Nuclear Physics B161 (1979) 283–293

LARGE TIME BEHAVIOUR OF CANONICAL QUANTIZED GAUGE THEORIES

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Received 20 February 1979
(Revised 15 August 1979)

Techniques of asymptotic analysis are used to obtain the leading large time behaviour of the canonical hamiltonian in the interaction picture. The structure of asymptotic spaces and symmetry realization modes of non-abelian gauge theories is discussed. An explicit (non-local) expression is obtained for the non-abelian Coulomb potential.

1. Introduction

In ref. [1] techniques of asymptotic analysis were used to study the large time behaviour of gauge theories in a covariant context. Through the construction of an asymptotic dynamics (in the sense of Kulish and Faddeev [2]) one characterized the asymptotic spaces and the symmetry realization modes of non-abelian gauge theories.

In this paper a similar analysis is carried out in the canonical quantization formalism. The general results concerning the structure of the asymptotic spaces are, of course, expected to be the same as in ref. [1]. However, because one may have, for the same physical theory, many different asymptotic dynamics (the only condition being that matrix elements at $|t| \to \infty$ are the same as in the exact theory), one might hope to obtain some additional information in the canonical formalism. In fact, the canonical asymptotic analysis yields an explicit expression for the non-abelian static (Coulomb) potential with a rich dynamical non-local structure.

In sect. 2 the basic notations are introduced and a particular splitting of the canonical momentum is specified which is appropriate for the analysis in the interaction picture and yields a simple solution of the constraint equation. The main result in this section is eq. (2.15) for the leading asymptotic behaviour of the static hamiltonian.

In sect. 3, drawing on the results of ref. [1], one discusses briefly the structure of the asymptotic spaces and in sect. 4 one obtains the static kernels (effective potentials) in situations of physical interest.

2. Aymptotic hamiltonian

Let L be the lagrangian of a non-abelian gauge theory in the first-order formalism [3]

$$L = \bar{\psi}(i\partial\!\!\!/ - gT^a B\!\!\!/_a - M)\,\psi + \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \tfrac{1}{2}F^a_{\mu\nu}(\partial^\mu B^\nu_a - \partial^\nu B^\mu_a - gf_{abc}B^\mu_b B^\nu_c)\,. \tag{2.1}$$

Separating L into its canonical parts:

$$L = \sum_i p_i\dot{q}^i - H(p, q) - \sum_a \lambda_a \xi^a(p, q)\,, \tag{2.2}$$

leads to the following hamiltonian density and constraint equation:

$$\begin{aligned}\mathcal{H}(x) = {}& \tfrac{1}{2}\pi^a_i\pi^a_i + \tfrac{1}{4}(\partial_i B^a_j - \partial_j B^a_i - gf_{abc}B^b_i B^c_j)(\partial^i B^j_a - \partial^j B^i_a - gf_{abc}B^i_b B^j_c) \\ & - \bar{\psi}(i\gamma^i\partial_i - M)\,\psi + g\bar{\psi}\gamma^i T^a\psi B^a_i\,,\end{aligned} \tag{2.3}$$

$$[\delta_{ac}\partial_i - gf_{abc}B^b_i]\,\pi^c_i = -g\psi^+T^a\psi\,, \tag{2.4}$$

with B^a_0 playing the role of the set of Lagrange multipliers λ_a.

At this stage the traditional procedure is to split the gauge field canonical momentum

$$\pi^a_i = -F^a_{0i} = -\partial_0 B^a_i + \partial_i B^a_0 + gf_{abc}B^b_0 B^c_i\,, \tag{2.5}$$

into transverse and longitudinal parts,

$$\pi^a_i = \partial_i\phi^a + \pi^{a\,\mathrm{T}}_i = \pi^{a\,\mathrm{L}}_i + \pi^{a\,\mathrm{T}}_i\,, \qquad \text{with } \partial_i\pi^{a\,\mathrm{T}}_i = 0\,,$$

use the constraint equation to eliminate $\pi^{a\,\mathrm{L}}_i$ and a gauge choice to separate the independent canonical variables in the set B^a_i. With this choice however, the relation between ϕ^a and B^a_i in the interaction picture is not straightforward and the asymptotic analysis becomes somewhat more involved. For our purpose it is more convenient to use the natural splitting of $\underset{\sim}{\pi}_i$ following from eq. (2.5) and set:

