were later identified as an underlying mathematical structure in topological field theories [8] and rational conformal field theories [9]. Other attempts to apply the notion of q-deformed algebras cover a wide range of different domains, from space–time symmetries [10–12] to gauge fields [13], to quantum chemistry [14].

In view of the actual and potential applications of q-deformation in the context of Lie algebras and superalgebras, it is interesting to ask whether the notion of q-deformation can also be extended to other (non-algebraic) mathematical structures. In this paper we try to extend this notion to stochastic processes. Our starting point is the well-known embedding of Brownian motion in the Fock space of bosonic free fields in one dimension [15,16]. Extending this correspondence to a time family of creation and annihilation operators satisfying a q-deformed algebra we establish a q-deformation of Brownian motion.

q-deformed creation and annihilation operators were defined by several authors [17–20]. They satisfy the algebra

$$aa^\dagger - q^{-1}a^\dagger a = q^N\,, \tag{1a}$$

$$aa^\dagger - qa^\dagger a = q^{-N}, \tag{1b}$$

where N is the number operator,

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a\,. \tag{2}$$

The operators a, $a^\dagger$ may be realized as infinite-dimensional matrices on a vector space by

$$a^\dagger|n\rangle = \sqrt{[n+1]}\,|n+1\rangle, \quad a|n\rangle = \sqrt{[n]}\,|n-1\rangle\,,$$
$$N|n\rangle = n|n\rangle\,, \tag{3}$$

where we used the notation

$$[X] = X_q = \frac{\sinh(X\ln q)}{\sinh(\ln q)}\,, \tag{4}$$

X being a number or an operator. q-deformation of

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single boson operators is invariant under the replacement $q \to q^{-1}$ and we write the algebra in an explicitly symmetric form which will be useful later on,

$$aa^\dagger - \tfrac{1}{2}(q+q^{-1})a^\dagger a = \tfrac{1}{2}(q^N + q^{-N}) \tag{5}$$

(notice that $q+q^{-1}=[2]$).

We now consider a family $\{a_\tau, a_\tau^\dagger\}$ of q-deformed operators labelled by a continuous time parameter and a scalar field

$$\phi_\tau = a_\tau + a_\tau^\dagger . \tag{6}$$

For the family $\{a_\tau, a_\tau^\dagger\}$ we generalize the relations (5) and (2) to

$$a_{\tau_1} a_{\tau_2}^\dagger - \tfrac{1}{2}(q+q^{-1}) a_{\tau_2}^\dagger a_{\tau_1} = \tfrac{1}{2}(q^N + q^{-N})\delta(\tau_1 - \tau_2) , \tag{7}$$

$$[N, a_\tau^\dagger] = a_\tau^\dagger , \quad [N, a_\tau] = -a_\tau . \tag{8}$$

This is the simplest extension of the relations to a family of q-deformed operators labelled by a continuous parameter. Other generalizations of (5) are possible, involving for example braid relations at different times. Notice also that, for our purposes of constructing a stochastic process, no assumptions are needed concerning the commutation properties of $a_{\tau_1} a_{\tau_2}$ and $a_{\tau_1}^\dagger a_{\tau_2}^\dagger$ at different times.

Smearing the fields with characteristic functions $\chi_{[0,t]}$ of the interval $[0, t]$,

$$a_q(t) = a_q(\chi_{[0,t]}) = \int_0^t \mathrm{d}\tau\, a_\tau , \tag{9a}$$

$$a_q^\dagger(t) = a_q^\dagger(\chi_{[0,t]}) = \int_0^t \mathrm{d}\tau\, a_\tau^\dagger , \tag{9b}$$

$$\phi_q(t) = \phi_q(\chi_{[0,t]}) = \int_0^t \mathrm{d}\tau\, \phi_\tau , \tag{10}$$

the algebraic relations become

$$a_q(t_1) a_q^\dagger(t_2) - \tfrac{1}{2}(q+q^{-1}) a_q^\dagger(t_2) a_q(t_1) = \tfrac{1}{2}(q^N + q^{-N}) \langle \tau_1 | \tau_2 \rangle , \tag{11}$$

where $\langle \tau_1 | \tau_2 \rangle = \min(t_1, t_2)$.