$$\begin{aligned}&\underset{\sim}{\pi}_i = \underset{\sim}{\pi}^{(1)}_i + \underset{\sim}{\pi}^{(2)}_i\,, \\ &\underset{\sim}{\pi}^{(1)}_i = -\partial_0\underset{\sim}{B}_i\,, \quad \underset{\sim}{\pi}^{(2)}_i = \partial_i\underset{\sim}{B}_0 + g\underset{\sim}{B}_0\times\underset{\sim}{B}_i\,.\end{aligned} \tag{2.6}$$

This and the choice of the radiation gauge * ($\partial_i\underset{\sim}{B}_i = \partial_i\underset{\sim}{\pi}^{(1)}_i = 0$) yield the following

* As recognized by several authors [4] the Coulomb gauge may lead to ambiguities in non-abelian theories. This is, in fact, true for any gauge and is related to the topological obstruction one meets in trying to prolong a local gauge section to a global one [5]. However, as long as one is concerned with small deformations around a classical solution this is not a serious problem because the existence of local gauge sections can be proved [6].

hamiltonian density in canonical form:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_r + \mathcal{H}_s ,$$

$$\mathcal{H}_0 = \bar{\psi}(-i\gamma^i\partial_i + M)\psi + \tfrac{1}{2}\underset{\sim}{\pi}_i^{(1)} \cdot \underset{\sim}{\pi}_i^{(1)} + \tfrac{1}{4}(\partial_i \underset{\sim}{B}_j - \partial_j \underset{\sim}{B}_i)^2 ,$$

$$\mathcal{H}_r = g\bar{\psi}\gamma^i \underset{\sim}{T}\psi \cdot \underset{\sim}{B}_i - \tfrac{1}{2}g(\partial_i \underset{\sim}{B}_j - \partial_j \underset{\sim}{B}_i) \cdot \underset{\sim}{B}_i \times \underset{\sim}{B}_j + \tfrac{1}{4}g^2(\underset{\sim}{B}_i \times \underset{\sim}{B}_j)^2 ,$$

$$\mathcal{H}_s = \tfrac{1}{2}\underset{\sim}{\pi}_i^{(2)} \cdot \underset{\sim}{\pi}_i^{(2)} + \underset{\sim}{\pi}_i^{(1)} \cdot \underset{\sim}{\pi}_i^{(2)} ,$$

where $\underset{\sim}{\pi}_i^{(2)}$ should be understood as standing for a solution of the equation

$$[\partial_i - g\underset{\sim}{B}_i \times]\, \underset{\sim}{\pi}_i^{(2)} = -g\psi^+ \underset{\sim}{T}\psi + g\underset{\sim}{B}_i \times \underset{\sim}{\pi}_i^{(1)} . \tag{2.7}$$

$\mathcal{H}_0$ is the free and $\mathcal{H}_r + \mathcal{H}_s$ the interaction hamiltonian densities. In the interaction picture the fields $\psi(x)$ and $\underset{\sim}{B}_i(x)$ have the following momentum expansions defined for all times:

$$\psi(x) = \sum_{\pm s} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E_p}} [\underset{\sim}{b}(ps)\, u(ps)\, e^{-ip\cdot x} + \underset{\sim}{d}^+(ps)\, v(ps)\, e^{ip\cdot x}] , \tag{2.8a}$$

$$\underset{\sim}{B}_i(x) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \int \frac{d^3k}{(2\omega_k)^2} \epsilon_i(k\lambda) [a(k\lambda)\, e^{-ik\cdot x} + \underset{\sim}{a}^+(k\lambda)\, e^{ik\cdot x}] . \tag{2.8b}$$

The normalization factor in $B_i(x)$ is chosen to make the $\underset{\sim}{a}(k)$ and $\underset{\sim}{a}^+(k)$ operators dimensionless. With $\underset{\sim}{\pi}_i^{(1)} = -\partial_0 \underset{\sim}{B}_i$ and (2.7) the hamiltonian $H(t)$ in the interaction picture becomes a functional of the momentum operators alone.

Denoting by $\underset{\sim}{J}$ the right-hand side of eq. (2.7) ,

$$J^a = -g\psi^+ T^a \psi + g f_{abc} B_i^b \pi_i^{(1)c} ,$$

a solution of (2.7) is

$$\pi_i^{(2)a} = \partial_i \left\{ \left(\frac{1}{1-M} \right)_{ab} \nabla^{-2} J^b \right\} . \tag{2.9}$$

The operator ∇^{-2} is well-defined as an integral operator and the formal matrix $1/(1-M)$ stands for the power series $1 + M + M^2 + \ldots$ with

$$M_{ab} = g f_{acb} \nabla^{-2} B_i^c \partial_i . \tag{2.10}$$

Choosing the solution (2.9) the term $\underset{\sim}{\pi}_i^{(1)} \cdot \underset{\sim}{\pi}_i^{(2)}$ in $\mathcal{H}_s$ may be dropped and $H_s(t) = \int d^3x\, \mathcal{H}_s$ becomes

$$H_s(t) = -\tfrac{1}{2} \int d^3x \left\{ \nabla^2 \left(\frac{1}{1-M} \right)_{ab} \nabla^{-2} J^b(x) \right\} \left\{ \left(\frac{1}{1-M} \right)_{ab'} \nabla^{-2} J^{b'}(x) \right\} . \tag{2.11}$$

Our goal now is to obtain the large time leading asymptotic behaviour of $H(t) = \int d^3x\, \mathcal{H}(x)$. Let us analyse the $H_s(t)$ term. Use the momentum expansions (2.8) to represent $J^b(x)$ as

$$J^b(x) = \int d^3p\, d^3p' \{F_1(p, p')\, e^{i(p-p')x} + F_2(p, p')\, e^{i(p+p')x} + F_3(p, p')\, e^{-i(p+p')x} + F_4(p, p')\, e^{i(p'-p)x}\} . \tag{2.12}$$

Using eqs. (2.8b) and (2.10) and the integral representation of ∇_x^{-2}, one obtains for the F_1F_1 contribution to $H_s(t)$:

$$-\frac{1}{2}(2\pi)^3 \int d^3p\, d^3q \Big\{ \delta_{bb'} \int d^3k \frac{1}{-|k|^2} F_1^b(p, p-k)\, F_1^{b'}(q, q+k) \exp[it[p^0 - (p-k)^0 + q^0 - (q+k)^0]] + 2g f_{bcb'} \int d^3k\, d^3s$$
$$\times \frac{1}{|k+\frac{1}{2}s|^2} B_j^c(s,t) \frac{ik_j}{|k-\frac{1}{2}s|^2} F_1^b(p, p-k-\tfrac{1}{2}s)\, F_1^{b'}(q, q+k-\tfrac{1}{2}s)$$
$$\times \exp[it[p^0 - (p-k-\tfrac{1}{2}s)^0 + q^0 - (q+k-\tfrac{1}{2}s)^0]] + \ldots \Big\} , \tag{2.13}$$

where

$$B_j^c(s,t) = \frac{1}{(2\pi)^{3/2}} \sum_\lambda \frac{1}{(2s^0)^2} \{\epsilon_j(s\lambda)\, a_c(s\lambda)\, e^{-is^0t} + \epsilon_j(-s\lambda)\, a_c^+(-s\lambda)\, e^{is^0t}\} .$$

In the infinite sum, (2.13), a typical nth order term is:

$$-(-i)^n g^n f_{bc'a'} f_{a'c''a''} \ldots f_{a_{n-1}c_nb'} \int d^3k\, d^3s_1 \ldots d^3s_n$$
$$\times \frac{1}{|k + \sum_{i=1}^{n} \frac{1}{2}s_i|^2} B_{j'}^{c'}(s_n, t) \frac{(k + \sum_{i=1}^{n-1} \frac{1}{2}s_i - \frac{1}{2}s_n)_{j'}}{|k + \sum_{i=1}^{n-1} \frac{1}{2}s_i - \frac{1}{2}s_n|^2}$$
$$\times \ldots \times B_{jn}^{cn}(s_1, t) \frac{(k - \Sigma \frac{1}{2}s_i)_{jn}}{|k - \Sigma \frac{1}{2}s_i|^2} F_1^b(p, p-k-\textstyle\sum \frac{1}{2}s_i)\, F_1^{b'}(q, q+k-\sum \frac{1}{2}s_i)$$
$$\times \exp[it\{p^0 - (p-k-\textstyle\sum\frac{1}{2}s_i)^0 + q^0 - (q+k-\sum\frac{1}{2}s_i)^0\}] . \tag{2.14}$$