We now use (11) to construct a q-deformation of Brownian motion. Let (Ω, F_t, μ, B_t) be the usual Brownian motion. Ω is the set of continuous functions vanishing at $t=0$, μ is the Wiener measure and F_t is the σ-ring generated by $\{B_s\colon 0 \leqslant s \leqslant t\}$. On the other hand let $(\mathscr{H}, A_t, \psi_0, \phi_1(t))$ be the free quantum field over $\mathrm{K} = \mathrm{L}^2([0, \infty), \mathbb{R})$. $\phi_1(t) = \phi_1(\chi_{[0,1]})$ (for $q=1$), $\mathscr{H}$ is the symmetric Fock space over K, ψ_0 is the Fock vacuum and A_t is the W*-algebra generated by $\{\phi_1(s)\colon 0 \leqslant s \leqslant t\}$. Then [16], interpreting B_t as a multiplication operator in $\mathrm{L}^2(\Omega, F_t, \mu)$, there is a unitary operator $V\colon \mathrm{L}^2(\Omega, F_t, \mu) \to \mathscr{H}$ such that $VB_tV^{-1} = \phi(t)$. That is $\phi(t)$ as a stochastic process with expectation

$$E(f(\phi(t))) = \langle \psi_0, f(\phi(t))\psi_0 \rangle \tag{12}$$

coincides with Brownian motion. For this identification of the free scalar field with Brownian motion it is useful to characterize the filtration A_t by the conditional expectation of Wick products [21]

$$E(:\phi_1(u_1) ... \phi_1(u_n): | A_t) = :\phi_1(\chi_{[0,t]} u_1) ... \phi_1(\chi_1(\chi_{[0,t]} u_n): . \tag{13}$$

Recall that the Wick products span the algebra generated by $\phi_1(u)$. Hence, by linearity, definition on Wick products suffices to define conditional expectations on the complete algebra.

We now use a minimal version of this correspondence to define q-deformed Brownian motion.

Definition. q-deformed Brownian motion is the process $(\Omega, F_t, \mu_{\psi_0}, \phi_q(t))$ where

(i) $\phi_q(t)$ is the operator defined by (9)–(11).

(ii) Expectations of field functionals $f(\phi_q)$ are obtained by

$$E(f(\phi_q)) = \langle \psi_0, f(\phi_q)\psi_0 \rangle ,$$

$$\psi_0 \text{ being defined by } a_q\psi_0 = 0 . \tag{14}$$

(iii) The filtration F_t is characterized by the conditional expectations of Wick products

$$E(:\phi_q(t_1) ... \phi_q(t_n): | F_s) = :\phi_q(\chi_{[0,s]}\chi_{[0,t_1]}) ... \phi_q(\chi_{[0,s]}\chi_{[0,t_n]}): . \tag{15}$$

Notice that the algebraic relations (11) allow all elements of the algebra generated by $\phi_q(t)$ to be reduced to Wick products, hence all conditional expectations may be computed. Notice also that in this minimal definition the family F_t of measurable events is fixed in advance and we avoid an explicit realization of the probability space Ω.

Theorem 1. q-deformed Brownian motion
(i) has zero mean, $E(\phi_q(t))=0$;
(ii) has variance $E(\phi_q(t)\phi_q(s))=\min(s,t)$;
(iii) has independent increments in the sense

$$E(\{\phi_q(t_1)-\phi_q(t_2)\}\{\phi_q(t_3)-\phi_q(t_4)\})$$
$$=E(\phi_q(t_1)-\phi_q(t_2))E(\phi_q(t_3)-\phi_q(t_4))=0\,,$$

if there is no overlap between the intervals $[t_1, t_2]$ and $[t_3, t_4]$;
(iv) is a martingale. Properties (i)–(iii) follow by a simple computation using (10), (11) and (14). The martingale property is a consequence of (15). If $s<t$

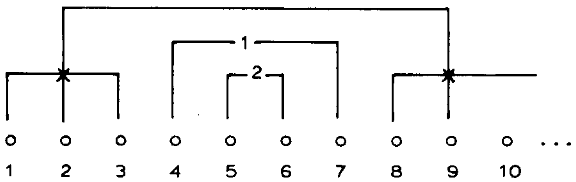
$$E(\phi_q(t)\,|\,F_s)=\phi_q(\chi_{[0,s]}\chi_{[0,t]})=\phi_q(s)\,.$$

Theorem 1 summarizes the similarities of q-deformed Brownian motion to the usual Brownian motion. The next result displays their main differences as well as an explicit characterization in terms of cumulants.