For each other n there are $n+1$ terms corresponding to different combinations of the minus signs in the s_i's. The leading large time behaviour of each term is obtained by doing n partial integrations on $d|s_1| \ldots d|s_n|$ followed by a residue evaluation of the $|\boldsymbol{k}| \simeq 0$ contribution. This is illustrated below for the 1st order term:

$$-\tfrac{1}{2} g f_{bcb'} \frac{1}{(2\pi)^{3/2}} \int \mathrm{d}\Omega_k \frac{\mathrm{d}k}{kt} \exp[it(\boldsymbol{\upsilon}_p - \boldsymbol{\upsilon}_q) \cdot \boldsymbol{k} + \mathrm{O}(k^2)] \int \mathrm{d}\Omega_s \hat{k}_j \sum_\lambda$$

$$\times \left\{ \frac{\epsilon_j(0^+\lambda)\, a_c(0\lambda)}{\frac{1}{2}(\boldsymbol{\upsilon}_p + \boldsymbol{\upsilon}_q) \cdot \hat{\boldsymbol{s}} - 1} + \frac{\epsilon_j(0^-\lambda)\, a_c^+(0\lambda)}{\frac{1}{2}(\boldsymbol{\upsilon}_p + \boldsymbol{\upsilon}_q) \cdot \hat{\boldsymbol{s}} + 1} \right\} F_1^b(p, p-k)\, F_1^{b'}(q, q+k)\,.$$

The angular integration in $\mathrm{d}\Omega_s$ requires some catuion. For each $\hat{s}$-direction one chooses 2 transverse polarizations $a_c(01)$ and $a_c(02)$. However, in the limit $s \to 0$ transverse polarizations lose operational meaning. One therefore projects out these operators in a fixed basis $\boldsymbol{a} = \{a_x, a_y, a_z\}$ where the z-direction is taken to be the direction of $\boldsymbol{V} = \frac{1}{2}(\boldsymbol{\upsilon}_p + \boldsymbol{\upsilon}_q)$. The $\mathrm{d}\Omega_s$ angular integration can then be carried out and the result is:

$$-\int \mathrm{d}\Omega_s \hat{k}_j \sum_\lambda \{...\} = \pi \{g_1(V)\, \hat{\boldsymbol{k}} + g_2(V)\, \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{V}} \hat{\boldsymbol{V}}\} \cdot \{\boldsymbol{a} - \boldsymbol{a}^+\}\,,$$

where

$$g_1(V) = \left(\frac{1}{V} + \frac{1}{V^3}\right) \ln \frac{1+V}{1-V} - \frac{1}{V^2}\,,$$

$$g_2(V) = \left(\frac{1}{V} - \frac{3}{V^3}\right) \ln \frac{1+V}{1-V} + \frac{6}{V^2}\,.$$

For future applications one registers the values $g_1(0) = \frac{8}{3}$ and $g_2(0) = 0$.

The operators $\boldsymbol{a}$ are the infrared limits of $a(k\lambda)$. For these "infrared boson operators" to satisfy a well-defined algebra the normalization chosen for $B_i(x)$ is essential, because then the dimensionless commutator

$$[a_c(k\lambda), a_b^+(k'\lambda')] = (2\omega_k)^3 \delta^3(k - k')\, \delta_{cb} \delta_{\lambda\lambda'}\,,$$

can, in the limit $k \to 0$, consistently lead to $[a_c^i, a_b^{j+}] = \delta^{ij} \delta_{cb}$.

For the $\mathrm{d}\Omega_k$ angular integration one defines a unit triad $\hat{\boldsymbol{t}}_i$ ($i = 1, 2, 3$) where $\hat{\boldsymbol{t}}_3 \equiv \hat{\boldsymbol{\upsilon}}_\mathrm{r} = \boldsymbol{\upsilon}_p - \boldsymbol{\upsilon}_q / |\boldsymbol{\upsilon}_p - \boldsymbol{\upsilon}_q|$ and $\hat{\boldsymbol{t}}_1$ and $\hat{\boldsymbol{t}}_2$ are two orthogonal directions transverse to the relative velocity $\boldsymbol{\upsilon}_\mathrm{r} = \boldsymbol{\upsilon}_p - \boldsymbol{\upsilon}_q$. After performing the $\mathrm{d}\Omega_k$ angular integration (up to $\mathrm{O}(k^2)$ contributions), a residue evaluation of the pole at $k = 0$ takes care of the $\mathrm{d}k$ integral. The leading large time behaviour of the 1st-order term is therefore,

$$\tfrac{1}{2} f_{bcb'} \frac{\pi^3}{(2\pi)^{3/2}} \frac{1}{|t|} \{g_1(V)\, \hat{\boldsymbol{t}}_3 + g_2(V)\, \hat{\boldsymbol{t}}_3 \cdot \hat{\boldsymbol{V}} \hat{\boldsymbol{V}}\} \cdot \{i a_c - i a_c^+\}\, F_1^b(p, p)\, F_1^{b'}(q, q)\,.$$

For the higher-order (in g) terms of (2.14) the integrations in $\mathrm{d}^3 s$ (at small fixed k) are identical to the 1st-order case. The main difference comes from the angular

$\mathrm{d}\Omega_k$ integration where now even powers of $\cos\phi_k$ and $\sin\phi_k$ lead to non-vanishing contributions for the projections on $\hat{t}_1$ and $\hat{t}_2$. For the $\mathrm{d}k$ integration one uses

$$\int_0^\infty \frac{\mathrm{d}k}{k^n}\int_{-1}^{1}\mathrm{d}x x^p \exp(itv_r kx) = \frac{i\pi\epsilon(t)(itv_r)^{n-1}}{(n+p)(n-1)!}\,,$$

for n and p integers of the same parity.

Finally one obtains for the leading large time behaviour of $H_s(t)$ the result,

$$H_s^a(t) = 8\pi^5\int \mathrm{d}^3p\,\mathrm{d}^3q\,\{F_1^b(p,p)+F_4^b(p,p)\}\frac{1}{v_r|t|}\left\{\mathbb{1}+\tfrac{1}{2}\sum_{\substack{n,k\\(n+k\neq 0)}}\frac{2n+k+1}{(2n+k-1)!}\right.$$

$$\left.\times\frac{(n+k-1)!}{n!k!}\times(\Gamma_1^2+\Gamma_2^2)^n(2\Gamma_3)^k\right\}_{bb'}\{F_1^{b'}(q,q)+F_4^{b'}(q,q)\}\,, \qquad (2.15)$$

where the Γ_i are the matrices

$$(\Gamma_i)_{ab} = -\frac{\pi}{8(2\pi)^{3/2}}gf_{acb}v_r\{g_1(V)\,\hat{t}_i+g_2(V)\,\hat{t}_i\cdot\hat{V}\hat{V}\}\cdot\{i\boldsymbol{a}_c - i\boldsymbol{a}_c^+\}\,, \qquad (2.16)$$

and

$$F_1^c(p,p)+F_4^c(p,p) = -\frac{g}{(2\pi)^3}\left\{\sum_s[\underset{\sim}{b}^+(ps)\,T^c\underset{\sim}{b}(ps)+\underset{\sim}{d}(ps)\,T^c\underset{\sim}{d}^+(ps)]\right.$$

$$\left.-gf_{cde}\sum_\lambda\frac{i}{16\omega_k^3}[a_d^+(k\lambda)\,a_e(k\lambda)-a_d(k\lambda)\,a_e^+(k\lambda)]\right\}\,. \qquad (2.17)$$

In eq. (2.15) account was taken of the fact that the F_1F_4 and F_4F_4 contributions have the same leading asymptotic behaviour as the F_1F_1 contribution. All other contributions (F_2F_2, F_2F_3, F_3F_3) retain, after the d^3s_i and d^3k integrations, exponential time factors which when integrating on d^3p or d^3q would lead to matrix elements of higher order in $1/|t|$.

In the asymptotic analysis of the $\mathcal{H}_r$ part of the hamiltonian one finds that the $1/|t|$ contributions vanish because of the transversality of the canonical gauge fields ★. Therefore eq. (2.15) defines the leading large time behaviour of the full interaction hamiltonian.