Theorem 2.
(i) q-deformed Brownian motion is not a Gaussian process.
(ii) The cumulants are

$$E_T(\phi_q(t_1)\ldots\phi_q(t_n))$$
$$=\sum c_{(i_1i_2)\ldots(i_{n-1}i_n)}\langle t_{i_1}|t_{i_2}\rangle\ldots\langle t_{i_{n-1}}|t_{i_n}\rangle\,, \qquad (16)$$

where $\langle t_i|t_j\rangle=\min(t_i,t_j)$, the sum is over all $\binom{n}{2}$ different partitions of the set $(t_1\ldots t_n)$ into pairs and the coefficients $c_{(i_1i_2)\ldots(i_{n-1}i_n)}$ are obtained by the following graphical rules.
(a) For each term in eq. (16) one draws the $\langle t_{i_k}|t_{i_{k+1}}\rangle$ contractions as follows,



(17)

(b) For each crossing of lines there is a factor $\frac{1}{2}(q+q^{-1})-1$ in $c_{(i_1i_2)\ldots(i_{n-1}i_n)}$.
(c) For each contraction contained at depth α inside other contractions there is a factor $\frac{1}{2}(q^\alpha+q^{-\alpha})-1$ in $c_{(i_1i_2)\ldots(i_{n-1}i_n)}$.
(d) If there are no crossings nor inner contractions the coefficient $c_{(i_1i_2)\ldots(i_{n-1}i_n)}$ vanishes.

In example (17) the contribution of the diagram to the coefficient is

$$\{\tfrac{1}{2}(q+q^{-1})-1\}^2\{\tfrac{1}{2}(q^2+q^{-2})-1\}$$
$$\times\{\tfrac{1}{2}(q+q^{-1})-1\}\,,$$

the first factor coming from the crossings and the last two from the inner contractions at levels 2 and 1.

A necessary and sufficient condition for a process to be Gaussian is that it possesses cumulants of all orders and that they vanish for orders higher than 2. Computing the four-time correlation one obtains using (11) and (14)

$$E(t_1t_2t_3t_4)=E(\phi_q(t_1)\phi_q(t_2)\phi_q(t_3)\phi_q(t_4))$$
$$=\langle t_1|t_2\rangle\langle t_3|t_4\rangle+\tfrac{1}{2}(q+q^{-1})\langle t_1|t_3\rangle\langle t_2|t_4\rangle$$
$$+\tfrac{1}{2}(q+q^{-1})\langle t_1|t_4\rangle\langle t_2|t_3\rangle\,,$$

implying that the cumulant

$$E_T(t_1t_2t_3t_4)=E(t_1t_2t_3t_4)-E(t_1t_2)E(t_3t_4)$$
$$-E(t_1t_3)E(t_2t_4)-E(t_1t_4)E(t_2t_3)$$

does not vanish. Hence the process is not Gaussian.

The explicit expression for the cumulants of arbitrary orders is obtained by systematic reduction of the expectation values using the algebraic relations (11). The "crossing lines" factor comes from the coefficient of the second term in the l.h.s. of (11) and the "inner contractions" factor from the r.h.s. together with (8).

As a final remark we point out that using the q-fermions b and $b^\dagger$ with

$$bb^\dagger+qb^\dagger b=q^M,$$
$$[M,b^\dagger]=b^\dagger,\quad [M,b]=-b\,, \qquad (18)$$

and a generalization along the lines of (6)–(11) one may construct a q-deformation of the non-commutative Clifford process [21]. We leave the details to the interested reader.

Note added

After this paper was circulated in preprint form we have learned that Bozejko and Speicher [22] have also proposed a deformed generalization of Brownian motion. Unlike the authors, instead of generalizing the q-deformed oscillator, they use as a starting point the quon commutation relations [23] which interpolate between bosons and fermions. The deformed process is therefore different, although some of the ideas are of course related. The authors are grateful to Professor O.W. Greenberg for calling their attention to Bozejko and Speicher's work.

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