3. Asymptotic spaces and symmetry modes

Eq. (2.15), containing the leading (non-integrable) large time part of the hamiltonian, can be used to define the asymptotic dynamics of the theory. As discussed

★ Similar terms in the covariant treatment of ref. [1] did not vanish because of the longitudinal component of $\epsilon_\mu(k\lambda)$ before the $\mu^2\to 0$ limit (see the equations following eq. (2.12) in ref. [1].

at some length in ref. [1] this consists in the construction of an asymptotic space of states which, although different from the actual states of the theory, have the same matrix elements in the limit of large times.

As in ref. [1] the infrared operator $i\boldsymbol{a}_c - i\boldsymbol{a}_c^+$ leads to a family of possible asymptotic vacua

$$|\lambda\rangle = \exp(-i\boldsymbol{\lambda}_c \cdot \boldsymbol{a}_c^+)|\Omega\rangle\,, \tag{3.1}$$

satisfying $(i\boldsymbol{a}_c - i\boldsymbol{a}_c^+)|\lambda\rangle = \dot{\lambda}_c|\lambda\rangle$. The vector $|\Omega\rangle$ is annihilated by $(i\boldsymbol{a}_c - i\boldsymbol{a}_c^+)$ and is formally related to the Fock space vacuum $|0\rangle$ by

$$|\Omega\rangle = \exp(\tfrac{1}{2}\boldsymbol{a}_c^+ \cdot \boldsymbol{a}_c^+)|0\rangle\,. \tag{3.2}$$

However, this representation is merely a formal one, because $|\Omega\rangle$ would be non-normalizable if $|0\rangle$ has a finite norm, revealing that the asymptotic spaces are necessarily outside the interaction picture Fock space.

A basis appropriate for the construction of the asymptotic spaces is obtained by operating with the interaction picture operators $\underset{\sim}{a}^+(k\lambda)$, $\underset{\sim}{b}^+(ps)$, $\underset{\sim}{d}^+(ps)$ on a particular $|\lambda\rangle$, generating a space $V_\lambda = \{|\lambda\rangle, \underset{\sim}{a}^+(k\lambda)\, ...\, \underset{\sim}{b}^+(p_1s_1) \cdots d^+(q_1r_1)\, ...\, |\lambda\rangle\}$. The asymptotic space would then be the subspace $\mathrm{H}_\lambda \subset \mathrm{V}_\lambda$ consisting of the vectors that obey the equation

$$i\frac{\partial}{\partial t}|\psi\rangle = :H_s^a(t):|\psi\rangle\,. \tag{3.3}$$

The situation is illustrated in fig. 1. In the manifold of possible asymptotic vacua one represents by lines the orbits obtained by the gauge group action. These orbits are then basis manifolds for a bundle whose fibers are the linear spaces V_λ. The asymptotic spaces are the time-dependent subspaces H_λ (solutions of eq. (3.3)) or coherent superpositions thereof. Even without solving eq. (3.3) one sees that the general symmetry-breaking conclusions are, as expected, the same as in ref. [1], namely:

(i) Except for the isolated point $\underset{\sim}{\lambda} = 0$, global transformations g of the non-abelian gauge group G map vectors in H_λ on vectors in a different space $\mathrm{H}_{g\lambda}$. Therefore a realization of the gauge theory with H_λ as the asymptotic space corresponds to a spontaneous symmetry breaking mode (induced by the infrared behaviour).

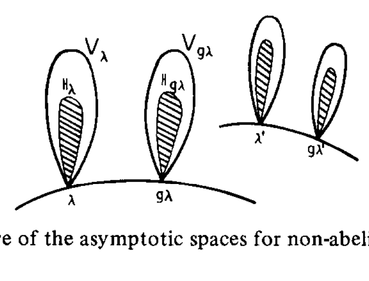

Fig. 1. Structure of the asymptotic spaces for non-abelian gauge theories.

(ii) The second possibility is a symmetric mode obtained by symmetry restoration. In this case instead of H_λ one uses the direct integral spaces $\int d\mu(g)\, H_{g\lambda}$ each vector being an unweighed superposition of group transformed states.

For the asymptotic spaces constructed in ref. [1] we have checked explicitly that in any one of the restored symmetry states the expectation value of any charge component always vanishes, i.e., charges are not observable quantities. This agrees with the general picture of non-abelian superselection rules [7] in gauge theories. In fact Gauss' law in gauge theories forces the group generators to commute with all local observables and their non-abelian nature bars the charges from being observable.

Superselection sectors are labelled by the Casimir operator's eigenvalues and in all but the singlet sector one has more than one vector corresponding to the same physical state. A description of non-singlet physical states must therefore depart greatly from the usual quantum-mechanical set-up, namely in that there is no complete commuting set of observables. Moreover, in the explicit construction of ref. [1] it was found that only the singlet states are eigenvalues of the charges.

However, if one allows for this departure of the usual quantum-mechanical framework, one cannot absolutely exclude the existence of non-singlet states. It is therefore of interest to see whether the asymptotic potentials derivable from eq. (2.15) put any extra constraints on non-singlet states.

Before analysing this question one should notice that, $\boldsymbol{\lambda}_c$ being a vector both in the internal gauge space and in 3-dimensional space, the most general realization of gauge theories could simultaneously imply spontaneous breaking of gauge and rotational (Lorentz) invariance. When, to agree with experimental fact, one applies the mechanism of symmetry restoration to recover rotational invariance, does one render the angular momentum quantities unobservable like the non-abelian charges? The answer, of course, is no, because rotations do not define superselection rules. In an explicit construction as in ref. [1] this corresponds to the fact that, whereas charges and their directions are defined only in relation to the quantization in the internal space, angular momentum directions are defined relative to an external frame tied to the observer *.

4. Effective potentials

Let the vector $\boldsymbol{\lambda}_a$ that labels the coherent vacuum have the form $\boldsymbol{\lambda}_a = \boldsymbol{\mu} l_a$. This form decouples internal and external (space-time) symmetry structures.

Operating on the asymptotic spaces constructed from the $|\lambda\rangle$ vacuum, the opera-

* Kibble [8] was probably the first to point out that the basic operational difference between superselection rules and regular symmetries is tied to the availability of external frames for the conserved quantities.

tor Γ_i becomes

$$(\Gamma_i)_{ab} = -\frac{\pi}{8(2\pi)^{3/2}} g f_{acb} l_c v_r \{g_1(V)\,\hat{t}_i \cdot \boldsymbol{\mu} + g_2(V)\,\hat{t}_i \cdot \hat{V}\hat{V} \cdot \boldsymbol{\mu}\}\,. \tag{4.1}$$

Experimental fact indicates that, whether the internal symmetry group is spontaneously broken or not, rotational invariance is always a good symmetry, i.e., one should always restore this symmetry by integrating the kernel $k(p, q)$,

$$k(p, q) = \frac{1}{v_r|t|}\left\{\mathbb{1} + \frac{1}{2}\sum_{\substack{n,k\\(n+k\neq 0)}} \frac{2n+k+1}{(2n+k-1)!}\,\frac{(n+k-1)!}{n!k!}(\Gamma_1^2+\Gamma_2^2)^n(2\Gamma_3)^k\right\}, \tag{4.2}$$

over all $\boldsymbol{\mu}$ directions. A straightforward computation using (4.1) yields

$$\begin{aligned}
K(p, q) &= \frac{1}{4\pi}\int \mathrm{d}\Omega_\mu\, k(p, q)\\
&= \frac{1}{v_r|t|}\left\{\mathbb{1} + \frac{1}{2}\sum_{\substack{n,k\\(n+k\neq 0)}} \frac{2n+2k+1}{(2n+2k-1)!}\,\frac{(n+2k-1)!}{n!(2k)!}\,\frac{4^k(g_1^2(V)\,\mu^2 M^2)^{n+k}}{(2n+2k+1)!!}\right.\\
&\quad \times \sum_{l=0}^{n}\binom{n}{l}\sum_{i=0}^{l+k}\sum_{j=-i}^{i}\binom{2l}{i+j}\binom{2k}{i-j}\left(1+\frac{g_2(V)}{g_1(V)}|\hat{V}\times\hat{\mathfrak{v}}_r|^2\right)^{i+j}\\
&\quad \times \left(1+\frac{g_2(V)}{g_1(V)}(\hat{V}\cdot\hat{\mathfrak{v}}_r)^2\right)^{2k-i-j}\\
&\quad \times \left.\left(\frac{g_2(V)}{g_1(V)}|\hat{V}\times\hat{\mathfrak{v}}_r|\,\hat{V}\cdot\hat{\mathfrak{v}}_r\right)^{2l-2j}(2l-2i+2k-1)!!(2i-1)!!(2n-2l-1)!!\right\},
\end{aligned} \tag{4.3}$$

where

$$\mu = |\boldsymbol{\mu}|\,, \qquad (M)_{ab} = -\frac{1}{16\sqrt{2\pi}} g f_{acb} l_c v_r\,, \qquad \hat{V} = V/|V|\,, \qquad \hat{\mathfrak{v}}_r = \mathfrak{v}_r/\mathfrak{v}_r\,,$$

$$V = \tfrac{1}{2}(\mathfrak{v}_p + \mathfrak{v}_q)\,, \qquad \mathfrak{v}_r = \mathfrak{v}_p - \mathfrak{v}_q\,.$$

Notice that because of the factors $\binom{2l}{i+j}$ and $\binom{2k}{i-j}$ the sum over j is constrained by $i - 2k \leqslant j \leqslant 2l - i$.

For the spontaneous symmetry breaking mode one chooses a particular fixed l_c and M_{ab} is a non-diagonal matrix.

For the restored symmetry mode one integrates over the hypersphere $|l|$ = const in internal space. The problem consists in finding the isotropic averages of the matrices $(f_{acb'}\hat{l}_c f_{b'c'b}\hat{l}_{c'})^N$ when $\hat{l}$ takes all possible directions in internal space. The trace of $f_{acb'}\hat{l}_c f_{b'c'b}\hat{l}_{c'}$ does not depend on the direction of $\hat{l}$; hence the easiest way to find the isotropic average consists in finding a particular direction (for example 3-direction for SU(2), 8-direction for SU(3)) for which this matrix is a multiple of an idempotent matrix, compute the trace of its Nth power and divide by the dimension of the internal space.

For SU(2) the result is

$$\overline{(f_{acb'}\hat{l}_c f_{b'c'b}\hat{l}_{c'})^N} = (-1)^N \tfrac{2}{3}\delta_{ab}\,, \tag{4.4a}$$

and for SU(3)

$$= (-1)^N (\tfrac{3}{4})^N \tfrac{1}{2}\delta_{ab}\,. \tag{4.4b}$$

The kernel, eq. (4.3), together with (4.4) for the unbroken internal symmetry case gives us the expression for the effective (long-distance) non-abelian Coulomb potential. The general expression (4.3) is fairly complex and so far we have not been able to represent its sums as standard functions. However for special geometrical conditions it simplifies substantially. For instance for two particles in the c.m. frame, i.e., $V = 0$, one obtains

$$K(p, q) = \frac{1}{v_r|t|}\left\{\mathbb{1} + \sum_{n=1}^{\infty}\frac{1}{(2n)!}(2\mu g_1(0)M)^{2n}\right\}. \tag{4.5}$$

Using eqs. (4.4) one finds for the effective non-abelian Coulomb potential in the c.m. frame in the symmetric case:

$$V_{\rm cm} = \frac{1}{r}[a + b\cos(\zeta v_r)]\,, \tag{4.6}$$

where $a = \frac{1}{3}$ $b = \frac{2}{3}$ for SU(2) and $a = b = \frac{1}{2}$ for SU(3).

The non-local factor $\cos(\zeta v_r)$ sums up the long-range effects of charged soft gluons to all orders in g. It therefore improves over previous attempts to obtain the static non-abelian potentials [9]. Although very different from the abelian Coulomb potential (namely in level multiplicity for large values of $g^2\zeta$) and potentially interesting for phenomenological applications, the fact that it is bound by $1/r$ means it cannot provide the same kind of "energetic confinement" of colour triplets that a linear or logarithmically rising potential could. However, this does not entirely rule out the possibility that non-abelian gauge theories might contain a dynamical mechanism favouring colour singlets. Even a potential that decreases for large separations can, through cooperative effects, lead to pair condensation, for example. Further insight in this issue would require the solution of eq. (3.3) with the general kernel (4.3).

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